Fast and Slow Strategies in Multiplication

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Abstract

In solving multiplication problems, children use both fast, retrieval-based, processes, and, slower computational processes. In the current study, we explore the possibility of disentangling these strategies using information contained in the observed response latencies using a method that is applicable in large data sets.

We used a tree-based item response-modeling framework (De Boeck and Partchev, 2012) to investigate whether the proposed qualitative distinctions in fast and slow strategies can be detected. We analyzed responses to two sets of multiplication items, totalling more than 500.000 responses, collected with an online computer-adaptive training environment for mathematics.

Results showed qualitative differences between the fast and the slow strategies. Building on these results, both item and person characteristics were differently related to fast and slow processes. These characteristics, resulting from substantive models of multiplication, allowed us to further describe the fast and slow strategies. Results emphasize the quantitative and qualitative differences between strategies used for solving multiplication problems, and provide possibilities for tailored feedback on learning multiplication.

Keywords: Item Response Theory, Response Times, Multiplication, Strategies

1. Introduction

The concept of strategy is central in the study of human problem solving. Important aspects of problem solving behavior such as accuracy, duration, and type of errors, are due to the choice of solution strategy. For instance, in solving arithmetic items, people may use either retrieval from memory or a computational strategy (Dowker, 2005; Ashcraft and Guillaume, 2009; LeFevre et al., 1996), where the former typically requires less time than the latter. In the case of basic multiplication (for example single-digit problems), detailed models for the retrieval process exist (Geary et al., 1986; Verguts and Fias, 2005), and several models for computational strategies have been developed as well (Lemaire and Siegler, 1995; Imbo et al., 2007). These models make different predictions about item difficulty and solution time (van der Ven et al., 2015).

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When measuring arithmetic ability by using psychometric tests, such as in IQ tests, individual differences in strategy choice are usually not taken into account. Arithmetic ability is ultimately tested by counting the number of correct items that participants solve in any particular test (e.g., Liu et al., 2008; Aunola et al., 2004). Different patterns of response times and errors are hence ignored when the aim is to compare individuals on a scale of arithmetic ability. Using the number of correct responses may be warranted when testing and comparing test takers, but may be inappropriate when concerned with studying development and understanding ability differences. In the latter case, different qualitative processes or strategies should be considered.

For example, an important developmental trend in learning arithmetic can be described by changes in strategy choice. Initially children will apply various slower computational strategies (Freudenthal, 1991). Over time, these computations become more sophisticated (Lemaire and Siegler, 1995). Through practicing multiplication, children will build up a network of associations between numbers. When this network is sufficiently strong, children will be able to confidently retrieve answers to items, and will tend to use faster retrieval from this network instead of a slower computational strategy (Siegler, 1988). Children with learning difficulties do not show this typical transition from computational to retrieval strategies (De Visscher and Noël, 2014; De Smedt et al., 2011). After years of practice, adults will rely predominantly on memory retrieval for single digit multiplication (LeFevre et al., 1996). Hence, the largest divide in strategy choice is whether children and adults use a retrieval strategy or a computational strategy.

In spite of the importance of the strategy concept, detecting strategies is still a major challenge in many areas of cognitive science. Verbal reports and neural imaging features are both correlated with strategy choice (Jost et al., 2004; Tenison et al., 2014; Price et al., 2013), but both also have pitfalls as strategy indicators. Verbal reporting, the most commonly accepted method of strategy detection, may interfere with the solution process and bias strategy choice (Kirk and Ashcraft, 2001; Reed et al., 2015). Another important problem with relying on verbal reporting for detecting strategy choice is that it is time-consuming to apply and thus not feasible in combination with large scale automatic assessment of arithmetic abilities. The latter problem also applies when using neural patterns to identify strategy choice. A third approach, whereby strategies are assessed through latencies combined with accuracy, is more promising in the context of large scale assessment of arithmetic problem solving as retrieval strategies are usually much faster than computational strategies (e.g., LeFevre et al., 1996). Hence, here we explore the possibilities of including response latencies in measurement models of arithmetic performance to disentangle possible qualitative differences between strategies.

In this paper we investigate whether the fast-slow model (Partchev and De Boeck, 2012; DiTrapani et al., 2016) allows for automatic analyses of strategy use in a large scale data set of arithmetic performance in children. In particular, we focus on multiplication problems as this is a well-studied domain. The fast-slow model is based on splitting the data into fast and slow responses and estimating separate abilities for each of the processes. A third process, based on the response latencies, indicates choice for the fast or slow process. The advantage of this type of psychometric model is that item and person effects are easily disentangled. This approach is intermediate between the purely psychometric approach of fitting IRT models to capture multiplication ability on a single latent trait (e.g., Liu et al., 2008; Aunola et al., 2004) and the purely cognitive approach of using computational models to predict response accuracy based on problem characteristics and strategies (partial abilities) (e.g., de la Torre and Douglas, 2008).

We will first introduce the fast-slow model, derive predictions for the case of multiplication, and then apply the model to a large data set. This data set includes a large set of responses collected with a popular Dutch online adaptive learning environment for mathematics; the Math Garden (Klinkenberg et al., 2011; Straatemeier, 2014).

1.1. The Fast-Slow Model

The fast-slow model is a tree-based item response theory (IRT) model (De Boeck and Partchev, 2012). The rationale of this model is that responses are governed by one of two processes, one fast and one slow, that can be separated by an additional observed variable, in this case the (recoded) response times. The response times are recoded to either fast (1) or slow (0), which serves as an approximation of the underlying process and is modelled as a latent speed dimension. This tree model can be formulated as follows, assuming that a (unidimensional) Rasch model (Rasch, 1960) holds in dimension d, where $d = 1, 2, 3$ denotes the speed-, fast- and slow dimension, respectively. In these dimensions respectively the probability of a fast response, a fast and correct or a slow and correct response are modelled using a Rasch model. In the Rasch model, the probability of a correct (or for the speed dimension a fast) response of a person p on an item i in dimension d is given by the logistic function:

$$P(x_{pid} = 1|\theta_{pd}, \beta_{id}) = \frac{exp(\theta_{pd} + \beta_{id})}{1 + exp(\theta_{pd} + \beta_{id})}, \tag{1}$$

where θ_{pd} denotes the ability of person p and β_{id} denotes the easiness of item i on dimension d. Hence, the full model has three sets of person parameters, and three sets of item parameters: θ_{p1} reflects the overall speed of a person, θ_{p2} reflects the ability to give a fast and correct response, and θ_{p3} reflects the ability to give a slow and correct response. Likewise, item easiness parameters correspond to the probability that items are answered fast versus slow (β_{p1}), the probability of a correct response given that the response was fast (β_{p2}), and the probability of a correct response given that the response was slow (β_{p3}). In line with De Boeck (2008), both $\boldsymbol{\theta_p} = (\theta_{p1}, \theta_{p2}, \theta_{p3})$ and $\boldsymbol{\beta_i} = (\beta_{i1}, \beta_{i2}, \beta_{i3})$ are treated as random variables with $\boldsymbol{\theta_p} \sim \mathcal{N}(\boldsymbol{\mu_\theta}, \boldsymbol{\Sigma_\theta})$ and $\boldsymbol{\beta_i} \sim \mathcal{N}(\boldsymbol{\mu_\beta}, \boldsymbol{\Sigma_\beta})$, constraining $\boldsymbol{\mu_\theta}$ to zero to identify the model (see Appendix B for a description of the model estimation procedure).

1.2. Empirical Predictions in Relation to Fast versus Slow Multiplication Processes

If fast and slow strategies are found to be qualitatively different, some item and person effects are expected to be differently related to fast and slow strategies. If these effects match common findings in the multiplication literature, the fast-slow model is a useful method to identify strategies at the individual level in a big data set.

1.2.1. Item effects

We focus on three prominent empirical effects; the problem-size effect, the tie-effect and effects of special operands, which are associated with systematic differences in accuracy and response times between items. Models of retrieval and computation strategies in simple multiplication have coined different explanations for these differences.

1) The problem size effect (Ashcraft and Guillaume, 2009) refers to the fact that items with large problem sizes are more difficult than items with smaller problem sizes. According to models of computational strategies this effect is due to the additional steps necessary for computing the answer (van der Ven et al., 2015; LeFevre et al., 1996). In retrieval based models this effect is explained by less frequent practice with items with large operands and therefore a less developed memory network (Ashcraft, 1995). Thus, no differences are expected between fast and slow processes with respect to the problem size effect.

2) The tie-effect (Miller et al., 1984; De Brauwer et al., 2006) implies that ties (items with an equal operand; e.g 7 x 7) are easier than other items. This effect is explained by more practice and easier storage in retrieval based models. Models of computational strategies do not predict a tie-effect since the computations involved in ties are the same as in non-tie items. Hence, a tie-effect is expected in the fast process, which is expected to be associated with retrieval, and no tie-effect is expected in the slow process which is expected to be associated with computational strategies.

3) The special operands effect refers to the finding that items with 1, 2, 5 or 9 as operands are easier than other items (Lemaire and Siegler, 1995). This effect follows from easier computations according to computational accounts, but is not predicted in models of retrieval. Hence, the effect of special operands is expected in the slow but not in the fast process.

1.2.2. Person effects

As explained in the introduction the development of simple multiplication skills involves a shift from computational strategies to retrieval. This shift is expected to be reflected in a higher number of fast responses for older compared to younger children, resulting in an effect of age on the latent speed dimension. A gender effect on speed is expected as well, due to individual differences in response styles. In addition and subtraction problems, boys provided more retrieval responses than girls, while girls were more likely to count with their fingers (Carr and Jessup, 1997). It is expected that boys have a higher probability to respond fast than girls.

2. Methods

2.1. Data sets: Items and Participants

Data are collected with the website Math Garden. Math Garden is an online adaptive learning environment for learning basic arithmetic, that is currently used by more than 200,000 children involving more than 1,500 schools in the Netherlands (see Appendix A). Math Garden provides a valuable data set, including accuracies and response times of a large group of children, on a large set of multiplication items.

For this study we selected responses of children collected between June 1, 2011 and June 1, 2015 on two subsets of all multiplication items: (1) all responses to items belonging to the multiplication tables from two up to nine (64 items in total), referred to as the single-digit data set, and (2) responses to the 150 most played items, referred to as the most-played data set. This second data set includes some of the items from the first subset and additionally includes multi-digit multiplication items (such as: 1×500, 7×100, 9×12, 803×10 and 80×6000). Items with a minimum of 200 encounters were selected, resulting in 145 items. Through analysing the second data set we investigated whether the results from the first data set can be generalised to a data set including responses to a broader set of items. Also, replicating the initial analyses using this second data set provides a check of the robustness of the results.

We discarded the first 90 responses that each child made to allow children to become acquainted with the task. Furthermore, because data were collected longitudinally and abilities tend to change over time we selected a time frame for a single assessment of a child's ability. This time-frame must contain sufficient data but should also be small enough to ensure a relatively stable ability, and was fixed to one week. Additionally, in order to set a minimum number of responses for this time frame, we selected data of children who completed at least 30 items within one week.[2] Only the child's first response to an item was selected (multiple responses for the same item within the time frame are possible). The total number of responses, children, items and percentage of missing responses for each data set are presented in Table 1. Note that the same children can be included in both data sets. Since the data were collected with an adaptive algorithm missing responses are missing by design, and can be seen as missing at random (MAR) since the missingness is conditional on the estimated ability (Rubin, 1976; Eggen and Verhelst, 2011).

In order to apply the model, the response times need to be dichotomized into fast or slow categories. In our analyses, we used three different approaches based on a median split: (1) a split on the overall response times distribution; (2) a within person split allocating 50% of the responses of each person to either fast or slow and (3) a within item split allocating 50% of the responses to each item to either fast or slow. The first split captures both person and item differences in speed, whereas the person (item) split only captures differences

[2] It was possible to make different choices for selecting data. However, using different inclusion criteria yielded comparable results, see Appendix C

Table 1: Data description

Item selection	N responses	N children	N items	% missing
Single digit data set	180,651	3,551	64	21
Most played data set	422,634	7,860	145	63

Note. The number of responses, children, items, and amount of missing data for the different constructed data sets. The missing data is introduced by the adaptive item selection.

between items (persons) in speed respectively. A comparison of the results of each of these split-methods provides information on the robustness of the results (see Appendix C).

2.2. Model Comparison

Within the fast-slow model, qualitative differences between fast and slow processes would be reflected by a different ordering of the item parameters, person parameters or both, in the fast compared to the slow component of the model. Hence, to test the hypothesis that these differences are present, the full fast-slow model with a set of item parameters for both the fast and the slow part was compared against three constrained versions of the model. This resulted in four different models: (1) the full model, (2) constrained item parameters: i.e., $\beta_{i,fast} = \beta_{i,slow}$, (3) constrained person parameters: i.e., $\theta_{p,fast} = \theta_{p,slow}$, and (4) both constrained item and person parameters. If one, or both, constraints resulted in a worse model fit (in terms of prediction; see next section), this would support the notion that indeed different processes were involved in the fast and the slow responses. However, from a measurement perspective different item parameters do not necessarily suggest that the person abilities are different, since these abilities could be highly correlated (the same holds for item parameters if person parameters are different).

Whenever a constraint was imposed we allowed for a difference in the overall mean and in the variances of the fast and slow item and/or person parameters. This reflects the idea that only a correlation between the fast and slow parameters that is significantly lower than one truly reflects a qualitative different process. For example, if fast retrieval responses are more often correct than slow computational responses it does not necessarily suggest that slow and fast responses have distinct response processes. It may be that for slower responses, retrieval is simply more difficult. However, if for some persons or items the slow responses are more often (in)correct than the fast responses, thereby influencing the correlations of these parameters, this would indeed suggest that a different response process is involved.

Cross-validation was used to assess the models' goodness-of-fit. For each person, data from one response (both the recoded response time and the accuracy) were selected for the test data. The remainder of the data were used to estimate (train) the model parameters, and the estimated models were subsequently used to predict the test data. This approach naturally prevents over-fitting the data with overly-complex models.

The test data formed between 1.4% and 3.0% of the total data in the different data sets but was still fairly large as, despite including one response per person, a large number of persons were included (see Table 1). Model predictions were based only on accuracy as the models did not differ in their analyses of response times.

Three cross-validation statistics were used, all three based on the deviation between the observed and the predicted response: the prediction accuracy (ACC), the root mean squared error (RMSE) and the log-likelihood (LL; Pelánek (2015); see Appendix B for a detailed description). In both RMSE and LL the continuous prediction of the probability of a correct response is analyzed. This results in a finer model comparison than the ACC, while the ACC provides a simpler interpretation of the goodness-of-fit. When interpreting the ACC and the LL, higher (less negative) values indicate better fit, while for the RMSE lower values indicate better fit.

3. Results

Since the results of the model comparisons were similar across the various dichotomizations, we limit the results section to the analyses from data sets where fast or slow was defined by the overall medium split (see Appendix C).

3.1. Data Description

The RT distributions of both data sets are presented in the left-panel of Figure 1. For the single-digit data set the median response time (RT) was 6.22 sec. 59% of the fast responses and 62% of the slow responses were correct. The lower percentage for the fast responses was related to the higher proportion of fast question-mark responses: 33% and 11% respectively for fast and slow responses. This is also shown by the relationship between RT and the probability of a question-mark response, plotted in the right-panel of Figure 1. In the most-played data set the median RT was 7.36 sec. 72% of the fast and 68% of the slow responses were correct.

3.2. Model Comparison

To estimate the model parameters we used 1,000 iterations and a burn-in of 100. Since some high auto-correlations were found we used every third iteration for the MAP estimates of the model parameters. Table 2 shows the fit measures for the estimated models. In line with our hypothesis, the results indicated that for both the single digit and most-played data set, the model with separate item difficulties and separate person abilities for the fast and slow dimension - the full model - provided a better fit that any of the constrained models in terms of ACC, RMSE and LL (see Table 2). This suggested that qualitatively different processes were involved in the fast compared to the slow processes for both the single-digit and the most-played data set.

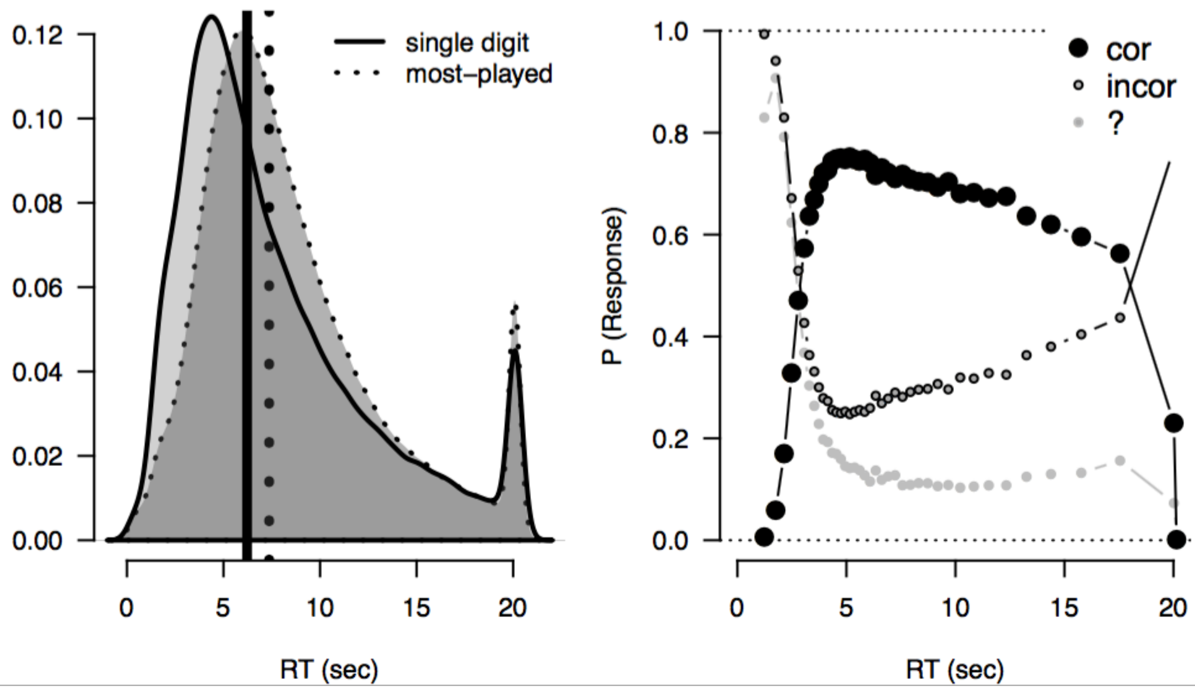

Figure 1: Data Description. The left-panel shows the RT distribution for the single-digit and most-played data set. The vertical lines (solid for single-digit and dotted for most-played data set) indicate the median of the RT distribution. The peak around 20 seconds is caused by the deadline in the game. The right-panel describes the proportions of a correct, incorrect and question-mark response for the different observed response times in the single-digit data set.

Table 2: Model fit based on cross-validation of the full and constrained fast-slow models in the single digit and most played item data set.

item selection	model	ACC	RMSE	LL
single digit	full model	**0.777**	**0.391**	**-2416**
	$\beta_{fast} = \beta_{slow}$	0.775	0.397	-2510
	$\theta_{fast} = \theta_{slow}$	0.773	0.398	-2518
	$\beta_{fast} = \beta_{slow}$ and $\theta_{fast} = \theta_{slow}$	0.772	0.397	-2489
most played	full model	**0.750**	**0.416**	**-5239**
	$\beta_{fast} = \beta_{slow}$	0.740	0.422	-5403
	$\theta_{fast} = \theta_{slow}$	0.742	0.420	-5351
	$\beta_{fast} = \beta_{slow}$ and $\theta_{fast} = \theta_{slow}$	0.737	0.421	-5375

Note. Results of the best fitting model are printed in bold.

These results indicate that the response times (split into fast and slow) distinguished between two qualitatively different response processes, both with respect to item and person parameters. In the following sections we will further describe the estimated parameters, and thereby investigate whether differences between the fast and slow strategies can be explained by retrieval and computational models of multiplication.

3.3. Fast vs Slow Correlations and Variances

The model comparison indicated that fast and slow item and person parameters are not perfectly correlated since the full model provided a better fit than any of the constrained models. However in both the single-digit and the most-played data set the correlations between β_{fast} and β_{slow} were very high: .969, and .896 respectively for the single-digit and most-played data set. The correlations between θ_{fast} and θ_{slow} were much lower (respectively .778, and .635). The lower correlations between person parameters might be explained by the smaller number of observations for the person parameters compared to the item parameters (which may have created more measurement error). The higher correlations in the single-digit data set compared to the most-played data set can be explained by a more unidimensional process underlying the responses of children in the single digit data set.

Furthermore, in the single-digit data set, higher variances in β_{fast} compared to β_{slow} were found ($\sigma_{\beta,fast} = 1.943$ and $\sigma_{\beta,slow} = 1.085$; Levene's test of equality of variance: $F(1,\ 62) = 30.07$, p ¡ .001). This was also the case in the most-played data set, however with smaller differences between fast and slow responses than in the single-digit data set ($\sigma_{\beta,fast} = 1.367$ and $\sigma_{\beta,slow} = .775$; Levene's test of equality of variance: $F(1,\ 145) = 52.58$, p ¡.001). The lower estimated variance in the slow process could suggest that there is more random variation, compared to structural variance, in the slow responses. This might be caused by a mixture of different slow strategies.

3.3.1. Item Analysis

In the next step in our analyses we regressed the item parameters on different item characteristics for both the slow and fast responses in the single-digit data set. We intended to replicate the effects of problem-size, tie and effects of special operands. Additionally, and most interestingly, here we were able to test for differential effects for fast and slow processing. Finding these differential effects would mean that predictors related to retrieval processes (tie-effect) and/or computational processes (special operands) are differently related to item parameters in fast compared to slow responses. To investigate these interaction effects we imputed the full original data set. To this end we generated a new set of responses based on the model estimated model parameters. We analysed the sum-scores over items for both fast and slow responses. This approach ensured that effects can be directly compared between different nodes.

In separate regression models we predicted the item parameters reflecting the fast and the slow accuracy and the probability of a fast response (speed). We used the BIC (Schwarz et al., 1978) for model selection,

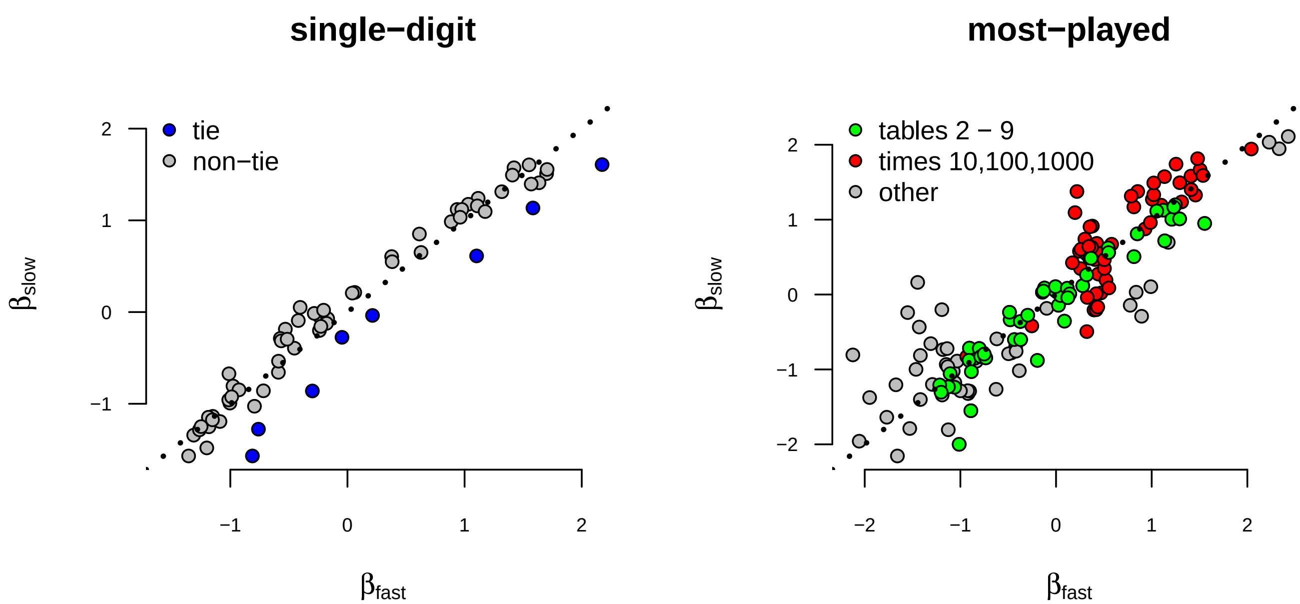

Figure 2: Relation between fast and slow item parameters in the single-digit and most played data set

Table 3: Regression of the item easiness parameters for fast and slow processes and speed (reflecting the probability of a fast response) in the single-digit data set.

	Fast			Slow			Speed		
predictor	B	SE	T-value	B	SE	T-value	B	SE	T-value
Intercept	2.851	0.374	7.632**	2.277	0.225	10.139**	0.034	0.167	0.204
problem-size	-0.351	0.032	-11.044**	-0.218	0.019	-11.289**	-0.039	0.013	-3.002*
tie	1.432	0.212	6.765**	ex	ex	ex	0.491	0.104	4.713*
times 2	2.408	0.207	11.607**	1.152	0.125	9.224**	0.757	0.100	7.577**
times 5	1.224	0.168	7.282**	0.762	0.101	7.543**	0.211	0.083	2.538*
times 9	0.817	0.203	4.016**	0.476	0.123	3.876**	ex	ex	ex

Note. * = $p < .05$, ** = $p < .001$; ex = excluded in the stepwise procedure using a backward stepwise procedure.

In line with the predictions, for the fast responses, we found a main effect of: (1) problem-size, indicating that items with large problem size were more difficult than items with small problem size; (2) ties, ties were easier than non-tie items; and (3) problems with special operands two, five and nine which were easier than other problems, see Table 3. These main effects explained in total 89.2% of the variance. For the slow responses, the effects of problem-size and the special operands were comparable (resulting in an explained variance of 88.6%). Interestingly, no differences between ties and non tie-items were found in the slow responses. This differential effect indicated that the difference between ties and non-ties was larger in the fast compared to the slow part. This effect is plotted in the left-panel of Figure 2, which shows that all tie items are below the diagonal that indicates $\beta_{fast} = \beta_{slow}$. Unexpectedly, effects of items with special operands were not differently related to the fast and the slow item parameters.

For the item speed parameters, a high β_{speed} indicated a high probability of a fast response. Thus Table 3 shows that responses to items with large problem sizes were often slow. Also, responses to items belonging to the two and five multiplication tables, and ties were often fast (see Table 3) indicating that these items were more often solved by retrieval rather than computational strategies. These effects explained in total 70.6% of the variance in the item speed parameters.

To conclude, the high explained variance indicates that the item difficulties could be largely understood by this set of item features. This supported the reliability of both the data and the model estimation. Moreover, although high correlations between fast and slow item parameters were found, the interaction between tie and node indicates that tie items tap into the differences between fast and slow processes. In line with the results of van der Ven et al. (2015), the tie-effect was more prominent in the fast responses.

Item characteristics were not regressed on item parameters in the multi-digit data set. However, the right-panel of Figure 2 clearly shows a positive relation between β_{fast} and β_{slow} in the multi-digit data set. Some items showed higher deviations. An exploratory look at the three items with the highest deviations where either $\beta_{fast} > \beta_{slow}$ or $\beta_{fast} < \beta_{slow}$ showed an interesting pattern. The items 11 x 6, 8 x 8 and 11 x 9 were easier when solved quickly compared to slowly, and the items 80 x 6000, 4 x 108, and 3000 x 80 were easier when solved slowly compared to quickly.

3.3.2. Person Analysis

In the second set of regression models we investigated whether person characteristics were differentially related to fast and slow abilities. For this analyses we only included children between 6 and 11 years old (N=4233; excluded 467), and children for which their age matched their grade (excluded 417 children for which their age deviated more than 1.5 year from the grade average). For the single-digit and multi-digit data sets the average ages of the selected children were 7.86 and 8.42 (sd 1.04 and 1.10) respectively, and 33% and 42% respectively were girls.

Table 4: Regression person ability parameters for fast and slow processes and speed (reflecting the probability of a fast response)

	Fast			Slow			Speed		
predictor	B	SE	T-value	B	SE	T-value	B	SE	T-value
Intercept	-0.038	0.019	-1.971	-0.023	0.012	-1.860	-0.173	0.021	-8.074*
age	0.432	0.020	21.508*	0.267	0.013	20.631*	0.221	0.013	17.077*
gender	ex	ex	ex	ex	ex	ex	0.198	0.026	7.478*
% ?	-1.184	0.020	-58.910*	-0.476	0.013	-36.817*	0.487	0.013	37.786*

Note. Boys are coded as 1 and girls as 0; %? = percentage of question-mark responses; * = $p < .001$;

ex = excluded in the stepwise procedure

All results, based on a stepwise backward procedure using BIC, are presented in Table 4. As expected we found a main effect of age. Older children were more able than young children in both fast and slow abilities. Second, no gender differences were found in both abilities. Third, children with more question-mark responses had a lower ability. However, this effect was smaller with slow compared to fast abilities. This highlights that the differences between the abilities measured by fast and slow responses can partly be explained by differences in how children relate to the question-mark answer option. These effects explain 54.9% and 36.7% of the variance for fast and slow abilities, respectively.

The regression model for differences in speed between children indicated that; (1) older children were faster than younger children, (2) boys were faster than girls and (3) children who provided more question-mark responses were faster than children who did not use the question-mark response as often (see Table 4). These effects explain 27.3% of the variance in the abilities between children.

3.4. Correlations between Speed and Accuracy

In this last section we explore the relations between speed and accuracy from an item and a person perspective in both the single-digit and most-played data set. We defined speed as the probability of a fast response, based on the overall split in response times.[3]

Item speed and accuracy correlated positively. In the single-digit data set the correlations between β_{speed} and β_{fast} and β_{slow} were .837 and .739. In the most-played data set, these correlations were respectively .694 and .440. The correlations between β_{speed} and β_{fast} are plotted in Figure 3.

We observed two interesting results. First, in the single-digit data set, the relationship between speed and accuracy showed an interesting pattern. A regression model with a breakpoint resulted in an explained variance of 83.8%, an increase of 13.5% compared to the explained variance of 70.4% of the linear regression

[3]The presented results were stable under the different RT splits; the within item split to investigate person speed and the within person split to investigate item speed.

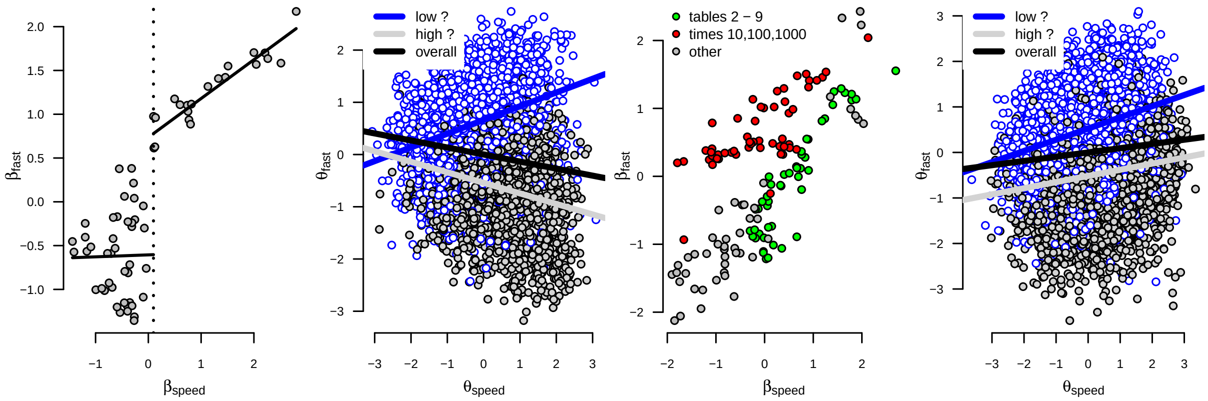

Figure 3: Relationship between speed and fast accuracy for items and persons in the single-digit data set (left two panels) and the most-played data set (right two panels). Low and high ? denotes the frequency of question mark usage.

model. Furthermore, the breakpoint could be confidently estimated at zero, indicated by a clear peak in explained variance compared to models with differently located non-zero breakpoints. This strongly suggested that, for items that were solved quickly ($\beta_{speed} > 0$), there was a strong relationship between speed and accuracy, whereas for items that were more often solved slowly this relation was absent. In line with results of the model comparison, this result signifies that fast strategies are qualitatively different from slow strategies. Secondly, an exploratory look at the item parameters in the most-played data set showed large differences between items including a times 10, 100 or 1000 operator and the other items, as visualized in Figure 3. These items were incorrect more often when answered quickly compared to other items.

For persons, a different pattern was found. In the single-digit data set, negative correlations between person overall speed and fast and slow abilities were found: -.125 and -.033 respectively. Thus, in contrast to expectations, children that were faster were more often incorrect. To test whether the negative correlation was related to differences between children in question-mark usage we calculated separate correlations for children who provided less or more than 13% question-marks (median split). We found a correlation of .306 for children who used fewer question-marks, indicating that for these children, the faster children were more able than the slower children, see the blue line in the second panel of Figure 3. This suggested that the negative relation is related to question-mark uses. Furthermore, all correlations found were positive (min = .191 and max = .373) when children were grouped by question mark use from zero to ninety percent in increments of ten percent. The same pattern was found in the most-played data set. To conclude, these results indicate that, when corrected for question-mark usage, children who are faster had higher fast and slow abilities.

4. Discussion

In this paper we investigated whether qualitatively different strategies in multiplication can be disentangled using information in observed response times. This approach allowed for an automatic assessment of strategy use appropriate for large-scale data. An application of the fast-slow model on a data set collected with a popular online learning program confirmed that a mixture of different strategies underlies the children's performance on multiplication items. Disentangling fast versus slow strategies improved understanding of children's observed responses.

Building on these results, additional analyses showed that specific item and person characteristics tap into the differences between these strategies. The aim of these analyses was to investigate differential effects between fast and slow strategies, as predicted by fast retrieval versus slow computational processes of multiplication (Siegler, 1988). On the item side, the difference between tie and non-tie items was more prominent in the fast responses compared to the slow responses. Against expectations, no differential effect was found for items of special operands (2, 5 and 9). This could be explained by having used rather crude methods to disentangle strategies. These methods may have allocated some retrieval responses as slow and some computational responses as fast, resulting in a lower power to find these effects. However, varying split methods differing in how responses were categorised as fast or slow show consistent results. Further methodological improvements are possible with developing better ways of splitting response times as the most important one. Ideally, the data itself determines the classification into fast and slow processes, resulting in a more optimal classification of responses to strategies (DiTrapani et al., 2016).

On the person side, older children (who are assumed to have more experience) provided more fast responses. Although older children can be faster in multiple ways, the results indicate that this developmental trend is partly due to a higher probability of a retrieval strategy for older compared to younger children. Additionally, although boys and girls did not differ with respect to both the fast and slow ability - in line with the results of Carr and Jessup (1997) - boys provided more fast responses than girls. This highlights that, as described by Siegler (1988), individual differences in strategy use are present between children. This difference in strategy use is also reflected by the different relationship between question-mark usage and fast versus slow abilities. Some children are more inclined than others to provide a question-mark response, and furthermore, fast question-marks are governed by a different strategy selection than slow question-marks.

These results confirm that children's strategies for solving mental multiplication items can be disentangled using a split in observed response times. Hence, as described by Siegler (2007) and Van der Ven et al. (2012), multiplication ability should be seen as a toolbox of different strategies, where both the ability of each child within a certain strategy and individual differences in strategy selection determine the observed performance. This study indicated that these processes, often studied in smaller and controlled experimental settings, also determine multiplication ability in a large-scale online learning platform, supporting the generalizability of the effects and the validity of the Math Garden.

4.1. Future Directions

It should be noted that the mixture of retrieval and computational processes underlying the responses in multiplication will depend on the testing conditions. In the Math Garden items were selected to match children's ability resulting in a mixture of different strategies. Presenting solely easy or hard items will change the mixture of strategies. Additionally, the test conditions were such that children perceived time-pressure. This evokes faster responses, and probably influences the strategies that were used (Hofman et al., 2015). Further research should investigate in what manner children's performances in high-stakes tests also depends on multiple processes. Additionally, next to response latencies, error types also contain information about the used strategy (Siegler, 1988). In a first minimal example Coomans et al. (2016) already showed that fast errors in response to multiplication items were different from slow errors. Utilizing both response latency and error types could provide additional confidence in estimating the used strategy.

This line of research will provide applied researchers, teachers and students with valuable information on strategies, without using time-intensive methods such as verbal protocols. First, it sheds light on what cognitive processes are involved in mathematics, and possible many other domains. Second, it enables tailored feedback about proficiency of strategies when learning multiplication, and thereby matches the aims for mathematics education. For instance, in the Netherlands education ultimately aims for understanding multiplication concepts and memorization of the single-digit tables of multiplication (SLO, 2009).

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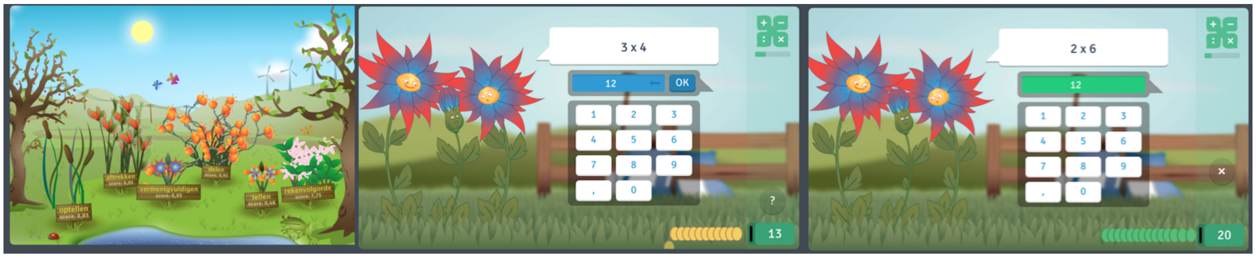

Figure A.4: Three screen shots of the Math Garden. The left panel shows the garden page where each plant represents a game measuring a different aspect of mathematics. The middle panel shows an open format multiplication item, where children use a numeric keypad to provide a response. The coins at the bottom represent points, and players lose one for each second that they do not provide a response. The coins turn green (or red; right-panel) in case of correct (incorrect) response, and are added to (subtracted from) the total.

Appendix A. Description of the Math Garden

Data are collected with the website Math Garden. After logging in, children arrive at a page showing a garden, where each plant represents a game that covers a domain of mathematics, see Figure A.4. The multiplication game starts after selecting the multiplication plant. In this game children are given 15 items that must be solved using the virtual numerical keypad, each with a time limit of 20 seconds. The time is visualized by disappearing coins (one is lost each second that they do not provide a response). If a correct response is given the coins are added to a money bag, whereas the coins are subtracted from the money bag if the response is incorrect. This explicit scoring-rule informs the users how to weight speed and accuracy. This scoring rule is called the High Speed High Stakes (HSHS) scoring rule (Klinkenberg et al., 2011; Maris and Van der Maas, 2012). This ensures that children perceive some time-pressure since they are motivated to provide fast responses, but they are discouraged from guessing due to the penalty of a fast but incorrect response. When a child does not know the answer (s)he can best wait the full 20 seconds. To prevent such waiting times the child can also use the question-mark button, in which case (s)he does not win or lose any coins. In our analyses, the question-mark responses were labeled as incorrect, and the percentage of question-mark responses was included as a person characteristic in secondary analyses.

With the HSHS scoring rule in the Math Garden the estimates of both the user ability and the item difficulty can be updated after each response.[4] Based on these estimates relevant items were selected for the child at a certain time point, such that children were expected to provide 60, 75 or 90% correct responses when playing at the hard, medium or easy difficulty level (for more details see Jansen et al., 2016).

[4]See Maris and Van der Maas (2012) for a detailed description of the psychometric properties of the HSHS rule and Klinkenberg et al. (2011); Straatemeier (2014) for a description of how the parameters are estimated using an elo (Elo, 1978) algorithm.

Appendix B. Model Estimation and Comparison

Appendix B.1. Estimation of the Fast-Slow Model

We adopt a Bayesian approach to estimate the parameters of our fast-slow model, and wish to quantify our uncertainty about these parameters in a joint posterior distribution: i.e., $f(\boldsymbol{\theta},\, \boldsymbol{\beta},\, \boldsymbol{\mu}_\beta,\, \boldsymbol{\Sigma}_\theta,\, \boldsymbol{\Sigma}_\beta \mid \text{data})$. To this aim, we need to specify a prior distribution for the population parameters $\{\boldsymbol{\mu}_\beta,\, \boldsymbol{\Sigma}_\theta,\, \boldsymbol{\Sigma}_\beta\}$. First, we specify a Jeffreys prior for the between dimension person covariance matrix $\boldsymbol{\Sigma}_\theta$ (Gelman et al., 2014, page 37)

$$f(\boldsymbol{\Sigma}_\theta) \propto |\boldsymbol{\Sigma}_\theta|^{-2},$$

where we assume that $\boldsymbol{\Sigma}_\theta$ is independent of $\{\boldsymbol{\mu}_\beta,\, \boldsymbol{\Sigma}_\beta\}$ a priori. Second, we constrain the off-diagonal elements from the between dimension item covariance matrix $\boldsymbol{\Sigma}_\beta$ to be zero and assign independent Jeffreys priors to the mean and variance for each dimension (Gelman et al., 2014, page 64), i.e.,

$$f(\mu_{\beta,\, d},\, \sigma^2_{\beta,\, d}) \propto \sigma^{-2}_{\beta,\, d}.$$

Constraining the off-diagonal elements of $\boldsymbol{\Sigma}_\beta$ to zero means that we a priori assume that the item parameter values are independent between dimensions. This is of course highly unlikely, but we have chosen to do this to favor convergence of our estimation procedure; the Gibbs sampler (Geman and Geman, 1984, See Appendix A for a description of the sampling procedures). Recent Bayesian theory shows that such a choice may shrink the posterior estimate of the between-dimension correlation to zero, but also that this shrinkage effect will be minor when there are many observations (Marsman et al., 2016). We expect the shrinkage of these correlations to be small in our analyses, and our suspicion was confirmed by some additional simulations.

Appendix B.2. Simulating from the full-conditional distributions

The full-conditional distribution of the between dimension person covariance matrix $\boldsymbol{\Sigma}_\theta$ is easily sampled from:

$$f(\boldsymbol{\Sigma}_\theta) \propto \text{Inverse–Wishart}_{n-1}(\mathbf{S}_\theta),$$

where n refers to the sample size and $\mathbf{S}_\theta$ to the 'sample' covariance matrix $Cov(\boldsymbol{\theta})$. Similarly, we find that the full-conditional distributions for $\{\mu_{\beta,\, d},\, \sigma^2_{\beta,\, d}\}$ are easy to sample from:

$$f(\mu_{\beta,\, d} \mid \sigma_{\beta,\, d},\, \boldsymbol{\beta}_d) \propto \mathcal{N}(\bar{\boldsymbol{\beta}}_d,\, \sigma^2_{\beta,\, d}/k)$$
$$f(\sigma_{\beta,\, d} \mid \boldsymbol{\beta}_d) \propto \text{Inverse-}\chi^2(k-1,\; \sum_{i=1}^{k} \beta^2_{id}/(k-1)),$$

where k refers to the number of items in our analyses.

Unfortunately, the full-conditional distributions of the person and the item parameters are not readily sampled from. Standard approaches, such as the Metropolis within Gibbs approach of (Patz and Junker, 1999a,b), are difficult to apply here due to the need of non-trivial fine-tuning that is required for each of the $n \times 3$ person and $k \times 3$ item parameters. This fine-tuning is particularly problematic as each of the persons responds to a possibly different set of items, and, similarly, each of the items has been responded to by a different set of persons.

To sample from the full-conditional distributions of the person and the item parameters we therefore utilize an independence chain Metropolis algorithm that was proposed by (Marsman et al., 2015).Their approach is particularly efficient when applied to the Rasch model and is simple to use with incomplete designs. [5]

Appendix B.3. Model Comparison

Three cross-validation statistics were used, all three based on the deviation between the observed and the predicted response in the test dataset. First, the prediction accuracy (ACC):

$$ACC = 1 - \frac{1}{n}\sum_{i=1}^{n} o_i - p_i \tag{B.1}$$

where, both o_i is the observed response and p_i is the predicted response and n is the number of responses in the test data. Here, p_i is either correct or incorrect based on the probabilities following from equation 1, and the maximum a-posteriori (MAP) estimates of $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$. The ACC reflects the percentage of correctly predicted responses by the model parameters.

The second cross-validation statistic was the root mean squared error (RMSE):

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(o_i - p_i)^2} \tag{B.2}$$

where, p_i is the predicted probability on a continuous scale between 0 and 1.

The third cross-validation statistic was the log-likelihood (LL), defined as follows:

$$LL = \sum_{i=1}^{n} o_i \log(p_i) + (1 - o_i)\log(1 - p_i). \tag{B.3}$$

Both the RMSE and the LL are presented since the LL provides a higher penalty to predictions that were confident and wrong (high deviation between o_i and p_i), whereas the RMSE provides an equal penalty for each deviation between o_i and p_i (Pelánek, 2015).

[5] Details about this algorithm as applied to the Rasch model can be found in (Marsman, 2014, pages 85–88).

Table C.5: Data description. The number of responses, children, items, and amount of missing data in the different constructed data sets.

item selection	time	min items	N responses	N children	N items	% missing
single digit	day	30	51,284	1,164	64	31
	week	30	180,651	3,551	64	21
most played	day	30	387,882	7,403	135	61
	week	30	422,634	7,860	145	63
	week	60	490,874	4,813	147	31

Appendix C. Robustness Analysis

To investigate the stability of the comparison of the full fast-slow model with the more constrained versions of the model we constructed multiple data sets and replicated the analyses presented in the paper.

Appendix C.1. Data Selection

For the single-digit items, we constructed two data sets based on the selection of children that completed at least thirty items within one day *or* within one week. For the most-played items we selected data of children that completed at least 30 items within one day *or* one week *or* sixty items within one week. These choices resulted in a total of five different data sets. Within each data set, we selected items with a minimum of 200 responses, and looked at the child's first response to an item (multiple responses can be given to the same item within a set of 30 or 60 items). The total number of responses, children, items and percentage of missing responses for each data set are presented in Table C.5

For each of the five data sets the response times were split using the overall median RT, within-person median RT and the within-item median RT. This resulted in a total of fifteen model comparisons.

Table C.6: Model fit based on cross-validation of the full and three constrained fast-slow models in the single digit data set.

time frame	min items	RT split	model	ACC	RMSE	LL
day	30	overall median	full model	0.788	0.374	-657
			$\beta_{fast} = \beta_{slow}$	0.789	0.381	-684
			$\theta_{fast} = \theta_{slow}$	0.786	0.378	-674
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.786	0.380	-676
		within persons	full model	0.783	0.391	-725
			$\beta_{fast} = \beta_{slow}$	0.783	0.401	-788
			$\theta_{fast} = \theta_{slow}$	0.769	0.405	-783
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.783	0.398	-750
		within items	full model	0.804	0.374	-656
			$\beta_{fast} = \beta_{slow}$	0.802	0.377	-690
			$\theta_{fast} = \theta_{slow}$	0.783	0.382	-685
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.784	0.383	-685
week	30	overall median	full model	0.777	0.391	-2416
			$\beta_{fast} = \beta_{slow}$	0.775	0.397	-2510
			$\theta_{fast} = \theta_{slow}$	0.773	0.398	-2518
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.772	0.397	-2489
		within persons	full model	0.776	0.387	-2353
			$\beta_{fast} = \beta_{slow}$	0.764	0.400	-2493
			$\theta_{fast} = \theta_{slow}$	0.756	0.402	-2513
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.769	0.397	-2465
		within items	full model	0.783	0.391	-2415
			$\beta_{fast} = \beta_{slow}$	0.778	0.397	-2519
			$\theta_{fast} = \theta_{slow}$	0.772	0.397	-2498
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.775	0.396	-2473

Appendix C.2. Model Comparison

Table C.7: Model fit based on cross-validation of the full and three constrained fast-slow models in the most played items data set.

time frame	min items	RT split	model	ACC	RMSE	LL
day	30	overall median	full model	0.760	0.410	-5131
			$\beta_{fast} = \beta_{slow}$	0.740	0.425	-5489
			$\theta_{fast} = \theta_{slow}$	0.750	0.417	-5287
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.747	0.417	-5286
		within persons	full model	0.766	0.404	-5025
			$\beta_{fast} = \beta_{slow}$	0.761	0.409	-5150
			$\theta_{fast} = \theta_{slow}$	0.752	0.415	-5283
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.750	0.417	-5328
		within items	full model	0.761	0.409	-5141
			$\beta_{fast} = \beta_{slow}$	0.757	0.415	-5342
			$\theta_{fast} = \theta_{slow}$	0.756	0.414	-5263
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.753	0.415	-5268
week	30	overall median	full model	0.750	0.416	-5239
			$\beta_{fast} = \beta_{slow}$	0.740	0.422	-5403
			$\theta_{fast} = \theta_{slow}$	0.742	0.420	-5351
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.737	0.421	-5375
		within persons	full model	0.750	0.413	-5168
			$\beta_{fast} = \beta_{slow}$	0.748	0.415	-5225
			$\theta_{fast} = \theta_{slow}$	0.736	0.421	-5363
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.730	0.425	-5449
		within items	full model	0.747	0.417	-5262
			$\beta_{fast} = \beta_{slow}$	0.734	0.425	-5436
			$\theta_{fast} = \theta_{slow}$	0.744	0.422	-5385
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.740	0.422	-5379
week	60	overall median	full model	0.777	0.391	-2416
			$\beta_{fast} = \beta_{slow}$	0.775	0.397	-2510
			$\theta_{fast} = \theta_{slow}$	0.773	0.398	-2518
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.772	0.397	-2489
		within persons	full model	0.776	0.387	-2353
			$\beta_{fast} = \beta_{slow}$	0.764	0.400	-2493
			$\theta_{fast} = \theta_{slow}$	0.756	0.402	-2513
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.769	0.397	-2465
		within items	full model	0.783	0.391	-2415
			$\beta_{fast} = \beta_{slow}$	0.778	0.397	-2519
			$\theta_{fast} = \theta_{slow}$	0.772	0.397	-2498
			$\beta_{fast} = \beta_{slow}$ & $\theta_{fast} = \theta_{slow}$	0.775	0.396	-2473