A Caveat on the Savage-Dickey Density Ratio: The Case of Computing Bayes Factors for Regression Parameters

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Author Note

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Abstract

The Savage-Dickey density ratio is a simple method for computing the Bayes factor for an equality constraint on one or more parameters of a statistical model. In regression analysis, this includes the important scenario of testing whether one or more of the covariates have an effect on the dependent variable. However, the Savage-Dickey ratio only provides the correct Bayes factor if the prior distribution of the nuisance parameters under the nested model is identical to the conditional prior under the full model given the equality constraint. This condition is violated for multiple regression models with a Jeffreys-Zellner-Siow (JZS) prior, which is often used as a default prior in psychology. Besides linear regression models, the limitation of the Savage-Dickey ratio is especially relevant when analytical solutions for the Bayes factor are not available. This is the case for generalized linear models, nonlinear models, or cognitive process models with regression extensions. As a remedy, the correct Bayes factor can be computed using a generalized version of the Savage-Dickey density ratio.

Keywords: Hypothesis test; Bayesian model selection; marginal likelihood; variable selection; JZS prior; general linear model.

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1 Introduction

Bayesian model selection provides many theoretical and pragmatic benefits for hypothesis testing (Wagenmakers, 2007). The Bayes factor, defined as the ratio of the marginal likelihoods of two models, quantifies the relative evidence for one over another model and thus provides an intuitive and principled measure for statistical inference. Besides other advantages, the Bayes factor allows for optional stopping during data collection (Rouder, 2014) and takes the relative complexity of models into account (Myung & Pitt, 1997). Moreover, prior distributions that satisfy certain theoretical requirements (Bayarri, Berger, Forte & García-Donato, 2012; Jeffreys, 1961) allow for a widespread reliance on Bayesian model selection as a default for hypothesis testing in psychology. However, from a practical perspective, computational methods are required to actually approximate the Bayes factor.

The Savage-Dickey density ratio (Dickey & Lientz, 1970) is a particularly simple and widely applicable method for computing Bayes factors that has recently gained popularity (Wagenmakers, Lodewyckx, Kuriyal & Grasman, 2010; Wetzels, Grasman & Wagenmakers, 2010). This computationally simple method can be used for any statistical model if the interest is in testing an equality constraint on one or more of the parameters.[1] The Savage-Dickey density ratio states that *under some conditions* the Bayes factor is equal to the ratio of the prior density and posterior density of the test-relevant parameter at the restriction, for instance, $\theta = 0$. As this simplification avoids solving complicated integrals, it is attractive to use. However, naïvely equating the Savage-Dickey density ratio to the Bayes factor can lead to incorrect results if the necessary conditions are violated. This problem can occur whenever the test-relevant parameters and the nuisance parameters are dependent. Importantly, this is the case in relatively common settings such as in multiple linear regression with default JZS priors (Liang, Paulo, Molina, Clyde

[1] In practice, the Savage-Dickey ratio is almost only used in settings where a single, one-dimensional parameter is tested.

& Berger, 2008) when some covariates are included in both the nested and the more general model. The problem extends to the Bayesian ANOVA (Rouder, Morey, Speckman & Province, 2012), generalized linear models (Li & Clyde, in press), and to cognitive process models with regression structures that link latent processes to behavioural data (see Boehm, Steingroever & Wagenmakers, 2018; Heck, Arnold & Arnold, 2018).

As a remedy, the present paper introduces the generalized Savage-Dickey density ratio proposed by Verdinelli and Wasserman (1995), which provides a density ratio that can be equated to the Bayes factor when the necessary conditions underlying the naïve Savage-Dickey are violated. This paper is organised as follows: Sections 2 and 3 give an overview of the naïve and the generalized Savage-Dickey density ratio, respectively. Section 4 contains the example of multiple linear regression with JZS priors, where the naïve Savage-Dickey density ratio fails to be equal to the Bayes factor, a problem that can be fixed with the generalized Savage-Dickey ratio. The paper is concluded with a discussion highlighting the scope of the issue.

2 The Savage-Dickey Density Ratio

The Savage-Dickey density ratio is a simple and widely applicable method for computing Bayes factors for nested models (for a detailed tutorial, see Wagenmakers et al., 2010). Given a statistical model defined by a likelihood function $f(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{\psi})$ for the data $\boldsymbol{x}$, we are interested in testing an equality constraint on the test-relevant parameters $\boldsymbol{\theta}$, which leads to the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$. For instance, in the regression model below, one is usually interested in testing whether the effect of a covariate on the dependent variable is zero (e.g., $\beta_k = 0$). When testing the equality constraint on the test-relevant parameters $\boldsymbol{\theta}$, the remaining parameters $\boldsymbol{\psi}$ are shared nuisance parameters that are not constrained under the null hypothesis (e.g., the residual variance). The Bayes factor quantifies the evidence for the null hypothesis over the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ and is defined as the ratio of the corresponding marginal likelihoods,

$$B_{01} = \frac{\int f(\boldsymbol{x} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0, \boldsymbol{\psi}) \pi_0(\boldsymbol{\psi}) \, \mathrm{d}\boldsymbol{\psi}}{\int f(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{\psi}) \pi_1(\boldsymbol{\theta}, \boldsymbol{\psi}) \, \mathrm{d}\boldsymbol{\theta} \, \mathrm{d}\boldsymbol{\psi}}, \tag{1}$$

where $\pi_1(\boldsymbol{\theta}, \boldsymbol{\psi})$ is the joint prior distribution of all parameters under the alternative hypothesis, whereas $\pi_0(\boldsymbol{\psi})$ is the prior distribution of the nuisance parameters under the null hypothesis. More generally, the Bayes factor B_{RP} represents the evidence of the data in favour of model $\mathcal{H}_R$ over model $\mathcal{H}_P$.

Often, it is intractable to compute the BF by evaluating the two integrals in Eq. (1) directly. As a remedy, the Savage-Dickey ratio provides a simple means of testing a point null hypothesis of the form $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ by computing the density ratio

$$B_{01} = \frac{\pi_1(\boldsymbol{\theta}_0 \mid \boldsymbol{x})}{\pi_1(\boldsymbol{\theta}_0)}, \tag{2}$$

where the denominator $\pi_1(\boldsymbol{\theta})$ is the marginal prior density of $\boldsymbol{\theta}$ under the full model. Eq. (2) implies that the Bayes factor B_{01} is equal to the ratio of the marginal posterior density of the test-relevant parameters (within the encompassing model) and the marginal prior density evaluated at the restriction. Moreover, the right-hand side of Eq. (2) implies that only information of the encompassing model is required.

Computing the Savage-Dickey density ratio in Eq. (2) is often easier than computing the integrals in Eq. (1). Usually, the prior distributions belong to well-known statistical families, in which case the denominator $\pi_1(\boldsymbol{\theta})$ can easily be evaluated at the restriction $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. Moreover, the numerator $\pi_1(\boldsymbol{\theta} \mid \boldsymbol{x})$ in Eq. (2) is the marginal posterior density of $\boldsymbol{\theta}$ under the full model. Often, it is difficult to derive this density function analytically. As a remedy, the marginal posterior density can be approximated for a wide range of models by drawing posterior samples $(\boldsymbol{\theta}^{(t)}, \boldsymbol{\psi}^{(t)})$ from the full model using Markov chain Monte Carlo (MCMC). In a second step, one can then easily approximate the posterior density based on the MCMC samples $\boldsymbol{\theta}^{(t)}$ at the critical value $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ (e.g., using kernel density estimation or a normal approximation; Wagenmakers et al., 2010; Wetzels, Raaijmakers, Jakab & Wagenmakers, 2009).

Figure 1 illustrates the Savage-Dickey ratio for the special case of testing whether a one-dimensional parameter θ is zero. For example, θ could be a single slope parameter β_1 in a regression model or a mean difference in a t-test (Wetzels et al., 2009). The dashed line shows the marginal prior density $\pi_1(\theta)$ under the full model, which is centred at zero

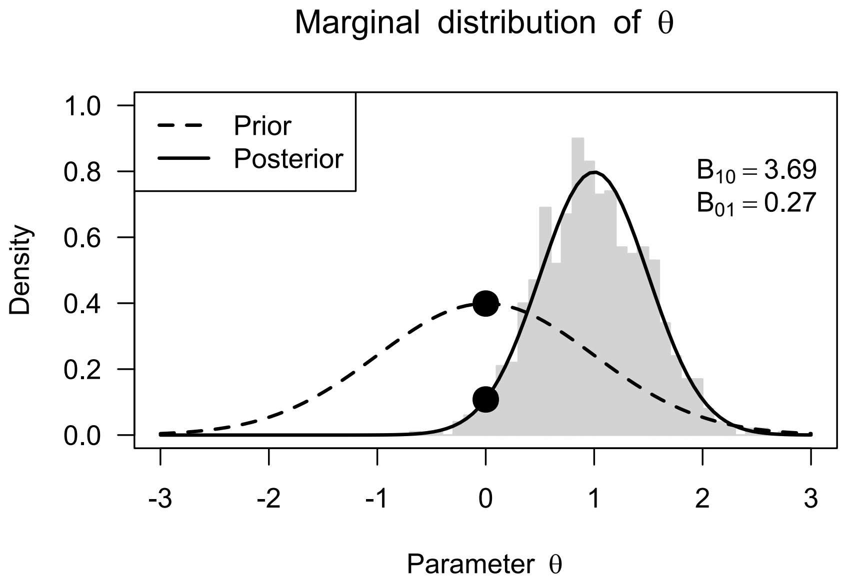

Figure 1. The naïve Savage-Dickey density ratio is defined as the ratio of posterior to prior density (solid and dashed line, respectively) at the critical value $\theta = 0$ (black points). The gray histogram shows the distribution of samples from the posterior distribution of the encompassing model, which are often required to approximate the marginal posterior density.

and has a relatively wide spread. In the t-test and in regression, the marginal prior on θ often is a univariate Cauchy distribution, for which closed-form solutions are available to evaluate the density at the critical point $\theta = 0$ (Ding, 2016). Moreover, the gray histogram in Figure 1 shows the distribution of the posterior samples $\theta^{(t)}$ for the test-relevant parameter from the full model $\mathcal{H}_1$, which are used to approximate the marginal posterior density $\pi_1(\theta \mid \boldsymbol{x})$ (solid line). Compared to the prior, the posterior distribution in Figure 1 is shifted from zero and more peaked, which implies that we gained information about the test-relevant parameter θ from the data. According to the Savage-Dickey ratio, the Bayes factor for $\mathcal{H}_1$ over $\mathcal{H}_0$ is $B_{10} = 0.399/0.108 = 3.69$, the ratio of prior to posterior density at the critical value $\theta_0 = 0$ (black points in Figure 1). This Bayes factor can be interpreted as moderate evidence that θ differs from zero. The illustration in Figure 1 shows that the Savage-Dickey ratio has an intuitive interpretation, because it quantifies the relative support for the constraint $\theta = \theta_0$ before and after observing the data $\boldsymbol{x}$.

3 The Generalized Savage-Dickey Density Ratio

Even though the naïve Savage-Dickey ratio can be applied for many models and scenarios, it is only valid under certain conditions. Specifically, the derivation of the method requires the assumption that the prior distribution of the nuisance parameters $\boldsymbol{\psi}$ under the nested model $\mathcal{H}_0$ is identical to the conditional prior distribution of $\boldsymbol{\psi}$ under $\mathcal{H}_1$ (Verdinelli & Wasserman, 1995):

$$\pi_0(\boldsymbol{\psi}) = \pi_1(\boldsymbol{\psi} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0). \tag{3}$$

Hence, when conditioning on the equality constraint $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, the full model $\mathcal{H}_1$ with the corresponding prior $\pi_1(\boldsymbol{\theta}, \boldsymbol{\psi})$ must reduce to the null hypothesis $\mathcal{H}_0$ with the prior $\pi_0(\boldsymbol{\psi})$ on the nuisance parameters. Put differently, Wagenmakers et al. (2010, p. 182) stated that it is "implicitly assumed that the common nuisance parameters fulfil exactly the same roles, whether they are part of $\mathcal{H}_0$ or $\mathcal{H}_1$." This assumption often holds since $\boldsymbol{\theta}$ is usually defined as an effect parameter (e.g., a difference in means), whereas $\boldsymbol{\psi}$ includes nuisance parameters such as residual variances that fulfil the same roles in both $\mathcal{H}_1$ and $\mathcal{H}_0$. In general, the assumption in Eq. (3) is automatically satisfied if the prior distributions for the test-relevant parameters and the nuisance parameters are independent, $\pi_1(\boldsymbol{\theta}, \boldsymbol{\psi}) = \pi_1(\boldsymbol{\theta})\pi_1(\boldsymbol{\psi})$, but it may not hold when the prior distributions are dependent. Section 4 shows that the assumption is violated in a common scenario in psychology, namely, when testing nested regression models with JZS priors. Hence, the naïve Savage-Dickey density ratio cannot be equated to the Bayes factor in this case.[2]

When the assumption underlying the naïve Savage-Dickey ratio does not hold, other methods are required to compute the Bayes factor. As a remedy, Verdinelli and Wasserman (1995) proposed the generalized Savage-Dicky density ratio, for which the

[2] There are other, measure-theoretic issues with the Savage-Dickey density ratio that have been discussed in the literature. First, Eq. (3) is not well defined according to mathematical measure theory, since conditional densities are not uniquely defined when conditioning on events with probability zero (Marin & Robert, 2010). However, the derivations below do not focus on the conditional probability *density function* but on the conditional *probability distribution* which is well defined (Ding, 2016). Second, the Borel–Kolmogorov paradox can cause the Savage-Dickey ratio to give different results depending on the specific parameterisation of a model (Wetzels et al., 2010) which are not a concern in the present paper.

critical assumption in Eq. (3) does not have to hold. According to this generalized approach, the Bayes factor is computed by multiplying the naïve Savage-Dickey density ratio (i.e., the ratio of the marginal posterior and prior densities under the full model at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$) by a correction term:

$$B_{01} = \frac{\pi_1(\boldsymbol{\theta}_0 \mid \boldsymbol{x})}{\pi_1(\boldsymbol{\theta}_0)} \, \mathrm{E}\left[\frac{\pi_0(\boldsymbol{\psi})}{\pi_1(\boldsymbol{\psi} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0)}\right] \tag{4}$$

where the expectation $\mathrm{E}[\dots]$ is defined with respect to the posterior distribution $\pi_1(\boldsymbol{\psi} \mid \boldsymbol{x}, \boldsymbol{\theta} = \boldsymbol{\theta}_0)$ of the nuisance parameters conditional on the equality constraint. As is evident from Eq. (4), the naïve Savage-Dickey density ratio is obtained as a special case if the assumption for its application in Eq. (3) holds. In this case, $\pi_0(\boldsymbol{\psi}) = \pi_1(\boldsymbol{\psi} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0)$, and thus, the multiplicative correction factor of the generalized Savage-Dickey ratio (i.e., the expectation in Eq. (4)) becomes one and can be dropped.[3]

Whereas the naïve Savage-Dickey density ratio depends only on the encompassing model $\mathcal{H}_1$, the correction factor in Eq. (4) depends on both the encompassing and the nested model $\mathcal{H}_0$. This expectation might be hard to compute analytically, but it can be well approximated with MCMC samples generated under the nested model. In practice, the first term of the generalized Savage-Dickey density ratio is computed identically as the naïve Savage-Dickey ratio by drawing posterior samples from the full model $\mathcal{H}_1$, which are then used to approximate the marginal posterior density at $\boldsymbol{\theta}_0$. In a second step, the approximation of the correction term requires to draw posterior samples $\boldsymbol{\psi}^{(t)}$ $(t = 1, \dots, T)$ from the nested model with the equality constraint $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ (Diciccio, Kass, Raftery & Wasserman, 1997). These samples are then used to obtain a Monte-Carlo estimate of the expectation,

$$\widehat{\mathrm{E}}[\dots] = \frac{1}{T} \sum_{t=1}^{T} \frac{\pi_0(\boldsymbol{\psi}^{(t)})}{\pi_1(\boldsymbol{\psi}^{(t)} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0)}. \tag{5}$$

Overall, the generalized Savage-Dickey density requires samples from both the nested and

[3] Verdinelli and Wasserman (1995) also provided the following (equivalent) representation of the generalized Savage-Dickey ratio: $B_{01} = \pi_1(\boldsymbol{\theta}_0 \mid \boldsymbol{x})\mathrm{E}\left[\pi_0(\boldsymbol{\psi})/\pi_1(\boldsymbol{\theta}_0, \boldsymbol{\psi})\right]$. This version may be easier to implement in practice since it requires only the *unconditional prior* density functions.

the encompassing model, which thus makes computations a bit slower. Note that Verdinelli and Wasserman (1995) also provide an approximation of the error for the resulting Bayes factor.

4 Computing Bayes Factors for Regression Parameters

In the following, the practical importance of checking the assumption underlying the naïve Savage-Dickey ratio is highlighted for the common scenario of multiple regression. The general linear model, which includes regression, the *t*-test, and ANOVA as special cases, is arguably the most important model class in psychology. Usually, the interest is in testing whether one or more of the regression parameters differ from zero (e.g., whether $\beta_k = 0$). To test the effect of a predictor in a regression model that also includes other covariates, it might be tempting to use the naïve Savage-Dickey density ratio for computing the corresponding Bayes factor. However, when choosing default (JZS) priors for the regression parameters (Jeffreys, 1961; Rouder & Morey, 2012; Zellner & Siow, 1980), the necessary assumption in Eq. (3) does not hold and thus the naïve Savage-Dickey method will result in an incorrect approximation of the Bayes factor.

4.1 The Jeffreys-Zellner-Siow (JZS) Prior

In multiple linear regression, the dependent variable $\boldsymbol{y} = (y_1, \ldots, y_N)'$ is modelled as a linear function of the design matrix $\mathbf{X}$, which includes P centred covariates (i.e., the P columns of $\mathbf{X}$ have a mean of zero):

$$y_i = \mu + \mathbf{X}_{i\cdot}\boldsymbol{\beta} + \epsilon_i, \tag{6}$$

where μ is the intercept and $\mathbf{X}_{i\cdot}$ the i-th row of the design matrix. Moreover, it is assumed that the residuals ϵ_i are independent and normally distributed with variance σ^2.

In the following, the main interest is in testing whether one or more (but not all) of the slope parameters differ from zero. For this purpose, the parameters $\boldsymbol{\beta}$ are partitioned into a vector $\boldsymbol{\beta}_K$ with K slope parameters that are constrained to be zero under the nested model and a vector $\boldsymbol{\beta}_R$ with the remaining $R = P - K$ slope parameters, which are freely estimated in both models. Using this notation, the alternative hypothesis states that all parameters have an effect on the dependent variable, and thus, the full, unconstrained model estimates all slope parameters freely, $\mathcal{H}_P : \boldsymbol{\beta} \neq \mathbf{0}_P$ (where $\mathbf{0}_P$ is the P-dimensional zero vector). In contrast, the nested model assumes that the test-relevant slope parameters $\boldsymbol{\beta}_K$ are zero whereas the remaining slope parameters $\boldsymbol{\beta}_R$ are free to vary, $\mathcal{H}_R : \boldsymbol{\beta}_K = \mathbf{0}_K, \boldsymbol{\beta}_R \neq \mathbf{0}_R$. To compare the two regression models, the Bayes factor B_{PR} represents the evidence of the data in favour of the more general model $\mathcal{H}_P$ with P covariates over the nested model $\mathcal{H}_R$ with $R = P - K$ covariates. Moreover, B_{P0} refers to the comparison between the linear regression model $\mathcal{H}_P$ and the global null model $\mathcal{H}_0 : \boldsymbol{\beta} = \mathbf{0}_P$, which assumes that *all* regression coefficients are zero and thus includes only the mean and the residual variance as free parameters.

In Bayesian model selection, the prior distribution on the parameters $(\mu, \sigma^2, \boldsymbol{\beta})$ must be carefully chosen (Jeffreys, 1961). When parameters are included in all models under consideration, they are often referred to as common parameters (Bayarri et al., 2012; Ly, Verhagen & Wagenmakers, 2016a). It can be shown that the prior distribution on such common parameters has a minor effect on the Bayes factor (Jeffreys, 1961; but also see Kass and Vaidyanathan, 1992). In multiple regression, the intercept μ and the residual variance σ^2 are common parameters, because they are included in the full model and also in all nested models that are obtained by dropping covariates (i.e., by constraining one or more of the slope parameters $\boldsymbol{\beta}$ to be zero). Hence, to derive a prior distribution for regression models, it is convenient to define an unconditional prior for the mean and the residual variance, and a conditional prior for the slope parameters,

$$\pi(\mu, \sigma^2, \boldsymbol{\beta}) = \pi(\boldsymbol{\beta} \mid \mu, \sigma^2)\, \pi(\mu, \sigma^2). \tag{7}$$

Using this representation, an improper, noninformative prior is chosen for the mean and the variance, $\pi(\mu, \sigma^2) \propto 1/\sigma^2$ (i.e., Jeffreys' prior; Liang et al., 2008; Rouder & Morey, 2012).

To test whether the subset of test-relevant parameters $\boldsymbol{\beta}_K$ is zero, Bayesian model selection requires a proper prior distribution on the slope parameters $\boldsymbol{\beta}$. When comparing only the nested model $\mathcal{H}_R$ and the full model $\mathcal{H}_P$, it would be sufficient to define a proper prior only on the test-relevant parameters $\boldsymbol{\beta}_K$ and use an improper prior on the remaining slope parameters $\boldsymbol{\beta}_R$. However, when comparing multiple nested regression models or when including the null model $\mathcal{H}_0$ in the model comparison, a proper prior on *all* slope parameters $\boldsymbol{\beta}$ is required (Consonni & Veronese, 2008). Since the prior on $\boldsymbol{\beta}$ determines the distribution of the test-relevant parameters $\boldsymbol{\beta}_K$ under the full model $\mathcal{H}_P$, it has a large impact on the resulting Bayes factor and needs to be chosen carefully (Jeffreys, 1961; Ly, Verhagen & Wagenmakers, 2016b).

A prior that satisfies several theoretical requirements is the Jeffreys-Zellner-Siow (JZS) prior (Jeffreys, 1961; Zellner & Siow, 1980). A detailed introduction and overview of the various advantages of this prior distribution is given by Rouder and Morey (2012) and Liang et al. (2008). Conditional on the residual variance σ^2, the JZS prior defines a multivariate Cauchy distribution for the slope parameters of the full model,

$$(\boldsymbol{\beta} \mid \sigma^2) \sim \mathcal{MVC}\left(\mathbf{0}_P, \gamma^2\sigma^2\mathbf{C}^{-1}\right), \tag{8}$$

which is defined by a location vector (here: the P-dimensional zero vector) and a scale matrix. The amount of scaling depends on the constant γ, which is chosen by the user a priori, the residual variance σ^2, and the matrix $\mathbf{C} = \mathbf{X}'\mathbf{X}/N$, which is the covariance matrix of the centred design matrix $\mathbf{X}$. In line with the general principle that the same type of prior distribution is assumed for each regression model under consideration (Liang et al., 2008; Rouder & Morey, 2012), the nested model $\mathcal{H}_R$ assumes an R-dimensional JZS prior for the non-constrained, shared regression parameters $\boldsymbol{\beta}_R$. By choosing a JZS prior for each regression model, the resulting prior distribution depends only on the design matrix $\mathbf{X}$ and not on the set of other models under consideration.

Some of the benefits of the JZS prior can directly be recognized from the definition in Eq. (8) (Rouder & Morey, 2012). First, the prior is symmetric and centred at zero in line with the predictive matching criterion (Bayarri et al., 2012). Substantively, this implies that positive and negative values of the slope parameters are a priori equally likely. Second, the prior is scale invariant, which means that the resulting Bayes factor is independent of the scaling of both the dependent variable and the covariates, meaning that results do not change if variables are measured in different units. This is achieved by scaling the multivariate Cauchy distribution by the residual variance σ^2 (a priori, a larger residual variance implies larger slopes) and by the inverse of the covariance matrix $\mathbf{C}$ (a priori, a covariate with a larger variance implies smaller slopes). Note that this approach of defining a scaled prior for the unstandardized coefficients $\boldsymbol{\beta}$ is equivalent to defining a prior for the standardized coefficients $\boldsymbol{\beta}^*$ (Rouder & Morey, 2012). Third, the scale parameter γ is fixed to a constant by the user, which allows to specify prior beliefs about the expected effect size. For instance, the `BayesFactor` package (Morey & Rouder, 2015) uses the default $\gamma = \sqrt{2}/4$, which reflects a prior belief of a medium effect size. For a single covariate $\boldsymbol{x}$, this choice implies that the standardized regression slope $\beta^* = \beta \cdot \mathrm{SD}(\boldsymbol{x})/\sigma$ has an a-priori probability of 60.8% to be in the range $[-.50, +.50]$. Rouder and Morey (2012) discuss further theoretical advantages of the JZS prior, including consistency in model selection (i.e., the Bayes factor in favour of the data-generating model goes to infinity as the number of observations N increases without bound) and consistency in information (i.e., the Bayes factor for an effect goes to infinity as the proportion of explained variance R^2 increases to one). Besides these theoretical advantages, Bayes factors for JZS priors can be computed with high precision relatively easily (Morey & Rouder, 2015) and have been adapted for the default t-test (Rouder, Speckman, Sun, Morey & Iverson, 2009) and ANOVA (Rouder et al., 2012).

4.2 The Savage-Dickey Density Ratio for JZS Priors

In regression models with JZS priors, the naïve Savage-Dickey density ratio cannot be used to test whether one or more of the predictors have an effect if the model also includes other covariates that are not tested. The problem arises for the Bayes factor B_{RP}, that is, when testing the full model $\mathcal{H}_P$ with $P \geq 2$ predictors against a nested model $\mathcal{H}_R : \boldsymbol{\beta}_K = \mathbf{0}_K$, which assumes that K parameters are zero (with $K < P$). Since it is clear that the null hypothesis is nested in the full model, it might be tempting to use the naïve Savage-Dickey ratio for computing the Bayes factor as proposed by Boehm et al.

(2018). However, with a JZS prior on $\boldsymbol{\beta}$, the test-relevant parameters $\boldsymbol{\beta}_K$ and the remaining slope parameters $\boldsymbol{\beta}_R$ are dependent. Such a dependence of the test-relevant and the nuisance parameters is a general warning sign that the condition in Eq. (3) may be violated (Wagenmakers et al., 2010).

In the present scenario, the necessary assumption underlying the naïve Savage-Dickey ratio in Eq. (3) is indeed violated, since the prior of the shared slope parameters $\boldsymbol{\beta}_R$ under the nested model $\mathcal{H}_R$ differs from the conditional prior for these parameters under the full model $\mathcal{H}_P$. To see why this is the case, recall that the full model assumes a P-dimensional JZS prior on all slope parameters $\boldsymbol{\beta}$, whereas the nested model $\mathcal{H}_R$ assumes an R-dimensional JZS prior for the non-constrained slope parameters $\boldsymbol{\beta}_R$. When checking the necessary assumption for the application of the naïve Savage-Dickey density ratio, we see that the prior on the parameters $\boldsymbol{\beta}_R$ under the nested model $\mathcal{H}_R$ (i.e., the left hand of Eq. (3)) is given by the R-dimensional Cauchy distribution

$$(\boldsymbol{\beta}_R \mid \mathcal{H}_R, \sigma^2) \sim \mathcal{MVC}(\mathbf{0}_R, \gamma^2\sigma^2\mathbf{C}_R^{-1}), \tag{9}$$

where $\mathbf{C}_R = \mathbf{X}_R'\mathbf{X}_R/N$ is the covariance matrix of the reduced design matrix $\mathbf{X}_R$ under the null hypothesis (which has $R = P - K$ columns).[4] However, on the right side of Eq. (3), the distribution of the remaining regression parameters $\boldsymbol{\beta}_R$ conditional on the equality constraint $\boldsymbol{\beta}_K = \mathbf{0}_K$ under the full model $\mathcal{H}_P$ is *not* a multivariate Cauchy distribution. As derived in the Appendix, the conditional prior is a multivariate t-distribution with $\text{df} = 1 + K$ degrees of freedom and a different scaling parameter,

$$(\boldsymbol{\beta}_R \mid \mathcal{H}_P, \boldsymbol{\beta}_K = \mathbf{0}_K, \sigma^2) \sim \mathcal{MVT}_{\text{df}=1+K}(\mathbf{0}_R, \tfrac{1}{1+K}\gamma^2\sigma^2\mathbf{C}_R^{-1}). \tag{10}$$

Figure 2 shows the probability density function of the Cauchy distribution (i.e., the JZS prior) in Eq. (9) and the t-distribution in Eq. (10) for one- and two-dimensional slope parameters $\boldsymbol{\beta}_R$, and for different number of equality constraints K. Compared to the JZS prior, the conditional t-distribution puts more probability mass on parameters

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[4] In Eq. (9) and (10), Jeffreys' prior $\pi(\mu, \sigma^2) \propto 1/\sigma^2$ is omitted to facilitate notation and readability.

close to zero, an effect that increases with the number of equality constraints K. This is due to the fact that the multivariate Cauchy distribution is a special case of the multivariate t-distribution with $\text{df} = 1$ (Ding, 2016). As the degrees of freedom increase to $\text{df} = 1 + K$, the t-distribution becomes more peaked. Moreover, whereas the expectation does not exist for the Cauchy distribution, the expectation does exist for the multivariate t-distribution with $\text{df} \geq 2$. Besides the larger degrees of freedom, the scaling matrix in Eq. (10) decreases by the factor $1/(1 + K)$, which amplifies the concentration of the probability mass close to zero. Overall, this comparison shows that the JZS prior in Eq. (9) clearly differs from the conditional t-distribution in Eq. (10), and thus, the assumption underlying the naïve Savage-Dickey ratio does not hold.

If the naïve Savage-Dickey density ratio is applied despite the violation of the assumption in Eq. (3), this results in an incorrect approximation of the Bayes factor. In this case, both upward and downward biases are possible as shown in the next section. Moreover, the error propagates if the naïve Savage-Dickey ratio is applied for the stepwise comparison of a set of nested linear models with JZS priors. For instance, given a set of regression models $\mathcal{H}_Q \subset \mathcal{H}_R \subset \mathcal{H}_P$ (e.g., linear models with one, two, and three predictors, respectively), the pairwise application of the naïve Savage-Dickey ratio leads to incorrect approximations of the Bayes factors B_{QR}, B_{RP}, and B_{QP}. Moreover, these three incorrect approximations will in general be inconsistent among themselves, meaning that the naïve Savage-Dickey ratios will violate the transitivity property of Bayes factors (i.e., $B_{QP} = B_{QR} \cdot B_{RP}$). This is due to the fact that the bias of the two Bayes factors B_{QR} and B_{RP} for testing $K = 1$ equality constraints will in general differ from the overall bias of the Bayes factor B_{QP} for testing $K = 2$ equality constraints. In sum, the incorrect application of the naïve Savage-Dickey density ratio for testing regression parameters with JZS priors results in severe consequences for statistical inference.

As a remedy, the generalized version of the Savage-Dickey ratio allows to compute the Bayes factor for nested regression models with JZS priors (Verdinelli & Wasserman, 1995). For this purpose, it is necessary to approximate the multiplicative correction term (i.e., the expectation in Eq. (5)) by drawing posterior samples $\boldsymbol{\psi}^{(t)} = (\mu^{(t)}, \sigma^{(t)}, \boldsymbol{\beta}_R^{(t)})$ from

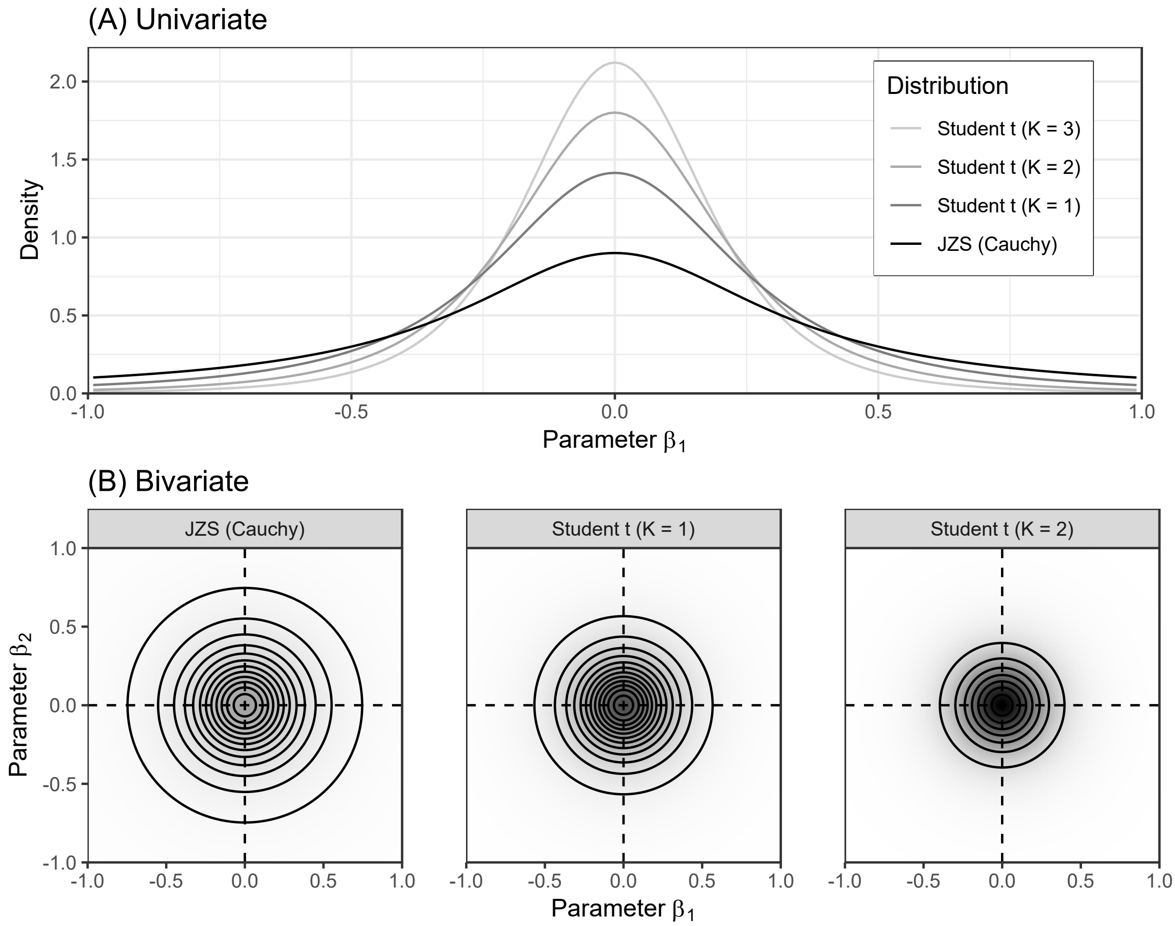

Figure 2. Comparison of different priors for slope parameters $\boldsymbol{\beta}_R$ that are not constrained when testing a hypothesis of the form $\boldsymbol{\beta}_K = \mathbf{0}_K$. Whereas the nested model $\mathcal{H}_R$ assumes a (univariate or bivariate) JZS prior, the full model $\mathcal{H}_P$ implies a t-distribution when conditioning on the K equality constraints on the test-relevant parameters. For all distributions, the scale parameter is set to $\gamma = \sqrt{2}/4$ (medium scale), the residual variance to $\sigma^2 = 1$, and the covariance of the predictors to the identity matrix $\mathbf{C}_R = \mathbf{1}_{R\times R}$.

the nested model $\mathcal{H}_R$ with the equality constraint $\boldsymbol{\beta}_K = \mathbf{0}_K$ (Diciccio et al., 1997). These samples are then used to evaluate the probability density function π_0 of the R-dimensional Cauchy distribution in Eq. (9) and the conditional density of the R-dimensional t-distribution with $\text{df} = 1 + K$ degrees of freedom in Eq. (10). Note that the vector of nuisance parameters $\boldsymbol{\psi}$ of the two models does not only include the shared slope parameters $\boldsymbol{\beta}_K$, but also the mean μ and the residual variance σ^2. However, the noninformative prior distribution on these parameters, $\pi(\mu, \sigma^2) \propto 1/\sigma^2$, is identical in both models and thus cancels out in the ratio in Eq. (5).

4.3 Comparison of Different Methods for Computing Bayes Factors

This section provides an empirical example showing that the naïve Savage-Dickey ratio results in an incorrect approximation of the Bayes factor for models with JZS priors (Rouder & Morey, 2012). The example focuses on standard linear regression models, since accurate Bayes factors are available as a reference (Liang et al., 2008; Morey & Rouder, 2015). As an illustration, we follow Rouder and Morey (2012) in reanalyzing a data set by Bailey and Geary (2009), which includes information about $N = 175$ hominid crania with an age between 1.9 million and 10 thousand years. Using a multiple linear regression, we test how the evolutionary development of the cranium capacity is affected by four covariates (i.e., local climate, global average temperature, parasite load, and population density). For the present example, the Bayes factor for a null effect of parasites was computed for a JZS prior with different scale parameters γ given that (1) three other predictors were already included in the regression (i.e., using model $\mathcal{H}_4$ with all four covariates as the encompassing model) and (2) one other predictor was already included (i.e., using model $\mathcal{H}_2$ with the two covariates parasites and population density as encompassing model).

As a first method, the `BayesFactor` package (Morey & Rouder, 2015) was used to obtain highly accurate Bayes factors for the linear regression models. The package relies on one-dimensional numerical integration for the mixture representation of the JZS prior by Liang et al. (2008) and thus served as a baseline for the remaining methods. Second, the Bayes factor was (incorrectly) approximated using the naïve Savage-Dickey ratio in Eq. (2), even though the necessary assumption for its application was violated. To approximate the marginal posterior density of β_1 (i.e., the effect of parasites), the software Stan (Carpenter et al., 2017) was used to draw posterior samples from the full model $\mathcal{H}_1$ (using 5,000 iterations from 8 chains after warmup). Third, the generalized Savage-Dickey ratio in Eq. (4) (Verdinelli & Wasserman, 1995) was computed as discussed above using posterior samples from both the unconstrained and the equality-constrained model. Note that this method has not yet been implemented in a package and thus requires a customized implementation of the Monte-Carlo estimate for the correction factor. As a forth method, the marginal likelihoods in Eq. (1) were approximated directly using warp-III bridge sampling (Gronau, Wagenmakers, Heck & Matzke, 2017; Meng & Schilling, 2002). This method is available via the R package `bridgesampling` (Gronau, Singmann & Wagenmakers, 2018), which only requires the fitted Stan objects of the nested and full model to approximate the Bayes factor. The R code to replicate all analyses is available in the supplementary material at the Open Science Framework (https://osf.io/5hpuc/).

Table 1 shows the resulting approximations of the Bayes factor for the JZS prior as well as the mean computation times for each method.[5] To assess the bias of the naïve Savage-Dickey ratio for different prior distributions, the analysis was repeated for different scale parameters γ. Note that the `BayesFactor` package uses scale parameters of $\sqrt{2}/4$, $1/2$, and $\sqrt{2}/2$ as a default for continuous predictors, which are often interpreted as medium, wide, and ultrawide effect sizes, respectively (Morey & Rouder, 2015). As expected, the generalized Savage-Dickey ratio (Verdinelli & Wasserman, 1995) and bridge sampling resulted in approximations of the Bayes factor that were in agreement with the highly accurate results of the `BayesFactor` package. Since the latter method relied on an efficient one-dimensional integration, computation times were much faster than for the generalized Savage-Dickey ratio or bridge sampling, which both required to draw posterior samples from two regression models. However, whereas the naïve Savage-Dickey ratio used only samples from the full model (which resulted in faster computation times), it provided an incorrect approximation of the Bayes factor. When testing the effect of parasites given that three other covariates were already included in the model (upper part of Table 1), the naïve Savage-Dickey ratio for the nested model versus the full model $\mathcal{H}_4$ showed a downward bias when $\gamma \leq \sqrt{2}/2$ and an upward bias otherwise. Moreover, when testing the effect of parasites given that only one other covariate was already included in the model (i.e., population density), the naïve Savage-Dickey ratio showed a downward bias for scale parameters up to $\gamma \leq 1$.

[5] The R script was not optimized for maximum speed. Therefore, the computation times (measured on an Intel i7-2600) should be interpreted with caution and do not necessarily generalize to other implementations of the methods.

To understand why the naïve Savage-Dickey ratio in Table 1 can be biased in either direction, it is instructive to compare the numerator and denominator in the correction term of the generalized Savage-Dickey ratio in Eq. (5). Essentially, the bias of the naïve Savage-Dickey ratio depends on the ratio between the JZS prior under the nested model $\mathcal{H}_R$ and the conditional t-distribution under the full model $\mathcal{H}_P$. The correction factor will be *larger* than one if the density based on the posterior samples $\boldsymbol{\beta}_R^{(t)}$ is on average higher for the JZS prior than for the conditional t-distribution. Since the JZS prior is wider than the t-distribution, this will be the case if the slope parameters in the nested model $\mathcal{H}_R$ are estimated to be relatively large. In contrast, the correction factor will be *smaller* than one if the density based on the posterior samples $\boldsymbol{\beta}_R^{(t)}$ is on average higher for the conditional t-distribution than for the JZS prior. Since the conditional t-distribution is more concentrated around zero, this will be the case if the slope parameter in the nested model $\mathcal{H}_R$ are estimated to have a small to negligible effect.

This line of reasoning shows that the direction of the bias depends on whether the JZS prior or the t-distribution results in a higher density for the posterior samples of the common slope parameters $\boldsymbol{\beta}_R$. For a given data set, a similar effect also occurs when varying the scale parameter γ: The conditional t-distribution will provide a better match than the JZS prior if γ assumes larger effects than observed (since the more concentrated t-distribution results in higher density values), but a worse match if γ is adequate for the observed effect size or assumes smaller effects than observed (since the more concentrated t-distribution results in smaller density values). Accordingly, the naïve Savage-Dickey ratio will in general overshoot the correct Bayes factor for the nested model $\mathcal{H}_R$ if γ is large and undershoot the correct Bayes factor if γ is small. Note that this is exactly the pattern that was observed for the smallest and largest scaling parameters $\gamma = \sqrt{2}/4$ and $\gamma = 1.5$ in Table 1. However, even though these theoretical considerations help to understand the qualitative nature of the discrepancy, it is not clear a priori for which scaling parameter γ the direction reverses.

Overall, the example showed that the naïve Savage-Dickey ratio results in an incorrect approximation of the Bayes factor for regression models with default (JZS) priors. If the naïve Savage-Dickey ratio is used nevertheless, this can result both in a downward or in an upward bias relative to the correct Bayes factor depending on the specific data set and the scale parameter γ. Importantly, the discrepancy is not negligible and can make the difference between "weak" and "moderate" evidence for a hypothesis according to the (arbitrarily chosen) verbal labels that are commonly used to interpret the size of Bayes factors (Jeffreys, 1961; Wagenmakers, 2007). Moreover, the present example of testing $K = 1$ slope parameter provides a lower bound for the error of the naïve Savage-Dickey method, since the discrepancy between the JZS prior and the conditional t-distribution increases for a larger number of equality constraints K (cf. Figure 2).

5 Discussion

The Savage-Dickey ratio provides a simple and intuitive approach for computing the Bayes factor for an equality constraint $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. However, the results are valid only if the prior distribution for the nuisance parameters $\boldsymbol{\psi}$ under the null hypothesis is identical to the conditional prior distribution under the full model given the equality constraint $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ (Eq. (3); Dickey & Lientz, 1970; Wagenmakers et al., 2010; Wetzels et al., 2010). This condition is automatically fulfilled whenever the test-relevant parameters are independent of the nuisance parameters. However, one should start to worry when the nuisance and test-relevant parameters are dependent. In this case, the condition in Eq. (3) may be violated, implying that the naïve Savage-Dickey density ratio cannot be used as an approximation of the Bayes factor. As a remedy, the generalized Savage-Dickey density ratio in Eq. (4) (Verdinelli & Wasserman, 1995) provides a correction term to approximate the Bayes factor.

Section 4 showed that the condition underlying the naïve Savage-Dickey density ratio is violated in a common scenario in psychology, that is, when testing whether one or more covariates have an effect in multiple regression models with default JZS priors (Rouder & Morey, 2012). It is not obvious that the assumption underlying the naïve Savage-Dickey ratio is violated in this case. For instance, Boehm et al. (2018) proposed the naïve Savage-Dickey ratio as a general method for testing the effect of one or more predictors in a multiple regression. In explaining the method, they correctly remarked that "the exact expression for the alternative hypothesis depends on the *marginal* prior for [the] standardized effect size under consideration, which in our case is a univariate Cauchy distribution" (p. 9, emphasis added). However, Boehm et al. (2018) did not check whether the *conditional* prior distribution on the nuisance parameters under the full model (i.e., those parameters that are shared by the nested model) is again a multivariate Cauchy of lower dimensionality (i.e., whether the right-hand side of Eq. (3) is the JZS prior). As shown in Section 4.2, this is not the case. Instead, the remaining, non-constrained regression parameters follow a multivariate t-distribution with degrees of freedom depending on the number of equality constraints that are tested. Hence, in this common scenario, the naïve Savage-Dickey density ratio is not equal to the Bayes factor.

In sum, the naïve Savage-Dickey ratio cannot be used to approximate the Bayes factor for nested regression models with JZS priors when other covariates are also included in the model. However, the method provides valid results when testing all regression parameters in a model at once. In this case, one is interested in the Bayes factor B_{P0} in favour of the full versus the null model with the equality constraint $\mathcal{H}_0 : \boldsymbol{\beta} = \mathbf{0}_P$. Since the null model includes only the mean μ and the residual variance σ^2 as free parameters, the necessary assumption for the naïve Savage-Dickey ratio is satisfied. However, to compute this density ratio in practice, the test of all P parameters would require to approximate the possibly high-dimensional posterior density, which is usually intractable (but see Morey, Rouder, Pratte & Speckman, 2011). Moreover, it is important to keep in mind that the assumption underlying the naïve Savage-Dickey may be satisfied in regression models with prior distributions other than the JZS prior. For instance, Zellner's (1980) g-prior assumes a multivariate normal instead of a Cauchy distribution for the regression parameters in a linear model. Aside from the difference in the distributional family, the g-prior has a similar structure as the JZS prior in Eq. (8) (in fact, the JZS prior is obtained as a mixture of g-priors; Liang et al., 2008). For the g-prior, the assumption underlying the naïve Savage-Dickey ratio holds, since the conditional distribution of a multivariate normal random variable is also a multivariate normal distribution (Consonni & Veronese, 2008; Ding, 2016). Thus, the naïve Savage-Dickey ratio results in a correct approximation of the Bayes factor for regression models with a g-prior.

5.1 Relevance for Models with JZS Priors

The assumptions underlying the naïve Savage-Dickey ratio are also important when computing Bayes factors for ANOVA (Rouder et al., 2012). As usual, the $P+1$ levels of a discrete, fixed-effects factor are encoded in the design matrix $\mathbf{X}$ of the general linear model via P orthogonal sum-to-zero contrasts. Moreover, for the Bayesian analysis, these contrasts should have equal variances to ensure that the prior remains symmetric and exchangeable (e.g., by scaling each contrast to have a variance of $1/P$; Rouder et al., 2012). For instance, Helmert contrasts for five levels are specified via

$$\mathbf{Q}_5 = \begin{pmatrix} 0.894 & 0 & 0 & 0 \\ -0.224 & 0.866 & 0 & 0 \\ -0.224 & -0.289 & 0.816 & 0 \\ -0.224 & -0.289 & -0.408 & 0.707 \\ -0.224 & -0.289 & -0.408 & -0.707 \end{pmatrix}. \tag{11}$$

Rouder et al. (2012) proposed a multivariate JZS prior on the corresponding parameters $\boldsymbol{\beta}$ similar as in multiple regression (Eq. (8) with $\mathbf{C} = \mathbf{I}_P$ being the identity matrix). Besides the overall test $\mathcal{H}_0 : \boldsymbol{\beta} = \mathbf{0}_P$, researchers are often interested in testing one or more of the contrasts separately. For instance, with respect to the Helmert contrasts in Eq. (11), the hypothesis $\mathcal{H}_R : \beta_4 = 0$ tests whether the means of the forth and fifth group are identical. Again, the naïve Savage-Dickey ratio cannot be used to compute the Bayes factor in this scenario, because the necessary assumption for its application are not met.

To overcome the limitation of the naïve Savage-Dickey ratio for the general linear model with JZS priors, one can resort to alternative approaches such as numerical integration (Liang et al., 2008). However, computational methods such as the Savage-Dickey ratio are especially relevant when approximating the Bayes factor for more complex models that assume non-linearity or non-normality of the residuals. Most prominently, generalized linear models allow for non-normal residual distributions of the dependent variable and assume a transformation of the linear predictor via a monotonic link function g (Nelder & Wedderburn, 1972),

$$\mathrm{E}[y_i] = g(\mu + \mathbf{X}_{i\cdot}\boldsymbol{\beta}). \tag{12}$$

For instance, logistic regression assumes that the responses y_i are binomially distributed with an expected value determined by a logit-link function. Moreover, regression structures as in Eq. (12) can also be included in cognitive process models to predict psychologically meaningful model parameters by external covariates (Boehm et al., 2018; Heck et al., 2018). In both generalized linear models and cognitive process models, the JZS prior in Eq. (8) can be adapted as a prior on the regression parameters for Bayesian model selection (e.g., Li & Clyde, in press). Since numerical integration is often infeasible, one might compute the naïve Savage-Dickey density ratio as an approximation of the Bayes factor to test whether a covariate has an effect (Boehm et al., 2018). However, similarly as in the example in Section 4, this will result in an incorrect Bayes factor if other covariates are also included in the model.

Another limitation of the naïve Savage-Dickey density ratio concerns models with simultaneous regression equations for multiple dependent variables or parameters. This is the case in structural equation models or in cognitive process models that assume separate regression structures on different model parameters (Heck et al., 2018). For such scenarios, researchers may assume a joint JZS prior for *all* regression parameters across multiple equations (Boehm et al., 2018). However, when testing whether a slope parameter is zero, this choice implies that the conditional prior of the remaining, non-constrained slope parameters is again a multivariate t-distribution (cf. Eq. (10)). Hence, the assumption underlying the naïve Savage-Dickey ratio is violated, even if each of the multiple regression equations features only a single slope parameter. Nevertheless, Boehm et al. (2018) applied the naïve Savage-Dickey ratio in such a scenario, thus obtaining an incorrect approximation of the Bayes factor.

5.2 Conclusion

Overall, researchers should be careful when using the Savage-Dickey density ratio. Even though its implementation is often straightforward, the application of this method requires that a necessary assumption is checked first, namely, that the prior of the nested model is identical to the conditional prior of the full model given the equality constraint on the test-relevant parameters. This condition automatically holds if the test-relevant and the nuisance parameters are independent. However, if the necessary assumption does not hold, the naïve Savage-Dickey ratio results in an incorrect approximation of the Bayes factor. As a remedy, the generalized Savage-Dickey ratio (Verdinelli & Wasserman, 1995) can be used to compute the correct Bayes factor. To approximate the necessary multiplicative correction factor in Eq. (4), this method also requires posterior samples from the nested model. Alternatively, marginal likelihoods for the models under consideration can be computed directly, for instance, via bridge sampling (Meng & Schilling, 2002). Whatever approach is used in practice, the present paper highlights the importance of checking the assumptions of a computational method to ensure that the approximation of the Bayes factor is correct.

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Table 1

Approximation of the Bayes factor for a null effect of parasites on cranial capacity (data by Bailey & Geary, 2009).

Method	JZS Scale Parameter γ					Time [sec]
	$\sqrt{2}/4$	$1/2$	$\sqrt{2}/2$	1	1.5	
$\mathcal{H}_4$: Parasites + Global + Local + Population density						
`BayesFactor` package	3.83	3.92	4.10	4.43	5.17	0.17
Naïve Savage-Dickey	2.04	2.89	4.07	5.79	8.52	9.76
Generalized Savage-Dickey	3.83	3.95	4.12	4.50	5.18	33.81
Bridge sampling	3.82	3.85	4.07	4.43	5.19	33.32
$\mathcal{H}_2$: Parasites + Population density						
`BayesFactor` package	4.06	4.19	4.46	4.94	5.97	0.04
Naïve Savage-Dickey	1.70	2.46	3.37	4.73	7.02	8.06
Generalized Savage-Dickey	4.04	4.28	4.42	4.89	5.88	29.21
Bridge sampling	4.04	4.18	4.44	4.91	5.94	30.19

Note. The Bayes factor for a null effect of parasites ($\beta_1 = 0$) is computed given that three other covariates are already included in the model $\mathcal{H}_4$ with all four predictors (upper part of the table) and given that one other predictor is already included in the model $\mathcal{H}_2$ with only two predictors (lower part). The naïve and the generalized Savage-Dickey density ratio (Verdinelli & Wasserman, 1995) used a normal approximation of the marginal posterior density. The computation time was averaged across the five scale parameters γ.

Appendix

Derivation of the Conditional Distribution

To improve readability, the following derivation uses a slightly different notation by referring to the test-relevant parameters as $\boldsymbol{\beta}_1$ and the remaining slope parameters as $\boldsymbol{\beta}_0$ (whereas above, these were termed $\boldsymbol{\beta}_K$ and $\boldsymbol{\beta}_R$, respectively). Conditional on the residual variance σ^2, the JZS prior defines a multivariate Cauchy distribution on the regression parameters, which is a special case of a multivariate t-distribution with $\nu = 1$ degrees of freedom,

$$\begin{pmatrix} \boldsymbol{\beta}_0 \\ \boldsymbol{\beta}_1 \end{pmatrix} \sim \mathcal{MVT}_{\mathrm{df}=\nu} \left(\begin{pmatrix} \boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{00} & \boldsymbol{\Sigma}_{01} \\ \boldsymbol{\Sigma}_{10} & \boldsymbol{\Sigma}_{11} \end{pmatrix} \right), \tag{13}$$

centred at zero ($\boldsymbol{\mu}_0 = \mathbf{0}_R$, $\boldsymbol{\mu}_1 = \mathbf{0}_K$) with the scale matrix $\boldsymbol{\Sigma}$ as defined in Eq. (8). Ding (2016, Conclusion One) showed that the conditional distribution is a multivariate t-distribution with $\nu + K$ degrees of freedom,

$$(\boldsymbol{\beta}_0 \mid \boldsymbol{\beta}_1 = \mathbf{0}_K) \sim \mathcal{MVT}_{\mathrm{df}=\nu+K} \left(\boldsymbol{\mu}_{0|1}, \tfrac{\nu+d}{\nu+K} \boldsymbol{\Sigma}_{00|1} \right). \tag{14}$$

In the present case, the conditional distribution is also centred, since

$$\boldsymbol{\mu}_{0|1} = \boldsymbol{\mu}_0 + \boldsymbol{\Sigma}_{01} \boldsymbol{\Sigma}_{11}^{-1} (\mathbf{0}_K - \boldsymbol{\mu}_1) = \mathbf{0}_R.$$

In general, the scaling factor of the conditional distribution in Eq. (14) depends on

$$d = (\mathbf{0}_K - \boldsymbol{\mu}_1)' \boldsymbol{\Sigma}_{00}^{-1} (\mathbf{0}_K - \boldsymbol{\mu}_1),$$

which equals zero for the JZS prior since $\boldsymbol{\mu}_1 = \mathbf{0}_K$. Finally, the conditional scale matrix $\boldsymbol{\Sigma}_{00|1}$ is the Schur complement

$$\boldsymbol{\Sigma}_{00|1} = \boldsymbol{\Sigma}_{00} - \boldsymbol{\Sigma}_{01} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{10}. \tag{15}$$

For the JZS prior, the full covariance matrix $\mathbf{C} = \mathbf{X}'\mathbf{X}/N$ reduces to the submatrix $\mathbf{C}_0 = \mathbf{X}_0'\mathbf{X}_0/N$ in Eq. (9), because (Consonni & Veronese, 2008, Section 4.1.2)

$$(\mathbf{X}_0'\mathbf{X}_0)^{-1} = [(\mathbf{X}'\mathbf{X})^{-1}]_{00} - [(\mathbf{X}'\mathbf{X})^{-1}]_{01}(\mathbf{X}'\mathbf{X})_{11}[(\mathbf{X}'\mathbf{X})^{-1}]_{10}. \tag{16}$$