Bridging Educational Priorities with Developmental Priorities: Towards a Developmental Theory of Instruction

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Abstract

This paper summarizes a theory of cognitive development and discusses its educational implications. The paper first outlines a set of principles that might allow tuning developmental priorities with educational priorities. It postulates, in contrast to several classic developmental theories, that developmental priorities change with development. It outlines the cognitive profile of four successive developmental cycles and presents evidence showing that developmental priorities change from interaction control in infancy to representational control in preschool to inferential control in primary school to logical truth control in adolescence. Studies are then summarized showing that the cognitive priorities of each cycle are the best predictors of school achievement in this or later cycles. Finally, we outline developmental changes in general problem-solving skills and show that learning in different domains, such as language and mathematics, depends on an interaction between the general cognitive processes dominating in each cycle and the state of the symbol systems associated with this domain. If command of any of these systems is deficient, specific learning deficiencies may emerge, as in dyslexia and dyscalculia. Principles for ameliorating these conditions are outlined.

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Introduction

This paper focuses on the relations between cognitive development and school learning. We propose a framework that may serve education in two important goals: support and enhance cognitive development through the years of school life and facilitate the attainment of the major learning goals for each school year across school subjects. Schools are challenging environments. Students are expected to learn many new skills and concepts over the years, ranging from early literacy and numeracy skills to complex concepts in science and mathematics. Anything taught at school is demanding at the beginning, even if automated later, such as reading or arithmetic. School classrooms are complex environments as well. They involve children who differ in ability or interest to learn, personality, and family background. Their teachers differ in education, abilities, personalities, teaching styles, and proficiency. Domains of learning, such as language, mathematics, and science, differ in their conceptual characteristics and demands for understanding. The hugeness of these differences makes it admirable that most students do learn a lot of what is planned. However, many students fall behind in learning at some points of their school life in at least some school subjects.

We advance a framework aiming to tune cognitive developmental priorities with educational priorities from preschool to adolescence. The aim is twofold: On the one hand, to satisfy cognitive developmental needs at successive phases in order to maximize the possibilities of each individual; ignoring developmental priorities may deprive learners from the support they need to consolidate developmental priorities to the level needed to move forward. On the other hand, to capitalize on the possibilities emerging from developmental priorities in order to maximize learning in each school subject; ignoring these priorities may cause difficulties and delays in grasping and consolidating the concepts and skills of interest (Demetriou & Spanoudis, 2018). We focus on general problem-solving processes and learning and problem solving in two domains, language and mathematics. Table 1 summarizes developmental priorities, educational priorities, and learning goals across age periods and domains,

Education has looked for direction and teaching practices in psychological theories since the early 20th century. The interaction between education and psychology met with both successes and failures. Here, we will not discuss or evaluate in detail psychological theories, which have been important for education in the past. However, we attempt a brief overview of the main ideas drawn from them and evaluate their contribution to the development of education of our time. We will then present a comprehensive model drawing on the successes and failures of the past and capitalizing on recent research.

Taking Stocks of Promising Models that Did not Fulfill All Promises

Boosting Intelligence and Cognitive Processes

The psychology of individual differences was the first to systematically influence education. The first intelligence test was designed in France by Binet and Simon (1916) as a method of identifying children facing difficulties to learn at school (Anastasi & Urbina, 1997). Subsequently, several programs were implemented to boost the intelligence of children in disadvantage. These programs targeted processes addressed by intelligence tests. The Head Start Program in the USA is a major endeavour planned to improve the learning skills of poor children (Neisser, 1998; Neisser et al., 1996). A similar program, the Abesedarian study, focused on children at risk. Training involved activities and problem solving in various domains of mathematics (i.e., measuring, counting, arithmetic operations) and language (i.e., children were exposed to new and rich vocabulary, elaborated syntax, decontextualized language about not present events and objects, etc.) (Campbell & Burchinal, 2008). The outcome of this research can easily be summarized. Significant gains of up to 7-8 IQ points (i.e., about half a standard deviation) were observed but this effect faded out soon after the end of the program; in three years after intervention gains dropped to a meager 1-2 points (Protzko, 2015, 2016).

Why are gains from intelligence training so fragile? A pessimist view espoused by theorists of individual differences (Jensen, 1998; Murray, 2020) would claim that general intelligence is a stable trait impervious to interventions. Interventions affect superficial skills, such as test-taking skills. Their effects degrade easily when they are removed. The environmentalist view is pessimistic about the individual from a different point of view. According to this view, interventions do change intelligence, reflecting adaptation to a new environment. However, their sustainability requires the presence of the benevolent environment that produced them. Individuals re-adjust to their initial low level if they leave the training environment because their gains are not useful anymore (Ceci, 1991; Ceci & Williams, 1997.

The developmental explanation, espoused here, ascribes the fade out effect to the fact that the studies designed to increase intelligence were developmentally insensitive. They ignored that the nature of intelligence changes with development. They boost a specific form of a cognitive ability that is important at a given age, but this same ability comes in different forms at later ages. This is because these interventions last for a relatively short period of time and address processes related to a specific development phase (e.g., early reading or arithmetic skills). A few years later, when re-examined and compared to non-trained age-mates, children are evaluated on forms of cognitive ability that were not trained. However, learning is developmentally sensitive. To succeed, it must capitalize on the present state of thought of the persons involved. To be sustainable, it must recycle to upgrade core reasoning, processing, and self-regulation processes of each successive developmental phase.

In recent years, learning studies focused on general-purpose cognitive processes shown by research to relate with general intelligence, such as processing speed, attention control, cognitive flexibility, and working memory (Carroll, 1993; Jensen, 1998; Kail, 2000; Kylonen & Christal, 1990). The assumption was that boosting these processes would transfer to general or fluid intelligence (e.g., Jaeggi et al., 2008; Protzko, 2015). The findings of this line of research are very similar to the findings above: these processes do improve because of training but gains *do not transfer* to intelligence, real world cognitive skills, or academic achievement (Melby-Lervag et al., 2016; Sala & Gobert, 2017; Shipstead et al., 2012).

Again, these studies largely ignored both the direction of causality in the relations between these processes and general intelligence (g) and their developmental role. Specifically, hierarchical models of the structure of intelligence, such as Carroll’s (1993) three-stratum model, assume that causality goes top-down, from g to component processes, rather than bottom-up, from component processes to g (Protzko, 2015, 2016). Transfer relies on the training of core markers of general intelligence shared by the component processes. Training specific processes recruited by general intelligence, when needed, such as inhibition or rehearsal, would not improve general intelligence, if core processes would remain unchanged. Change would occur only if training central processes that are used to regulate inhibition or working memory, such as relational or awareness processes that may be activated to control inhibition or working memory. Also, attention control and working memory contribute to the formation of general intelligence but their contribution varies with development. They are more important at the beginning rather than the end of developmental cycles and they recede in importance after adolescence (Demetiou et al., 2013, 2014). Also, several training programs involved young university students (Jaegi et al., 2008). Obviously, training executive processes at this age would not generalize to general intelligence because these processes are secondary in its current functioning, which primarily draws on advanced reasoning. If one wants to increase the intelligence of university students one must focus on what is developmentally important at this age phase.

Capitalizing on Developmental Theories

The theories of Piaget (1970), Vygotsky, (1986), and Bruner (1973) influenced education significantly in the second half of 20th century. At that time, these theories were instrumental in raising awareness that children view the world differently from adults and that their view must be respected, if efficient learning is to occur. This implies recognizing developmental constraints of learning: to be learned, concepts and skills must be presented at the level of abstraction and complexity that is appropriate for the developmental level of the student. For instance, using arbitrary symbol systems standing for abstract entities may present an impossible task at the beginning of primary school. As a result, these theories motivated extensive research on cognitive development and learning which, in turn, influenced the organization of primary and secondary school curricula (Kuhn, 1979). Finally, these theories raised awareness about the role of children's active involvement in learning, generating the constructive approach to learning. According to this approach, learning is more efficient and stable if children themselves are active in constructing and discovering knowledge rather than passively listening to a teacher.

Eventually, however, the popularity of these theories declined because they failed to improve day to day teaching and learning at school. There were several reasons for their decline. One of them is that these theories were too global to direct the formation of educational priorities and programs at the level of detail needed by education for day to day teaching or even successive school grades. For instance, in mathematics, children need to learn, grade by grade, a vast array of concepts and skills, such as understanding and operating with different types of number (e.g., integers, decimals, fractions, algebra, etc.). Learning each of these concepts and skills often requires specific representational and integrative processes. These are often ignored by general theories. Thus, it is not a surprise that grasping Piagetian concepts, such as conservation of number, is not related to learning and doing arithmetic (Hiebert, Carpenter, & Moser, 1982; Lemoyne & Favreau, 1981; Pennington, Wallach, & Wallach, 1980).

Also, these theories underestimated the complexity of development. For instance, Piaget's theory (1970) conceived of cognitive development as a linear progression towards an ideal end state, formal operations. As a result, it emphasized one aspect of this progression, reasoning, and downplayed others, such as self-regulation and representation. For instance, early in primary school children do not know how to adjust their learning according to tasks (Annevirta & Vauras, 2001; Digmath, Guettner, & Langfeldt, 2008). Thus, memorizing material may fall short of the effort required because students think, wrongly, that what it is in front of their eyes now would be available in memory later. Therefore, stressing construction and discovery learning is not enough to ensure the learning of the concepts involved. Learning often requires assistance and systematic scaffolding that would direct the student to grasp new concepts and drop old ones. Accordingly, extensive research suggests that guided discovery is much more effective than unguided discovery learning in helping students to learn new concepts and transfer them in different contexts (Mayer, 2004).

Capitalizing on Conceptual Change

The conceptual change approach to learning and cognitive development emerged in the 80s as a reaction to the weakness of the theories above to account for understanding of specific concepts (Carey, 1985; Vosniadou, 2013). In a way, this approach turned priorities upside down, emphasizing domain-specificity and downplaying general abilities or processes. According to this approach, children construct models about the world as they interact with various phenomena, such as physical, astronomical, biological phenomena, etc. These models are subject to revision and change like scientific theories. They survive in as long as they serve their purpose to explain incoming information relevant to their domain; if new information cannot be understood or contradicts predictions, they are revised to avoid mistakes in the future.

Learning in this context is based on several principles: Children must "do" science actively–observe, experiment, test hypotheses, compare with held beliefs, reflect and revise, if evidence does not accord with expectations. Teachers must take children’s mental models seriously and build environments for them where they can express their concepts about a phenomenon, manipulate objects according to their expectations, and compare findings with expectations and expert models, and revise them accordingly. However, research suggests that students often do not learn the scientific concepts we teach them. Wrong models and misconceptions persist, even among specialists, because restructuring dominant concepts in favor of scientific concepts requires processing and inferential possibilities far exceeding students' current developmental capabilities (King, 2010).

Fundamental Principles for Tuning Education with Developmental Priorities

In summary, the research reviewed above suggested that enhancing intelligence is possible, but gains are not sustainable over time. Training specific processes is also possible, but gains are not transferable to the core of intelligence. Letting children be active and constructive when they learn is useful, but learning is weak if not properly directed and focused on the specifics of interest. This weak influence of learning experiments comes in contrast with the influence of formal schooling. There is strong evidence that each extra year of schooling causes an increase of 1-5 units of IQ (Gustafsson, 2008: Ritchie & Tucker-Drob, 2018) and accelerates cognitive development (Kyriakides & Luyten, 2009). These influences underlie the secular increase in intelligence, known as the Flynn effect (Flynn, 2009). According to this effect, there has been an increase in average intelligence by about 25 IQ points over the 20th century. The model to be presented below suggests that this is due to three aspects of schooling: First, it is long lasting, recurrent, and repetitive; i.e., it covers early childhood to late adolescence and many concepts and skills often recycle over different grades, getting increasingly complexified and expressed in different contents and contexts. Second, it addresses both general and specific mechanisms in the context of different subjects, promoting reflection, bridging of concepts, abstraction, and reconceptualization. For instance, abstracting relations and re-stating them into more general language may occur in very different subjects such as language and mathematics. Third, even if not very systematically, it familiarizes with the use of different symbol or representation systems, such as mathematical, linguistic, visual, or musical notation. This helps decontextualize mental representations and transformation from its actual contents. This pattern of effects suggests that to be sustainable, learning addressed to central intellectual mechanisms must be developmentally relevant and recursive over developmental cycles. To be transferable, it must directly address the common core at each developmental phase. To be self-enhancing, it must be able to operate beyond the known and familiar. This may occur in the context of educationally important concepts or skills.

Therefore, a major challenge here is to distill what is most important in school from each of these sources, refine it, and give it back better targeted and programmed. The aim is a balanced model integrating the three traditions outlined above. In this model, the constructive approach to learning is coupled with developmentally sensitive methods for scaffolding learning at successive school grades throughout school life, capitalizing on developmental priorities and possibilities of successive phases of life. Concepts are taught at the representational resolution, inferential power, abstraction refinement, and flexibility in conceptual revision that is possible at different developmental periods.

Table 1. Developmental priorities, educational priorities, and learning across developmental cycles

Age/cycle	Developmental priorities	Symbol systems	Problem-solving priorities	Language priorities	Priorities in mathematics	Learning difficulties
Early Preschool	Attention control.	Mastering language and understanding the role of symbols.	Explore action-based solutions.		Early number sense.	
Late preschool	Perceptual awareness.	Building connections between symbol systems.	Learn how to build a representation for a problem, plan, evaluate.	Acquire basic phonological awareness; pre-writing activities.	Representational mapping and differentiation, map number words on number digits, relevant images of objects. Grasp cardinality and quantitative comparisons.	Attention control and coordination between a represented goal and action.
Early primary	Inferential control, Rule induction, inductive reasoning.	Symbol and representational integration.	Fill in lags in information by inference.	Encoding; learn that letters stand for sounds, syllables for blocks of sounds, which form words, the main units of meaning. Combine words into meaningful textbase representations. Writing.	Exploration of the patterns between 1-digit numbers, tens and decades in order to grasp the rules underlying structure of place value system	Dyslexia, dyscalculia
Late primary	Inferential awareness.	Understanding that symbol systems are the exchangeable	Specify when available knowledge or solutions are not enough and look	Abstract different lines of meaning in texts. Evaluate the rationale behind	Master rules of specifying numeric relations and patterns and transform them to	Crystallization of mental processes and knowledge into ready-to-draw on mental

			for or invent new ones.	a text, even where the text covers multiple events or themes and the overall rationale is not apparent unless analyzed at several levels.	each other. Recognition of relations between numbers must extent from integers to rational numbers, complex numbers	skills and concepts
Early secondary	Truth and consistency control, Principles, deductive reasoning		Think for all solutions possible and then reject less good solutions and choose promising ones according to criteria.	Fluent reading and story production in writing. Understand abstract ideas from texts in different domains, such as newspapers, scientific texts, literature,	Grasp the algebraic principle of number as a variable that can take any value and that operations on numbers depends on their precise nature	Difficulties in forming a personal symbol system for handling and manipulating abstract concepts
Late secondary	Awareness of logical constraints, accurate self-representation		Be creative and original, choosing best solutions even against personal biases or believes.	Read texts in different domains, mastering domain-specific special language.	Model complex problem situations using formal mathematical language, specify similarities and differences between different problem situations, and reflect on, evaluate, compute fractions	

		of different denominators and decimals,

Cognitive Architecture, Development, and School Learning

We outline a framework aiming to tune cognitive developmental priorities with educational priorities from preschool to adolescence. The aim is twofold: on the one hand, to satisfy cognitive developmental needs at successive phases in order to maximize the possibilities of each individual; on the other hand, to capitalize on the possibilities emerging from developmental priorities in order to maximize learning in each school subject. This framework is based on how the mind is organized and how it develops. The basic principles of cognitive architecture and development are summarized below.

Cognitive Architecture

A general factor. Research in the psychology of individual differences strongly suggests that *learning and understanding are constrained by a powerful factor of general cognitive ability.* Through the years, this factor has come under various names. For psychology of individual differences, which discovered and studied this factor, it is general intelligence or g. Tests of intelligence, such as the WISC or the Raven test, measure this factor (Carroll, 1993; Jensen, 1998; Spearman, 1904). This factor stands for the processes following:

1. Allows identifying things of interest amidst the variety of objects and events usually present in the environment and focusing on them, resisting distraction.
2. Hold these things in mind and relate them with other things of relevance or with past knowledge for as long as required to make sense of them, given their context.
3. Make connections by inference, if information is missing, and conclude if they fit or deviate from what was known or believed so far.

4. Make choices, interpretations, or decisions best serving current priorities or interests.
5. Capitalize on feedback from the results of choices, decisions, or actions, to avoid mistakes in the future.

In more technical language, this factor is defined by a mixture of (i) processing speed, (ii) attention control and flexibility, (iii) working memory, (iv) inductive reasoning allowing generalization and extrapolation of information and deductive reasoning allowing to validly fill in gaps in information based on inference, and (v) cognizance, (i.e., awareness of perceptual and inferential origins of knowledge, awareness of cognitive processes, and self-evaluation) (Demetriou et al., 2018a, 2019b; Makris et al., 2017; Spanoudis et al., 2015).

This factor has been detected in standard tests of academic performance, such as the SAT (Coyle, 2015) or International Educational Competitions, such as PISA, TIMSS, and PIRLS (Rindermann, 2007) even though with some controversy. There is ample evidence that this factor strongly influences school learning from preschool to university. Psychometric g or equivalent measures of cognitive ability derived from developmental theories accounts for the lion's share of school performance, varying from about 30% to 70% of the variance of school grades or scores in standardized tests of academic achievement (Demetriou et al., 2019a, 2019b; 2020; Gustafsson, 2008; Gustafsson & Balke, 1993; Kaufman, Reynolds, Liu, Reynolds, McGrew, 2012; Roth et al., 2015). Also, g predicts many life outcomes, such as job selection, performance, and income (Strenze, 2015), and eminence in many domains, including science and prestigious professions (Bernstein, Lubinski, & Benbow, 2019).

Learning is representation-specific and domain-specific. There is ample research suggesting that *learning and understanding are also constrained by the representational and procedural specificities of different domains of information and knowledge in the world.* For instance, verbal, quantitative, spatial, and social information are differently represented and inter-related. Words are patterns of sound connected by grammatical and syntactical rules to form sentences bearing meaning for the speakers of a language (Dehaene, 2010). Numbers stand for aggregations of objects or events that may increase, decrease, or re-distribute according to various rules (Butterworth, 2005, 2010; Dehaene, 2011; Gelman, 1986).

It is a truism that language or arithmetic cannot be learned without the general processes above. To parse words in one's own language and grasp the rules of grammar or syntax inter-relating them, children must decipher recurring patterns of sound, keep them in memory and match them on memories of them, recognize similarities and differences between word forms (e.g., some-times words end in -ed). and induce relations with actions (e.g., when ending in -ed they refer to actions which took place in the past) (Dehaene, 2010). To grasp number, children must discriminate individual objects, recognize their spatial or other relations organizing them into amounts (of something), keep these patterns in memory and compare them so that their relations standing for numerical operations may be induced (Dehaebe, 2011; Gelman, 1986). Recent research suggests that even very early sensitivity to number, which was ascribed to an innate "number sense", reflects the operation of general mechanisms that capture key characteristics of numerocity in the perceptual field, such as a particular aggregation of objects which gives rise to numerical interpretations (Testolin, Zou, & McClelland, 2020). On the other hand, to function efficiently, general mental processes need to have the representational and relational units in each domain available to mind (see Demetriou & Spanoudis, 2018; Demetriou et al., 2018). If, for any reason, units in a domain, such as words in language or amounts in arithmetic, cannot be recognized or represented accurately, and mentally processed, performance in this domain would suffer, even if general mechanisms operate well other domains.

Cognitive Profiles and Priorities Change with Development

Developmental profiles. Psychometrically speaking, the general factor, g, is always present, regardless of age or the use of conventional tests of intelligence or developmental tasks as Piagetian tasks (Case, Demetriou, Platsidou, & Kazi, 2001; Carroll, Kohlberg, & DeVries, 1984; Demetriou et al., 2017). However, there is equally strong evidence that *the profile of cognitive processes underlying g changes with development.* A study of the precise combination of these processes in marking g in successive periods of life (i.e., 4-7, 7-11, 11-17, and 20-80 years of age showed that *g in each period was marked by a different combination of the processes specified above* (Demetriou et al., 2017).

Specifically, the major marker of *g* in preschool was attentional control and awareness of the perceptual origins of knowledge. The refinement, interlinking, and lexicalization of episodic representations at the end of infancy make preschoolers highly symbolic: they are interested in symbols, they learn them fast, they massively use them in their interactions with objects and persons, and they have some awareness of their dual nature as *real* entities (e.g., a photograph or video) and *representations of* something else (e.g., the persons in the photograph or video) (DeLoache, 2000). This representational insight ("I can think of my parents", "I can think of my toys", "I can try my thoughts out") poses an important challenge to the toddler: representational control: holding representations active and in focus so that information in the senses is encoded, processed, chosen or ignored, according to its relevance to the currently focused goal so that an action sequence may implement a mental plan. *Thus, control of attention is the major developmental priority of this cycle.* This is an important developmental task because it allows more complex cognitive tasks, such as sustaining a plan of activity, using tools in sake of a goal, exploration of objects, categorization, etc. Notably, the development of domain-general selective attention contributes to the development of category representations, because it allows to search for and process objects or properties related to a category under formation (Deng & Slutsky, 2015).

Attention control involves awareness. For instance, to focus on this rather than another object requires one to be aware that one has eyes on a head which can be turned to objects at will. In fact, toddlers are aware of representations they have in mind, such as memories of events observed earlier (Paulus, Proust, & Sodian, 2013). Also, they are aware that perception and representations are connected: we know what we see, hear, touch, etc. This makes Theory of Mind (ToM) possible (Spanoudis et al., 2015). Children have a ToM when they understand that one's actions relate to one's own representations which derive from one's own perception (Demetriou et al., 2018a; Kazi et al., 2019). Flexibility and abstraction are linked at this age: when children abstract a pattern from several stimuli are also flexible in switching across them. Representing and implementing an executive program is possible because it is based on the ability to abstract a common pattern across situations (e.g., all objects but one lie horizontally) and hold this pattern as a general representation that may guide action (Kharitonova & Munacata, 2011).

Finally, reasoning emerges in this cycle when event sequences are meta-represented as implicative associations involving a conceived necessary connection between two events. At first, it appears as a meta-representation of “if A then B” relations, laying the ground for modus ponens reasoning. For instance, a 3-year-old seeing the dad’s car outside the house concludes that the dad is in, but he cannot explain why. At 5 years, explanation is possible based on awareness of the specific representations involved (e.g., “we know that everyday dad’s car is here when he is in the house too”; personal communication with grandchildren, AD, and children, SG). Thus, reasoning in this cycle is secondary to the priorities of attention control and related awareness.

In primary school, from 7 through 12 years, the main markers of g were inductive reasoning as indexed by Raven matrices, simple deductive reasoning, and awareness of inferential processes. With attentional control established, links between representations take priority. This complexity presents a new developmental challenge: identify relations between representations and organize them so that they can be called upon efficiently in sake of understanding and interaction. Thus, *inferential control is the major developmental priority in this cycle*, because it is a process dealing with relations between representations. Inductive and analogical inference is the major tool in rule induction and inter-linking (Gentner & Hoyos, 2017). This allows mental fluency that is evident in analogical reasoning required in arithmetic problem-solving and Raven matrices. Solving these problems indicates that inference is fluid enough to access individual representations (e.g., numbers in an arithmetic task or figures in a Raven matrix) align them and identify relations between them (e.g., numbers double, figures increase by size) and bind them together according to underlying relations (e.g., size increase goes with decrease in color shade). Reasoning consolidates in this cycle as it is evident in explicit deductive reasoning showing a grasp of the relations between reasoning schemes. For instance, they now understand that “if A then B” necessarily implies that “if not B then not A”. They understand that if an event A causes an effect B, then if the effect is not present the causal event is not present either.

The development of reasoning in this cycle comes with increasing awareness of the inferential processes themselves: awareness of underlying inferential processes (e.g., one may know something because one reasoned about it) and awareness of cognitive processes involved in different reasoning tasks (e.g., to find your way you need to imagine a road; to find a sum you need to combine numbers) emerge at 8-10 years of age (Kazi et al., 2012, 2019; Spanoudis et al., 2015), indicating that inferential choices become the object of reflection in sake of optimizing conclusions. In turn, this generates awareness of inferential control: inference may take alternative roads depending on the representations connected and how they are connected. This awareness may be used to arrange representations according to the rule at hand, such as sequences of executive acts to formulate an action plan, extrapolate dimensions in inductive reasoning tasks according to the relations involved, or deduce conclusions from premises in deductive reasoning tasks (Kazi et al., 2019). Overall, in primary school, multimodal representations are organized into rule-based hierarchies of concepts increasingly predicated by language or other symbol systems. Thus, rules themselves stand out as powerful representations in the fashion that specific representations are the object of mental processing in the previous cycle. In conclusion, reasoning and related awareness acquire priority in this cycle. Attention control recedes as a predictor it approaches ceiling.

In adolescence, from 11 to 17 years, the main markers of g were advanced deductive reasoning, mathematical reasoning, awareness of logical constraints of reasoning, and awareness of mental processes as such. It seems that *a major priority of this cycle is mastering processes allowing to evaluate knowledge and choices for cohesion, validity, and truth*. Logic, for those who reach this cycle, becomes important because it is a powerful tool for specifying truth and validity of conceptual and inferential processes (Demetriou & Spanoudis, 2018; Demetrtiou et al., 2017, 2018). This requires the construction of principles underlying and constraining relations between rules or concepts. Therefore, control in this cycle is a logical metaprocess that defines constraints for acceptable and non-acceptable inferences. For instance, adolescents resist logical fallacies associated with the basic schemes of deductive reasoning. They understand that "if A then B" does not imply that "if B then A" because B may occur for reasons other than A. Thus, logical reasoning dominates in this period. Truth control is also an epistemic metaprocess: this specifies when descriptions of reality may be accepted as true knowledge of that reality. For instance, a statement about a relation may be accepted as true (e.g., food A affects health) only when confounding variables (e.g., exercise, other foods, etc.) were controlled according to acceptable practices of control; even then it is understood that some unknown variable may falsify this relation in the future.

Principle-based thought in adolescence is obvious in different domains. Adolescents solve complex Raven matrices requiring abstracting a principle underlying several seemingly different transformations. They grasp analogical relations within and across different hierarchies. For instance, they can specify multiple relations connecting societal institutions, such as family, education, and government (e.g., father, teacher, and president play equivalent roles in different organizations). Thus, control in this cycle is based on a suppositional-generative program enabling co-activation of conceptual spaces, evaluation against each other and formation of personal preferences and long-term life plans, such as choosing a course of studies. The suppositional stance brings disparate representational spaces under principles accepted as true so that values of truth and validity may vary according to the principle currently used. This achievement allows consistency in reasoning, because a single principle overwrites different contexts as it shifts processing from contextual information to their underlying relations.

Constructing principles about principles may never end, expanding in a never-ending hierarchy, in a Gödelian fashion. These representations are remote from actual objects and their properties. To be represented as such they may need ad hoc symbols, as in mathematics and science. Scientific terminology and related symbol systems stand for abstract relations or high-level multidimensional concepts, such as mass, energy, velocity, etc. This kind of representation may be a prerequisite to learn abstract scientific categories and their underlying principles, regardless of discipline, such as logical or mathematical in science or legal and moral in social or political relations.

The reader may have noted that the cycles above coincide, by and large, with developmental levels as specified by a long tradition of research and theorizing on intellectual development (e.g., Bruner, 1973; Case, 1985; Piaget, 1970; Shayer & Adey, 2002). At the surface, this seems to be the case. However, these cycles index "regions of change" rather than hard boundaries. They indicate time windows over which different developmental priorities dominate until they are mastered. In fact, each cycle involves a first phase of emergence, when the new representational unit is established and a phase of consolidation, when representations are interlinked, yielding representational complexed. Each phase is indexed by different processes, such as speed of processing at the beginning and working memory at the end (Demetriou et al., 2013; Demetriou et al, 2014). When established, they give way to the next priority. Developmental priorities satisfy different adaptive needs, such as sustaining interaction with objects, sustaining attention to regulate one's own action according to goals, integrating and filling in gaps in information and knowledge, and checking for truth and cohesion in the four cycles, respectively.

How Change Happens

A factor of change on top of g. Attention is drawn to the fact that developmental priorities changed across cycles but one: cognizance was always present, each time taking the form of awareness of the processes dominating in each cycle. This makes cognizance a strong candidate for the transition force underlying change across cycles. A recent longitudinal study differentiated the state of ability at first testing from the momentum of change from first to second testing, which took place two years later. To implement this differentiation, a general second-order factor was associated with all first-order domain-specific factors standing for performance on attention control, working memory, reasoning, and cognizance. To capture change as such, a second general factor was associated with the difference between performance at first and second testing. This factor was used to predict change in each process from first to second testing, in addition to the factor standing for performance at first testing. It was found that the factor of change was more important in predicting change in various executive and reasoning processes (Kazi et al., 2019). Developmental momentum was driving change across different mental processes. The further one was from the ceiling of a cycle the more one tended to change across the board to reach the ceiling. Further modeling showed that cognizance was the leading factor in transitions from a lower to a higher level of inductive and deductive reasoning. In other words, awareness of the processes dominating in each developmental cycle was instrumental in catalyzing their change. It is notable that Tucker-Drob, Bradmaier, and Lindenberger (2019) recently found a general factor for cognitive change that strengthens with advancing age.

It seems that striving to meet the developmental priorities of a cycle causes learning related to these priorities. At preschool, trying to attain attention control results into learning the role of perceptual processes and related acts. For instance, focusing on a stimulus of interest and avoiding looking at another attractive stimulus may make the child realize that information comes from the senses and that these may be partly under control, if focusing on them and acting accordingly, such as persisting seeing a goal related stimulus and ignoring another. In turn this may make the child realize that differences between persons in perceptual access to a specific stimulus cause them to have different knowledge and mental states about it. Later, striving to attain inferential control by combining information may make children realize that inference fills in gaps in information. Also, that relating is a process for looking for similarities and differences in concern to a property of interest.

Training general cognitive processes. Several studies examined if changing specific aspects of general cognitive processes would increase performance in specific domains and generalize to other domains as well. These studies involved inductive and deductive reasoning and related awareness of the processes involved. One study examined if training relational thought and related awareness in mathematics would improve performance in several aspects of mathematical problem solving and generalize to other aspects of intelligence (Papageorgiou, Spanoudis, Christou, & Demetriou, 2016). This study involved 10-11-years-old children. Children were instructed to identify the dimensions underlying the various mathematical reasoning tasks involving number series varying on several patterns (e.g., double, triple, half, one fourth) and mathematical analogies, explicitly conceive of their similarities and differences, group them according to organizational rules, and build the problem-solving skills associated with each. Thus, they were required to explicitly meta-represent both problem structures and processes as well as their associations. The emphasis was on formative concepts like “attributes”, “relations”, “similarity”, “dissimilarity or difference” and their instantiation in the various problem types.

The training group and a matched control group were examined, in addition to mathematical problem solving, on various aspects of attention control, working memory, and reasoning (deductive, analogical, spatial, causal-scientific, and mathematical). The change in the domain of mathematical reasoning was considerable soon after the end of the intervention (effect size =.38) and remained significant (effect size = .20), although somewhat weaker, about six months later. However, the gains did transfer to domain-free analogical reasoning tasks (effect size = .20) and, to a lesser extent, to other domains, such as deductive reasoning (effect size = .12). Gains in deductive reasoning were stable from second to third testing (effect size = .13), implying transcription of gains into more formal inferential processes. Also, there was a strong effect on working memory (.93) and a less strong but significant effect on inductive reasoning (.38) and attention control (.10) which were preserved at delayed posttest. Obviously, these effects indicate that cognizance mediated in the transfer of gains in relational thought to processes residing in the executive control level. This pattern lends support to the assumption advanced above that training general intelligence processes travels top-down rather than bottom-up when affecting core processes that may improve the control of general processes that serve specific needs in overall processing, such as attention control and working memory.

Another study examined if building awareness of logical schemes and facility in transforming them into relevant mental models would catalyze transition from rule-based to principle-based reasoning (Christoforides et al., 2016). Specifically, this study trained 8- and 11-years old children, allocated in a control, a limited instruction, and a full instruction group, to become aware of the logical characteristics of the four basic logical schemes of conditional reasoning: modus ponens, modus tollens, affirming the consequent, and denying the antecedent; also we trained children to build and mentally process mental models appropriate for each scheme and explicitly represent their relations (e.g., that affirming the consequent is not the opposite of modus ponens and denying the antecedent is not the opposite of modus tollens). The aim was to examine if enhancing cognizance via conscious inferential activity about these schemes and processes would result into transition from rule-based to principle-based deductive reasoning. At the same time, we investigated whether possible progress depends on attention control and working memory. The limited instruction group learned the notion of logical contradiction and the logical structure of the schemes involved. The full instruction group learned, additionally, to adopt an analytical approach to logical arguments as contrasted to their "every-day" use in language, differentiate between the stated and the possibly implied meaning of propositions, recognize logical contradiction and truth in propositions and reality, and grasp the notions of logical necessity and sufficiency.

In terms of spontaneous developmental time, this short training program pulled children up by an almost full developmental phase, especially in the full instruction group. That is, trained third graders handled problems at the level of principle-based reasoning *if aided by context*; sixth graders moved to this level regardless of content and context. Specifically, this intervention enabled both age groups to master the fallacies of affirming the consequent (knowing that when A occurs B also occurs does not allow any inference about A when knowing that B occurred) and denying the antecedent (under this condition, knowing that A did not occur does not allow any inference about B). The key to this success was awareness of the inferential identity of each scheme and the principle of logical consistency. The limited instruction group trained in these two aspects of inferential awareness performed close to the full instruction group. Overall, awareness almost fully mediated the influence of training on deductive inference. However, awareness as such improved significantly only in the full instruction group and was highly dependent on attention control and working memory. In short, third graders grasped the logical principles implicitly; sixth graders grasped the principles explicitly and performed accordingly. These findings show that when children think about reasoning they become aware of it; when they become aware of reasoning, they better handle other reasoning or cognitive tasks. Interestingly, those high in executive processes profit more from awareness training. When profiting, this generalizes to other processes as well.

Learning is Developmentally Specific

It was noted above that g is always a major predictor of school performance (Gustafsson, 2008; Ritchie & Tucker-Drob, 2018). It is important to stress, however, that the contribution of various mental processes to the prediction of school performance varies as a function of their relative importance in successive developmental cycles: *the best cognitive predictors of academic achievement in a developmental cycle or in later cycles are the developmental priorities of this cycle.* Specifically, attention control (~23% of variance), working memory (~20%), and cognitive awareness (~43% of variance) at 4-6 years, but not reasoning, are strong predictors of school performance in mathematics and language at 8-10 years (Demetriou, Kazali, Kazi, & Spanoudis, 2020, in press). Interestingly, this study also showed that the factor standing for the momentum of change as such, additionally to g, was a strong predictor of school achievement in primary school. Working memory (46%), cognitive flexibility in rule-shifting (6%), and reasoning (4%) are the best predictors of achievement in language, mathematics, and science in primary school. In adolescence, language ability (36% of variance), reasoning (20% of variance), and self-evaluation (5%) are the best predictors of achievement in mathematics, language, and science (Demetriou et al., 2019a). Also, in adolescence, a personality factor, conscientiousness (10%), was a significant predictor (Demetriou et al., 2019b). Perhaps, conscientiousness is the mature expression of the executive control factors operating in preschool and primary school.

There are good reasons for this pattern of relations. Executive processes are more relevant than other processes, such as reasoning, for learning in preschool and early primary school because they reflect the ability to conform to school demands and demonstrate the effort and focus needed to master the numeracy and literacy skills taught at this level of education (Nelson et al., 2017). Additionally, these processes are needed to carry on the representational integration required for learning the two central subjects in early primary school, namely reading and arithmetic learning. There is extensive research showing that inhibitory control, flexibility in shifting, and representational awareness, including phonological and numerical awareness, at preschool, account for learning to read and do arithmetic in primary school (Chung & Ho, 2010; Clark, Pritchard, & Woodward, 2010; Vanbinst, van Bergen, Ghesquière, & De Smedt, 2020). Obviously, grasping the perceptual and inferential origins of knowledge and its dependence on one's personal access to information facilitates school learning because it allows cognitive self-management to meet the demands of complex school tasks and directs the child to examine others' knowledge, including teachers or other children. In adolescence, the emergence of abstract reasoning and differentiated self-awareness and self-evaluation coincides with the increased semantic, syntactic, and abstract reasoning demands of the concepts and problem-solving taught at secondary school (Demetriou et al., 2018, 2020).

General Problem Solving

There is a problem to be solved when attaining a mental or behavioral goal is not automatically possible. Broadly speaking, problem solving is figuring out how a goal is to be attained. Thus, problem-solving involves several discrete components:

1. Using available knowledge to understand what the problem is about.
2. Specifying where in the problem available knowledge is not enough for attaining the goal.
3. Exploring if variations of existent knowledge or strategies may meet the goal.
4. Search for other knowledge, information, or strategies that may serve the purpose.
5. Invent an answer or solution by inferentially extrapolating from present knowledge so that lags of information and knowledge may be filled in, creatively and validly.

6. Integrating old and new knowledge and strategies into a solution that may be evaluated for adequacy and accuracy vis-à-vis the goal.

Cognitive developmental priorities of successive developmental cycles constrain the situations presenting problems to be solved and the type of solutions that may be attempted (Stadler, Becker, Gödker, Leutner, & Greiff, 2015). Mastering interaction with objects is a major source of problems in infancy. The lack of differentiated representations that may be used to explicitly specify a goal and assemble a sequence of actions for solution constrain the infant to trial-and-error strategies for problem solving, allowing gradual modification of an initial behavior, such as using a spoon to eat (Keen, 2011). In preschool, mastering executive control is a major source of problems to be solved. Preschoolers need to realize that alternative representations may stand for the same thing and for each other. To solve problems, they need to learn how to build a representation for a problem (e.g., jumping from the sofa on the floor), plan their actions (e.g., contraction of feet and hand in a specific way before jumping), and evaluate (e.g., if it was precise, if it was painful, etc.) in order to improve next time (Zelazo, Carter, Resnick, & Frye, 1997). Thus, mastering strategies to resist temptation or distraction, stay focused on goal, and try to produce a solution is important in preschool. Learning problems associated with attention control, as in learning to read and learning arithmetic, may be ameliorated by early programs aiming to train attention control and related perceptual awareness in relation to self-control of behavioral action and then associate with school-related activities.

In primary school, children need to learn that they can “decipher” missing representations from other ones, once their common referent is known. Thus, they must learn how to recruit available knowledge, looking for new knowledge if available knowledge does not suffice, and use reasoning to relate, bridge, and fill in lacks in representations vis-à-vis a problem (as in arithmetic problem solving, text comprehension, etc.) are important for problem solving.

Later, in adolescence, adolescents must realize that they may generate sets of representations for a given object provided the constraints given and that they may be mapped onto a reality and checked for consistency with it. At this age, problem solving matches its traditional definition postulating that it is "… successful interaction with task environments that are dynamic (i.e., change as a function of the user's interventions and/or as a function of time) and in which some, if not all, of the environment's regularities can only be revealed by successful exploration and integration of the information gained in that process." (Frensch & Funke, 1995, p. 14). So defined, problem solving is moderately related with the general intelligence (.43; Stadler et al., 2015). At this phase of development, problem solving requires of c*ritical thinking:* the ability to embed cognitive functioning into real-life contexts and make decisions, drawing on the information available together with an evaluation of possible outcomes and their possible value both for the present and the future as well. Critical thinking requires viewing representations, concepts, believes, and models from the point of view of each other and specify alternative conditions under which they might be acceptable.

It is argued that no one can really be critical outside a specific conceptual system, because a critical approach is applied on established facts or ideas that need to be changed, improved, or abandoned. Being critical is also constrained by the current state of problems and the specific needs dominating at a specific time in a specific social group. However, there is research showing that general cognitive ability and the ability to decontextualize were more flexible in using prior knowledge overcoming personal biases to solve a problem originally (Sa, West, & Stanovitch, 1999). Thus, educating critical thinking would first enable thinkers to map the domain of inquiry and draw upon the concepts already possessed in concern to it (see Demetriou & Spanoudis, 2018). Once the domain is specified and concepts mapped, the information available must be evaluated for relevance, given the problem domain and goals. Information and possible solutions must be examined for cohesion and adequacy by reasoning. Alternatives, past or new, must be evaluated and possible advantages and disadvantages for each solution must be specified. Also, relations with personal beliefs and possible personal biases must be specified. The aim would be to examine if solutions are preferred for consistency with personal bias rather than for general value or quality that may run against personal preferences or biases. Ideally, critical thinkers are able to choose solutions that are better, nicer, or broader than extant solutions, even if this implies abandoning old beliefs, extant theories, or dominant views (see Stanovich, 2011).

Learning in Language and Mathematics

We noted above that changes in the nature of cognitive ability relate with changes in important markers of it, such as the symbol systems than may be mastered. Here we focus on two domains of school learning that are important throughout school life: language and mathematics. Systematic teaching in both domains starts at preschool and continues through college under a development hierarchical perspective. There is significant common ground between the two domains but also large differences. On the one hand, both domains are highly symbolic, requiring children to learn an arbitrary code, where units may be specified at various levels and composed according various, hierarchically organized, rules. Learning at school is organized, to a large extent, according to each system's hierarchical organization from simple to complex.

On the other hand, the two systems differ extensively in their representational functions, scope, precision, specificity, and familiarity of use. Language is a universal omnipresent representational system and children are exposed to it since birth, if not earlier (Dehane, 2010). Although the construction of mathematical concepts and the development of mathematical relations start intuitively from the infancy, the understanding of the symbolic mathematical system becomes part of everyday life much later, when formal schooling begins (nowadays in most cases at the age of 3 to 5). Mathematics mostly draws on its own notational and representational systems which differ extensively from language. The rules and principles underlying transformation and composition of representations in the two systems are drastically different (e.g., semantics, grammar and syntax in language, symbols, arithmetic operations and rules in mathematics). At the first stages children learn how to express verbally and in writing their thoughts, recognizing communicational dispositions or constraints (e.g., the writing of a formal letter). In mathematics they find and present examples or counter-examples in order to verify or reject a statement under an experimental perspective and then they have to construct a formal mathematical proof by using the appropriate symbols and structure. Thus, from the point of view of education, it is interesting to specify similarities and differences of learning in these two domains during development.

Cognitive developmental research suggests that changes in the content and teaching methods of school mathematics may result into significant improvements. For instance, presentation of several concepts at school may come earlier than in current programs, if properly organized and technically supported. For example, using a fairy tale with an elevator going underground may enable children grasp intuitively the concept of negative number, a concept which is currently formally introduced at 6th grade of primary education. The introduction of concepts before formal teaching, based on the findings of research, may enable the construction of a solid intuitive foundation for teaching high demand concepts.

The present theory suggests that efficient learning in each of the domains depends of Language and Mathematics on tuning the demands of learning in each school subject with developmental priorities of the age concerned. Specific learning difficulties would ensue if curricular demands and children's possibilities diverge. For instance, delays in mastering the developmental priorities in preschool (attention control, representational awareness, and a minimal flexibility in analyzing and inter-relating representations) would hinder learning the symbolic skills required for reading or arithmetic in early primary school. Deficient attention control would disable children to accurately register and encode letters or numbers; deficient representational awareness would hinder them to understand that written words stand for the words of oral language or numerals stand for number words and related quantities. Delays in mastering rule-based thought in primary school, such as difficulty in rule induction, may cause difficulties in reading comprehension that is based on grammatical and syntactical rules or in implementing arithmetic operations on numbers. Therefore, deficiencies must be diagnosed in time, removed to the extent possible to render students ready to learn, and systematically direct learning to strengthen mental development.

Language

Reading involves three hierarchical levels (Kintsch, 1988, 1994; Kintsch & Rawson, 2007) that align with the cycles of development described above. The first is the encoding level that is dominated by perceptual attentional processes. Users of alphabetic writing systems learn that letters stand for sounds, composed into syllables standing for blocks of sounds, which form words which are the main units of meaning. These are composed into sentences according to grammatical and syntactic rules which signify meaning. At this level, letters are recognized and integrated into words which are themselves integrated into sentences.

Learning at this level starts in preschool and extends over the first two grades of primary school. At preschool children start to recognize script or words as symbols for speech and they are involved in pre-writing activities aiming at mastering the basic skills required for writing. Learning to read (and write) is one of the major goals of the first two primary school grades. In terms of the present model, learning to recognize letters and compose them into syllables and words is heavily based on three two major developmental priorities of the cycle of realistic representations: (i) Construction of new representations to stand for different realities (i.e., written words to stand for spoken words. (ii) Attention control is important at the very early stages of learning to read, when focusing and shifting are required for the integration of letters into words and words into sentences. Several studies showed that deficiencies in control of attention allowing systematic spatial search and orienting at the early stages of reading hinder learning to read even after IQ, hyperactivity, and other behavioral problems are control (Franceschini, Gori, Ruffino, Pedrolli, & Facoeti, 2012; Rabiner & Coie, 2000). These difficulties are present in different languages, such as Arabic (Friedmann & Haddad-Hanna, 2104) and Chinese (Chung & Ho, 2010). (iii) Representational awareness needed to recognize words as signifiers of objects or actions that may be focused on mentally and combined. Representational awareness, expressed here as phonological awareness and comprehension monitoring allows children to monitor, reflect on and evaluate their comprehension to reprocess and re-visit the text if necessary. In line with this assumption, McCardle, Scarborough, Catts, (2001) found longitudinally a mutual causal relation between phonological awareness and reading achievement: phonological awareness enhanced literacy development which resulted into further growth of phonological awareness. Kim (2015) showed recently that comprehension monitoring and theory of mind were the strongest predictors of reading performance at the age of six years. Also, the relation between working memory and listening comprehension was mediated by comprehension monitoring and theory of mind.

General representational awareness is needed as a top-down guide directing visual search and the integration of mental units into meaningful symbolic ensembles, be they alphabetically composed words, Chinese logographs, or numbers. In alphabetic languages, representational awareness takes the form of phonological awareness, because this is what is required to direct the composition of letters into units corresponding to words. In Chinese, the deficit affects morphological awareness, which "is conceptualized as the ability to distinguish meanings among morpheme homophones or as the ability to manipulate and access morphemes in words with two or more morphemes" (Chung & Ho, 2010, p. 217). In Chinese, morphological rather than phonological awareness is important because readers must distinguish between meanings of homophones and morpheme construction in logographs where semantic radicals and phonetic radicals combine to denote meanings and pronunciation. Deficits in these processes cause delays in learning in both domains. Therefore, the important factor in first-level learning to read is general representational awareness rather than phonological awareness.

The second level goes beyond the technical level of the recognition and production of words to the abstraction of meaning from them. At this level a textbase representation is constructed, drawing on language and cognitive processing mechanisms. This textbase representation is based on vocabulary and syntactic knowledge enabling the reader to process the meaning of words and phrases in the text. Textbase representation may be literally based on the words involved and it may be incoherent. The third level goes beyond words. At this level, the reader distils propositions from sentences allowing the abstraction of a situation model; this is “a mental model of the situation described by the text” (Kintsch & Rawson, 2007, p. 211). At this level readers form causal inferences about the actions and relations in the story, drawing on past knowledge or reasoning to fill in gaps in information or grasp nuances not directly spelled out by the words involved.

Reading at the second level starts at second primary school grade and it may continue until fourth grade for many children. In terms of the present model, this level draws primarily on rule-based thought; the third level is based on both, rule-based and principle-based thought. As expected, reading comprehension by 8-yr-old third graders at the second level is predicted by working memory and fluid intelligence García-Madruga, Vila Gómez-Veiga, Duque, & Elosúa, 2014). Demands of the third level may draw on both rule-based and principle-based thought. In line with this assumption, we recently found that advanced principle-based thought is highly related with advanced language ability (Demetriou et al., 2019).

These abilities are addressed by international assessments of literacy, such as the PISA (2012). In these assessments it is expected that, ideally, children must be able, by second primary grade, to link multiple pieces of information to draw inferences about events and relations and integrate text-wide information in order to identify the main themes in a text. By sixth grade, they must be able to evaluate the rationale behind a text, even where the text covers multiple events or themes and the overall rationale is not apparent unless analyzed at several levels. By third secondary school grade they must be able to understand abstract ideas from texts in different domains, such as newspapers, scientific texts, literature, etc. It is noted that only about 15% of third graders in the OECD countries operate at this advanced level.

Mathematics

Internationally, the mathematical curriculum includes five major domains (number-operations, geometry, measurement, algebra, and statistics-probabilities), from preschool to the end of secondary school. The National Council for the Teaching of Mathematics (NCTM, 2000) specifies mathematical targets for each grade level that must be mastered across these five domains in five major realms of cognitive activity: problem solving, reasoning and proof, communication, connections, and representations. Ideally, students are expected to create and use representations fluently and flexibly in order to organize and communicate mathematical ideas, transform them from one domain to another, and interpret physical, social, and mathematical phenomena. Due to space considerations, in this paper we focus on numbers and operations, a core domain underlying all domains of mathematical thinking.

An Approximate Number System (ANS; Dehaene, 1997) is the background for the development and learning of mathematics in the fashion that natural language is the background for the development of reading and writing. The core of the ANS is subitization, i.e., automatic perception of numerocity of object sets involving up to 3-4 elements. Subitization is present in infancy (and other animals as well); infants also recognize the basics of arithmetic operations, such as the addition or subtraction of elements within the subitization limit. This ability develops into a mental number line in early childhood, which is the pivot of the ANS. Numbers on the mental number line are ordered from left to right according to magnitude. The number line allows approximate comparisons between numbers; the accuracy of these comparisons decreases with increasing number magnitude or decreasing distance between numbers. For instance, it is easier to judge that 27 is larger than 23 than to judge if 727 is larger than 723.

The APN and the mental number line develop throughout childhood and adolescence (Dehaene, 1997; Siegler & Braithwaite, 2016). The mental number line is established between 3-5 years, covering only small numbers between 1 and 10. In this cycle, children can also map number words on small sets of up to 3 dots. They cannot map number words on arrays of 4-6 dots, dots with digits or number words with digits, nor do they compute arithmetic operations (Benoit, Lehalle, Molina, Tijus, & Jouen, 2013). Obviously, they have a global representation of quantities within the subitization limit associated with corresponding number words as an ensemble. The number line extends to 100 between 5 and 7 years. At this phase, representations from the different representational spaces become accessible as distinct mental entities that can be aligned. For instance, 4-year-olds map both number words and number digits with arrays of up to 6 elements but they do not map number words on digits. At 5 years children map all representations with each other for all sizes. Also, they can compute additions and abstractions on numbers smaller than 10, using their fingers as tools for counting. Counting includes both reciting a series of numbers and understanding a symbol as an index of a quantity.

Curriculum in mathematics for preschool and early primary school must develop number sense: a general understanding of numbers, operations on them, and the use of strategies for solving problems involving numbers (Way, 2011). Individuals with good number sense exhibit an intuitive grasp of numbers, they can count, grasp quantitative relations and their mutual constraints, they can plan operations on them and flexibly shift between them, and they can associate numbers and operations with symbolic representations (Mohini & Jacinta, 2010; Ontario Curriculum, 2016). Young children, even before the age of 3 years, may demonstrate a number sense: they may try to count recalling number names from memory, and, if asked what number comes after 4, they may count from 1, e.g. 1, 2, 3, 4, 5, 6, … or inconsistently, e.g., 1, 3, 4, 8, … Also, they may count objects without tagging each time they counted, thereby counting different amounts each time even if in reference to the same object set, and they fail to define the total number, indicating lack of the concept of cardinality. Hence, the main objective of the school curriculum at the age of 3-4 years is to connect the quantitative value of a number and the counting sequence. A first understanding of the relationship between numbers is mastering global quantitative comparisons, such as “more”, “less” and “same as”. Understanding quantity comes next, when they grasp cardinality: i.e., they understand that the last number stated when counting a set of objects represents the number of objects in the set. At the ages of 4-6 years, emphasis is given on aligning and connecting complementary representations of number, such as number names (e.g., one, two, three … ten), corresponding symbols (e.g., 1, 2, 3, …, 10, respectively), and the quantities represented (e.g., one thing, two things, three things, …, ten things, respectively.

The number line extends to 100 between 5 and 7 years. At this phase, representations from different representational spaces become accessible as distinct mental entities that can be aligned but alignment is not fluid yet. For instance, 4-year-olds map both number words and number digits with arrays of up to 6 elements but they do not map number words on digits. At 5 years children map all representations with each other for all sizes. Therefore, children at this age need experiences in using a variety of tools and resources (e.g. number lines, base ten blocks, mathstories) in order to realize that the number system is pattern-based (3+5=8 and 83+5=88). Children need to use manipulatives and pictures to represent mathematical concepts before the introduction of symbols, thereby building the necessary links between the mathematical concepts and the symbolic language which may stand for them. The introduction of symbols as a mean of communication and representation is important for preschool education, because it allows the consolidation of representational thought. A quantity of 7 objects can be represented by 7 objects, by 7 pictures, by 7 images, or by 7 points on a line. Also, they can compute additions and subtractions on numbers smaller than 10, using their fingers as tools for counting or other manipulatives or images. The use of different representations for the same concept facilitates representational awareness and abstraction underlying rule-induction.

At the first primary school grade, emphasis must shift to the exploration of the patterns between 1-digit numbers, tens and decades in order to grasp the rules underlying structure of place value system (Kilpatrick, Swafford & Findell, 2001). For instance, TEN is a very important anchor for children at this age because it helps them to remember the combinations that make 10

(3+7) and also that 2-digit numbers, such as 12, is divided into 10 and 2, and 25 may be decomposed into 2 X 10 and 5 units. Number sense is enriched by the value of place value. A numeral represents the number symbol, the word, the placement on a number line, a quantity and a place value position (e.g., 2 can mean 2, 20, 200 in respect to the place value). Thus, first grade children have to grasp that a general rule connects different aspects of number; also, that knowing this rule allows transforming numbers to each other, according to this rule. Grasping this rule requires that children in Grade 1 and 2 need opportunities to experience counting to 100 and to establish a link between the numbers and their visual representations as numerals. They are expected to use their knowledge about the order of numbers on the number line in order to conduct basic calculations. Ideally, second grade primary school students must be able to calculate the cost of items which may be bought with a given sum of money and can calculate the best estimate of the sum or difference of two two-digit numbers. They also must show understanding of the associative property of addition; the connection between two-step word problems and their corresponding numerical expressions.

Between 7 and 12 years the mental number line extends massively, going up to 1000 at fourth Grade and 10000 at fifth Grade. This expansion is associated with a grasp of the rules underlying the relations between numbers and how they may be combined and transformed into each other. Thus, in Grade 2, children are expected to understand that if 6 x 8 = 48 then consequently 7 x 8 is 48 + 8. In Grade 3, numeric relations and patterns need to be used in order to develop relevant strategies. For instance, they can think of “68-29” by relating it with “69-30” and make the calculation easier. Subtraction must be understood as the inverse of addition and division as the inverse of multiplication. Also, division may be demonstrated as repeated subtraction and multiplication repeated addition.

In the following grades counting may extend to very large numbers. Normally, at this phase, children must be able to extrapolate their early grasp of the relations between numbers to large numbers (e.g., the relationship between 2 and 5 is similar to the relationship between 22 and 25 and between 1232 and 1835). Obviously, early number sense becomes highly refined at the final grades of primary education. However, children must be given practice to develop strategies for mental calculations that would generate exact numerical results and relations beyond the familiar small numbers (Yang, Reys & Reys, 2007). At this phase, the recognition of relations between numbers must extent from integers to rational numbers, complex numbers, etc.

Fluid mathematical thought requires that consolidation of number sense with integers must expand to include rational numbers. Jordan et al. (2016) showed longitudinally that children' ability to accurately place numbers on a number line at Grade 3 was the most important predictor of their understanding of the concept of fraction at Grade 4. Fraction involves an understanding that whole numbers are divided into equal parts according to the numerator and the denominator. Thus, the shift from whole numbers to rational numbers must expand from the activities with discrete quantities to activities with continuous quantities. Building competence to operate on fractions is required for success in more advanced mathematics (Booth & Newton, 2012). Siegler et al. (2013) found that fraction knowledge at 5^{th} grade uniquely predicted algebraic understanding and overall mathematical performance in high school.

Understanding fractions is a demanding process. It is easier for children in early primary school grades (first to third grade) to represent ½ as a part of the area of a surface than to represent ½ of a set of objects (1/2 of a pizza than ½ of a set of apples). Also, it is easier for them to understand that ½ is larger than 1/3 if they translate them into component parts of real objects. However, they face significant difficulty to think of fractions in terms of symbols before the end of primary school (from 4^{th} to 6^{th} grade). The emphasis of teaching from Grade 4 to 6 must focus on understanding fraction equivalence and magnitude including comparing and ordering fractions. These abilities must extend to operations with decimals and applications involving proportional reasoning (Fennell & Karp, 2016), including ratios and proportions. It needs to be stressed that teaching fractions after students consolidate their understanding of whole numbers may cause serious misconceptions about fractions (Namkung, Fuschs & Koziol, 2018; Ni & Zhou, 2005), because rules for operating on fractions differ from rules for operating on whole numbers. Thus, curriculum must enable children at the end of primary school to grasp and differentiate the rules for operating on the two forms of number.

In fact, grasping the relations between operations on whole numbers and operations on fractions requires grasping the algebraic principle of number as a variable that can take any value and that operations on numbers depends on their precise nature (e.g., whole numbers vs fractions). This is attained in adolescence when the mental number line is conceived as a sequence of any numbers to infinity. Thus, in secondary school adolescents can solve problems which require this general conception of number and the assumption of constraints that define the value of number instantiations, given their relations as specified in mathematical propositions. For instance, they can understand that "L + M + N = L + P + N" , assuming that M

= P (Demetriou et al., 1996; Demetriou & Kyriakides, 2006). As a result, number can be explored as such, defined in alternative ways (e.g., natural, real, imaginary number, etc.) which can then be compared for consistency (Dehaene, 2001). By third secondary school grade, adolescents must be able to model complex problem situations using formal mathematical language, specify similarities and differences between different problem situations, and reflect on, evaluate, compute fractions of different denominators and decimals, and communicate their work to others (Shiel, Kavanagh, & Millar, 2014; Shiel, Kelleher, McKeown, & Denner, 2016).

It is interesting to specify the cognitive parameters of the development of mathematical thinking as outlined above. Research shows that attention control and phonological awareness are the main predictors of learning and performance in arithmetic. Specifically, performance on inhibitory control, flexibility in shifting and planning during preschool was found to account for substantial variability in fluency in executing arithmetic addition and subtraction and reading and comprehension. These associations between executive functions and arithmetic persisted even after controlling for individual differences in general cognitive ability and reading achievement (Clark, Pritchard, & Woodward, 2010). Also, the relation between executive functions at 4 and learning mathematics at 6 years was similar to the relation between these functions and reading. Recently, recent research showed that phonological awareness predicted both early reading and early arithmetic; inversely, numerical recognition predicted performance in both domains (Vanbinst, van Bergen, Ghesquière, & De Smedt, 2020). This pattern suggests that the critical factor is representational awareness in general rather than awareness about a specific notational system.

Dyslexia and Dyscalculia: Understanding Developmental Learning Difficulties

A recent meta-analysis of 680 studies, involving 793 independent samples and more than 370,000 participants showed that fluid intelligence (basically reasoning) was moderately related to reading ($r = .38$) and mathematics ($r = .41$) and this relation increased over time; also, learning to read and do arithmetic relies on fluid intelligence early on and, as schooling progresses, learning in these two subjects boosts fluid intelligence (Peng, Wang, Wang, & Lin, 2019). This relation is reflected in the fact that difficulties in each domain do relate with difficulties in the other domain, indicating dependence on common representational-processing mechanisms. For instance, there is recent longitudinal evidence showing that during the transition from preschool to kindergarten, from 4.5 to. 6.5 years, executive functions interact with early numeracy and literacy skills, themselves also interacting with each other (Schmitt, Geldhof, Purpura, Duncan, & McClelland, 2017). There is also evidence that in this age period listening comprehension predicts numeracy skills (Aunio et al., 2019). We show below that deficiencies in general developmental priorities handicap learning in the aspects of each domain concerned.

However, the two domains have very special characteristics as well which are related to special learning difficulties in each, which incapacitate many children. In the domain of language about 20% of children in early primary school face strong difficulty in learning to read and write; a proportion of them, about 5-10% meet the requirements of dyslexia, a serious condition interfering with every aspect of school life in children with normal intelligence. These children face serious problems in phonological processing, which underlies the translation of letters into sounds and their integration into meaningful words (Siegel, 2006).

About the same number of children face similar difficulties in learning in mathematics. Specifically, about 3.5-6.5% of children present developmental dyscalculia (Reigosa-Crespo, Butterworth, Estevez, Rodrıguez, Santos, Torres, Suarez, & Lage, 2012). These children "have a poor intuitive sense of quantity, … poor understanding of more and less, and slow learning of Arabic numerals, number words, and their meanings" (Chu, vanMarle, & Geary, 2013, p.9). For instance, they have difficulty in enumerating small sets of up to 9 elements, compare small magnitudes, such as 5 to 7 and 7 to 5 between, and do simple mental arithmetic by adding or subtracting numbers between 1 and 9. Butterworth (2005) advanced the "defective number module hypothesis". According to this hypothesis, dyscalculia is caused by a deficit in numerocity coding. Numerocity coding is mapping symbols onto representations of quantities. This implies an exact representation of one as a quantity of one, two as a quantity of two, three as a quantity of three, etc. A deficit in coding numerocity as precise magnitudes would render learning to count difficult because counting words would lack the exact corresponding representations to be associated with. In turn, this would hinder the functioning of both the ANS and the learning of rules underlying the relations between quantities. For instance, there is evidence that numbers on the number line overlap in dyscalculic children, causing difficulties in number comparisons (Mussolin, Mejias, & Noël, 2010).

It is important to examine if reading difficulties, including dyslexia, and mathematics difficulties, including dyscalculia, share a common representational deficit or if each is caused by representational difficulties specific to each domain. Empirical evidence supports the specific deficit hypothesis. Specifically, in reading difficulties, the phonological system does not have the resolution required for letter recognition and their composition in words. In arithmetic difficulties, the numerocity coding system is not precise enough to allow building representations for different quantities. Several studies presented evidence in support of this double dissociation hypothesis. Children with dyscalculia have problems to associate Arabic numerals with their representations of magnitudes but no problems in associating letters with phonemes; dyslexics faced problems with letter and digit recognition and naming but no problem with magnitude processing, symbolic or mon-symbolic (Rubinstein & Henik, 2006). Notably, children with both dyslexia and dyscalculia face cumulatively the problems of both groups (Landerl, Fussenegger, Moll, & Willburger, 2009). Also, there is evidence that word inhibition as examined in the Stroop task (name the ink color of color words written in a different ink color) suffers in dyslexics and number inhibition (e.g., recognize which number is bigger, 8 or 5) suffers in dyscalculics. Noticeably, however, graph inhibition, that is the recognition of a geometric shape in the context of more complex geometric figures suffers in both groups, implying a more general attention control problem that is aggravated in domain-specific symbol systems in each group (Wang, Tasi, & Yang, 2012).

It seems that difficulties in learning written language and mathematics, and probably other domains, such as spatial or social relations, lie on a root-like structure where separate roots converge to a common trunk. Specifically, learning in each domain is based on both, the state of the encoding and representation of domain-specific information and general representational and processing functions. All domains need both domain-specific and general representational ability to operate above a certain level for learning to occur smoothly. Severe domain-specific representational difficulties may seriously and persistently interfere with learning in the domain concerned, such as phonological and magnitude representation in the domains of language and arithmetic, even if general representational and processing functions are generally intact. However, smooth and efficient learning in all domains requires both, domain-specific and general representational functions to operate above a certain limit. This is the implication of the findings summarized above that learning in both language and arithmetic depends on both general executive processes, such as attention control processes, and awareness processes, if indexed by domain-specific awareness, such as phonological awareness, which relates to learning to read but also to learning arithmetic.

Conclusions

We summarized a theory cognitive development aiming to unify cognitive developmental, psychometric, and clinical theories of intellectual development and learning. This theory postulates that cognitive development occurs via several cycles of representational expansion and reorganization, which allow increasingly accurate multidimensional representation of the world. Each new form of representation poses a new problem of mental control, such as attention at preschool, inferential in childhood, and logical control in adolescence. In other words, mental control gradually shifts from perceptual and action systems to inferential and logical systems. Mental awareness is an important component of this process, reflecting the cognitive processes coming under control in each cycle.

Control is exercised according to the symbol systems used; symbol systems stand for different levels of mental complexity and express variably the same aspects of reality or different levels of complexity of reality. For instance, oral language, written language, mathematics, etc. Commanding a system requires facility with the symbol systems involved; for instance, acoustic patterns standing for spoken words, visual patterns standing for written words, visual patterns standing for quantities, etc. If grasping the external patterns and representing them is deficient for any reason, their learning and use would be also deficient, as in speech delays at the transition from infancy to early preschool or reading and arithmetic difficulties at the transition from preschool to secondary school.

This state of affairs is indicated by individual differences in mastering major developmental tasks at transitions between developmental cycles. All three major transitions are associated with fast learning of a new symbol system: speech at the transition between episodic and realistic representations, reading and writing at the transition between realistic representations and rule-based thought, and highly specific idiosyncratic symbol systems at the transition between rule-based and principle-based thought. From an evolutionary point of view, the three systems are separated by thousands of years. Human language emerged with homo sapiens, if not earlier, at about 200,000 years BC; reading and writing is about 5,000 years old; domain-specific symbol systems, such as mathematical notation in modern mathematics, is only a few hundred years old. Mastering each system becomes increasingly difficult and learning difficulties increasingly likely; special education is increasingly needed to master each symbol systems. All children speak at about 2-3 years of age without any education, unless there are specific difficulties or delays after the age of 3 years. Most children learn to read and write at 6-7 years, but this is only possible if they are educated to do so. People command the symbolic systems of mathematics or different sciences in their fullness only after very long and demanding university education.

In all three cases, however, difficulties arise when central and domain-specific processes are not well tuned to each other to allow coping with representational load needed to command the new symbol system. Speech delays relative to the typical speech age of 2 years may occur either because are limited in their capacity to handle language complexity during encoding (Panagos et al., 1979) or because of specific phonological encoding problems (Paul & Shriberg, 1982). Reading or arithmetic delays relative to the typical reading age of 6-7 years may occur because of attention control and representational awareness deficits or because of specific letter or quantity encoding difficulties. Difficulties in learning high level mathematics and sciences concepts in secondary school and the university may relate to delays in commanding specific symbol systems that stand for principle-level concepts and using them for operating on these concepts (Susac, Bubic, Vrbanc, Planinic, 2014).

Remedial programs must include both, training in the symbol systems affected and in how the general processes may be used on them. Children must be trained to command the phonology of their language in speech delays or the visual-phonological structure of a written system. This must also involve the inferential mechanisms needed to integrate sound patterns and production or written symbols and sentence production into semantically meaningful structures. At a higher level, learning must enhance awareness of representational units and their inferential connections. Training metaphorical, analogical, and deductive reasoning so that thinkers can access and systematically operate on the processes involved may be a means to this end. This may occur for the processes mastered in each cycle. For instance, in episodic though, training how to focus on, notice, handle, and represent similarities across objects and categories may be given priority. In representational thought instruction would have to focus on managing perceptual systems and action in reference to varying goals; explicit use of different means to symbolize concepts and actions by means, such as language, photographs, drawings, and noting their advantages and disadvantages would be important for mastering mental representation. In sake of advancing rule-based thought, instruction should focus on underlying relations connecting representations. In principle-based thought it should focus on general principles specifying truth and validity and the value of using special symbol systems that may shield thinking from context or content limitations and constraints. These changes are associated with education in problem solving and critical thinking. Below we outline the implications of this model for both.

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