Intuitive proportion judgment in number-line estimation:

Converging evidence from multiple tasks

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Abstract

How children's understanding of numerical magnitudes changes over the course of development remains a key question in the study of numerical cognition. In an ongoing debate about the source of developmental change, some argue that children maintain and access different mental representations of number, with evidence coming largely from common number-line estimation tasks. In contrast, others argue that a theoretical framework based on psychophysical models of proportion estimation accounts for typical performance on these tasks. The present study explores children's (n=71) and adults' (n=27) performance on two number-line tasks, both the "number to position" or NP task and the inverse "position to number" or PN task. Estimates on both tasks are consistent with the predictions of the proportion estimation account and do not support for the hypothesis that a fundamental shift in mental representations underlies developmental change in numerical estimation and, in turn, mathematical ability. Converging evidence across tasks also calls into question the utility of bounded number-line tasks as an evaluation of mental representations of number.

Keywords: numerical cognition; estimation; proportion judgment; number-line

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Children's numerical thinking and reasoning change considerably over the course of development. Recent work in cognitive development has mapped out one aspect of these changes in detail: a clear and consistent developmental sequence that appears in children's numerical performance across multiple age groups, tasks, and timescales (e.g. Siegler, Thompson, & Opfer, 2009). Roughly speaking, this work has shown that the relation between children's numerical estimates and the to-be-estimated numbers is well described by a logarithmic function for relatively young children, but better described by a linear function for older children (see Siegler, Thompson, & Opfer, 2009 and Slusser, Santiago, & Barth, 2013 for reviews).

This developmental sequence has been identified largely through data from estimation tasks, such as number-line tasks, that involve translating numerical magnitudes into spatial positions (or vice versa). In a typical number-line estimation task, participants mark the placement of a given value, such as 43, relative to two marked endpoints, such as 0 and 100, on an otherwise unmarked number line. In addition to revealing a clear developmental sequence in estimation patterns, such tasks may reflect children's familiarity with certain number ranges (Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008) and their ability to reason about single- versus double-digit numbers (Moeller, Pixner, Kaufman, & Nuerk, 2009). Performance patterns may also be predictive of children's understanding of basic arithmetic operations (Booth & Siegler, 2008).

Number-line estimation has also been thought to reveal how numbers are mentally represented and how these representations change over development (e.g. Siegler & Opfer, 2003; Siegler, Thompson, & Opfer, 2009). In fact, the change in children's estimation patterns has commonly been taken to indicate a shift from the use of logarithmic to linear mental representations of number. According to this theoretical framework, children can access multiple types of coexisting mental number representations. A child may produce more logarithmic estimates for a less familiar numerical range and more linear estimates for a more familiar range (Siegler & Opfer, 2003), a finding that has been interpreted to mean that children draw upon a linear representatios of number when dealing with more familiar numerical ranges, and upon a logarithmic representation for less familiar ranges. By this account, children come to rely more consistently on a linear representation of number with time and experience, which then supports more accurate estimation (e.g. Siegler & Opfer, 2003; Siegler, Thompson, & Opfer, 2009).

This view of the development of numerical representation has been influential across disciplines, with demonstrated links to formal education and mathematics learning. For example, children characterized as linear estimators do better on standardized math tests and other measures of mathematical ability (Ashcraft & Moore, 2012; Booth & Siegler, 2006) and children with mathematical learning disability (MLD) produce linear estimates later than comparison groups (Geary, Hoard, Nugent, & Byrd-Craven, 2008) – behaviors that have been attributed to differences in children's numerical magnitude representations. These findings have also led researchers to develop effective interventions to improve formal math performance (e.g. Siegler & Ramani, 2008). This work is therefore relevant both to theories of mathematical cognition and development and to education research and practice.

Converging evidence from multiple research groups, however, has recently shown that a different theoretical explanation offers a better explanation of performance on cognitive tasks, including number-line estimation tasks, that are commonly used to assess learning and development in numerical thinking (Barth & Paladino, 2011; Barth, Slusser, Kanjlia, Garcia, Taggart, & Chase, 2016; Cohen & Blanc-Goldhammer, 2011; Cohen & Sarnecka, 2014; Rouder & Geary, 2014; Slusser et al., 2013; Sullivan, Juhasz, Slattery, & Barth, 2011; see also Chesney & Matthews, 2013; Hurst, Monahan, Heller, & Cordes, 2014). These and other findings have fostered an ongoing debate, calling into question the hypothesis that a shift from logarithmic to linear mental representations of number underlies developmental change in numerical estimation and, in turn, observed improvements in formal math (see Barth, Slusser, Cohen, & Paladino, 2011; Opfer, Siegler, & Young, 2011).

The alternative theory, based on a psychophysical model of proportion estimation (Hollands & Dyre, 2000; Spence, 1990), posits that the proportional structure of the typical number-line task must be taken into account when attempting to understand task performance. Developed for tasks involving judgments of perceptual magnitude, this model of proportion estimation is similarly applicable in this context given that typical 0 to *N* number-line estimation tasks ask participants to 1) retrieve the mental magnitudes represented by the given numeral and the marked upper endpoint, 2) arrive at an estimate of the proportion of the two magnitudes [1], and then 3) produce a corresponding spatial proportion by marking the number line in the appropriate position. Thus, the task requires the estimation of a smaller magnitude (the value presented) *relative to* a larger one (the value given at the upper endpoint of the line), eliciting an estimate of a numerical proportion rather than an isolated numerical magnitude. Recent research has shown that the proportion estimation model is an excellent predictor of typical numerical estimation patterns and even explains systematic patterns of bias in estimation data that cannot be accounted for by the log-to-linear shift hypothesis (see Slusser et al., 2013 for a detailed discussion).

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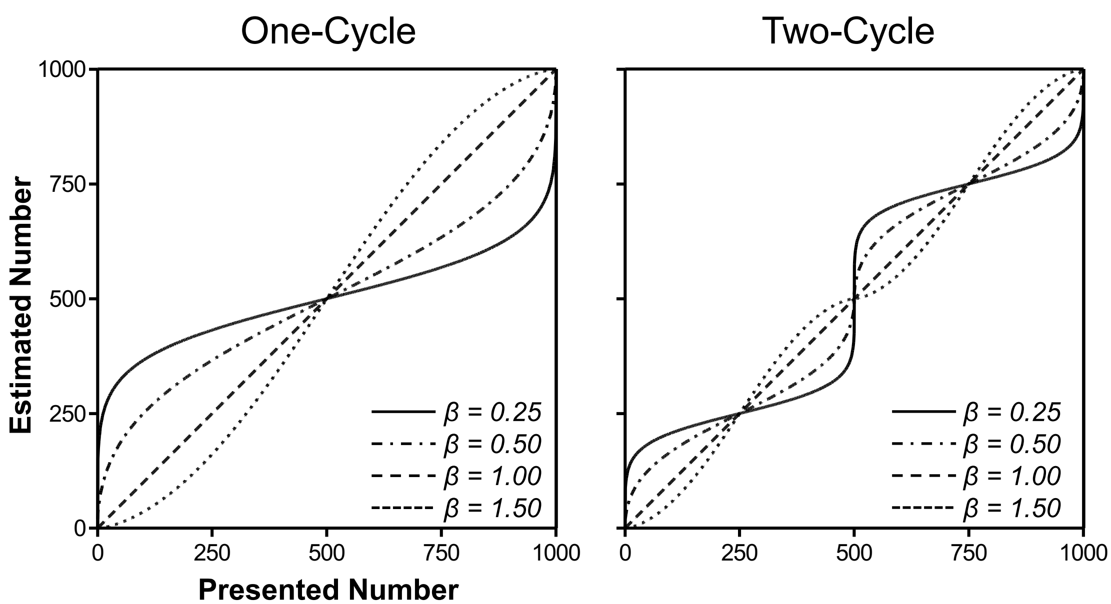
[1] The model is consistent with the use of more than one cognitive process to arrive at an estimate; it does not provide evidence for the use of any one specific process (Spence 1990; see also Cohen & Sarnecka, 2014).

Proportion estimation models are derived from Stevens' Law, which describes the relationship between the estimated or perceived magnitude of a stimulus and its actual magnitude as a power function $y = \alpha x^{\beta}$. The exponent β theoretically quantifies the bias associated with estimating a particular type of stimulus magnitude, such as brightness, area, or length (see Teghtsoonian, 2012, for a nuanced discussion), and α is a fixed scaling parameter. Spence (1990) showed that estimates of proportions (estimation of a part relative to a whole, or of a part relative to another part) should take the form of S-shaped or inverse S-shaped curves, depending on the particular value of β in question. When this model is applied to a typical 0-1000 number line task, estimates are predicted by $y = x^{\beta} / (x^{\beta} + (1000 - x)^{\beta})$ (Figure 1, left panel).

Figure 1. Predictions of the standard proportion estimation model on a 0-1000 number line task.



This model was later generalized to account for cases in which the observer makes use of additional reference points (for example, estimating the proportion of a cylinder that is partially filled with liquid by judging the liquid's level relative to a halfway point, rather than relative to the entire height of the cylinder; Hollands & Dyre, 2000). This work showed that the use of such reference points 1) produces a pattern of estimates with multiple S-shaped or inverse S-shaped curves or "cycles" (hence the term "cyclical power model" of proportion estimation; see Figure 1, right panel), and 2) increases overall accuracy in estimation even when values of β remain constant (Figure 1, compare right panel to left). Cyclical patterns of estimation bias consistent with this model are observed in children's number-line estimates (Barth & Paladino, 2011; Barth et al., 2016; Rouder & Geary, 2014; Slusser et al., 2013), as they are in adults' estimates of proportions using various continua (Cohen & Blanc-Goldhammer, 2011; Hollands & Dyre, 2000).

Quantitative models of proportion estimation can account for many of the behavioral phenomena that have been interpreted as evidence for a representational shift in children's numerical thinking. For example, a cross-sectional study by Slusser et al. (2013) found that models associated with the proportion judgment framework accounted for developmental differences in estimation performance in 5- through 10-year-olds, and that they outperformed models associated with the log-to-linear shift view at both the group and individual levels (see also Barth & Paladino, 2011). In contrast to a theory of representational change, the proportion judgment framework (Slusser et al., 2013) delineates two main sources of change that account for the variability observed across development. First, values of the β parameter typically increase with age, with values near one corresponding to highly accurate estimation patterns (see Figure 1). β values far less than one correspond to highly biased estimation patterns (which arise when, for example, children tend to overestimate smaller numbers and underestimate larger numbers on a typical number-line estimation task). Conversely, β values greater than one (which have been observed in older children or adults) correspond to estimates that follow an inverse pattern (see Figure 1), with underestimation of smaller numbers and overestimation of larger numbers on a typical number-line task (Cohen & Blanc-Goldhammer, 2011; Slusser et al., 2013). A second source of variability stems from changes in children's use of reference points. Specifically, older children are more likely to use an inferred midpoint in addition to the two endpoints as reference points, resulting in performance patterns resembling the two-cycle model (Figure 1, right panel). Younger children, on the other hand, are less likely to use an inferred midpoint and tend to rely only on the two endpoints, resulting in estimates resembling a one-cycle model (Figure 1, left panel). Some of the youngest children produce estimates consistent with an "unbounded" model (i.e., the standard power model) as they seem not to use an upper reference point at all [2].

Recent longitudinal work examining number-line estimation and its variability in a large sample of children has provided further support for the idea that proportion judgment underlies children's performance (Rouder & Geary, 2014). The proportion estimation framework also explains performance differences across more- vs. less-familiar numerical ranges (Slusser et al., 2013), the effects of feedback on children's number-line estimates (Barth et al., 2016; cf. Opfer & Siegler, 2007), and adults' numerical estimation performance (Cohen & Blanc-Goldhammer, 2011; Sullivan et al., 2011). Furthermore, performance on (non-numerical) spatial position reproduction tasks also shows reliable patterns of bias consistent with the predictions of these proportional models (Barth, Lesser, Taggart, & Slusser, 2015).

One question that remains, however, is whether this theoretical framework extends to estimation patterns observed on the inverse version of the typical number-line estimation task. Most of the hundreds of published number-line estimation studies make use of a "number to position" or NP task, in which participants are given a target number and must place a

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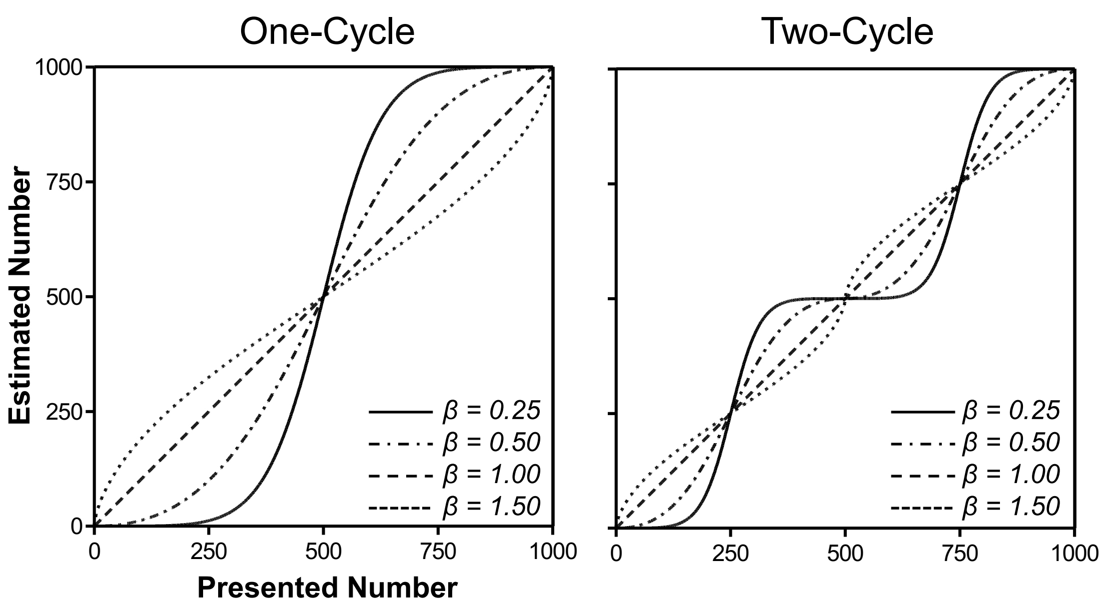
[2] This performance pattern could result from a number of different procedural strategies (Slusser et al., 2013) and may arise when children approach the number-line as an essentially "unbounded" estimation task (Cohen & Sarnecka, 2014).

corresponding mark on a number line. Very few studies have used a "position to number" or PN task, in which participants are given a number line with a marked position and must produce a corresponding number (cf. Ashcraft & Moore, 2012; Iuculano & Butterworth, 2011; Siegler & Opfer, 2003). With their initial introduction of the representational shift account, Siegler and Opfer (2003) noted that participants' performance on the PN task is related to their performance on the NP task: when the given values (numbers corresponding to the marked positions) are plotted on the x-axis and the estimated values (participants' estimates of the numbers corresponding to the marked positions) are plotted on the y-axis, roughly inverse patterns of performance result. For example, the estimates of kindergartners on a 0-100 line can be fit by a logarithmic model for the NP task (with overestimation of smaller values and underestimation of larger ones), and by an exponential model for the PN task (with underestimation of smaller values and overestimation of larger ones). As with the NP task, older children (such as second graders presented with a 0-100 number-line) tend to produce PN estimates that are fit better by a linear model (Siegler & Opfer, 2003).

Contrary to prior claims in the numerical development literature, however, the representational shift account is not unique in its predictions of related performance patterns on NP tasks and their inverse, PN tasks. Earlier work on perceptual proportion estimation has shown that tasks requiring translation between different types of proportions also yield patterns of performance related to their inverse tasks. For example, when adults were shown two spheres and asked to provide a percentage to describe the volume of one sphere relative to the total volume of both, they overestimated smaller proportions and underestimated larger ones, resulting in estimation patterns well described by proportion estimation models. When performing an inverse task – adjusting one sphere so that its volume, relative to the total volume of both, would correspond to a given percentage – participants underestimated smaller proportions and overestimated larger ones, with adjustments well described by inverse versions of these models (Hollands, Tanaka, and Dyre, 2002). Thus, the proportion estimation framework is at least qualitatively consistent with previous PN data: it predicts that over- and underestimation patterns seen previously for the standard NP number-line task (e.g. Barth & Paladino, 2011; Slusser et al., 2013) will generally be inverted for the PN task [3].

Figure 2. Predictions of the inverse proportion estimation model on a 0-1000 number line task.



Quantitatively, performance on the PN task should be predicted by inverse versions of the proportion estimation model described above. Thus, children who produce estimates relative to both endpoints (resulting in a standard one-cycle pattern on the typical NP task, Figure 1 left panel) should produce roughly the inverse pattern (Figure 2, left panel) on the PN task ($y = x^{(1/\beta)}/$

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[3] One previous study has applied proportion models to PN data (Ashcraft & Moore, 2012). Though results were arguably consistent with the proportion models' predictions, they were interpreted as supporting the logarithmic-to-linear shift hypothesis due, in part, to the use of inappropriate model comparison methods. We return to this in the Discussion.

$(x^{(1/\beta)} + (1000 - x)^{(1/\beta)})$. Children who produce estimates that resemble the standard two-cycle pattern in the typical NP task (Figure 1, right panel) should produce a roughly inverse pattern on the PN task (Figure 2, right panel). And some young children with insufficient numerical knowledge may not make effective use of the upper endpoint at all when forming estimates, thus producing a standard unbounded pattern on the typical NP task akin to a simple power function; these children should produce roughly the inverse pattern on the PN task ($y = \alpha x^{(1/\beta)}$). Simply put, estimation patterns on the inverse PN task are essentially reflecting the standard number-line (NP) predictions across the $y=x$ line, such that where there is overestimation for the NP task, there should generally be underestimation for the PN task, or vice versa (Figure 2). However, because reference point choice is strategic, different tasks may lead to different uses of reference points (Hollands & Dyre, 2000; Hollands et al., 2002).

Here we ask whether the specific predictions made by the proportion estimation framework can quantitatively account for children's and adults' performance on both the PN and NP tasks. To get a sense of how performance improves over development, this study applies psychophysical models of proportion estimation to the NP and PN placements of children and adults. Younger children (6 -7 years old) were presented with a 0-100 number line, older children (8-10 years old) were presented with a 0-1000 number line, and adults were presented with a 0-100000 number line. To evaluate the reliability of the proportion estimation framework, participants' performance on the PN task were evaluated in relation to their performance on the NP task. We hypothesized that performance patterns would be consistent with the predictions of the proportion estimation framework: that the PN and NP tasks would produce roughly inverse patterns of performance, the inverse and standard versions of the proportional models would provide a strong description of the data, and variability across development would be well accounted for by the two main sources of change identified by the framework. Findings will have implications for the way that we assess and interpret children's performance on these tasks and may be influential in the development of new methods to support and facilitate children's numerical and mathematical learning.

Methods

Participants

Children. Seventy-nine 6- to 10-year-old children participated in the study. Seven children failed to complete both tasks and were excluded from the analyses presented here. One additional child was excluded because of developmental delays (parental report). This left a total of 71 children (31 female and 40 male, mean age 8;6). Most children were recruited through a database of families residing in the central CT area. No questions were asked about socio-economic status, race, or ethnicity, but children were presumably representative of the community from which they were drawn. In this community, 84% of adults have a high-school diploma and 30% have a bachelor's degree. Most residents identify as white (80%), black (12%) or Asian (3%) (U.S. Census Bureau, 2000).

Adults. Twenty-seven adults also completed the study (14 females, 11 males, 2 no report; mean age 21 years). All adult participants were college students recruited through the Wesleyan University introductory psychology subject pool in exchange for course credit. No questions were asked about socio-economic status, race, or ethnicity.

Stimuli

Participants were presented with a series of pages (approximately 28 x 11 cm), each with a 23 cm line printed in the center of the page. The left end of the line was marked with 0 and the right end of the line was marked with 100, 1000, or 100000 (depending on the age group, see below). Each participant completed the position to number (PN) and the number to position (NP) tasks. For the PN task, the target position on each line was indicated by a 1.3 cm vertical hash mark. For the NP task, the target number was printed 2.5 cm above the center of the number line (for similar stimuli and design see Booth & Siegler, 2006; Opfer & Siegler, 2007; Siegler & Opfer, 2003; Slusser, Santiago, & Barth, 2013; Thompson & Opfer, 2010).

Design

Participants always completed the PN task first. The number range varied according to the participants' age. Six- and seven-year-olds were tested with number lines bounded by 0 and 100; eight-, nine-, and ten-year-olds were tested with 0-1000 number lines; and adults were tested with 0-100000 number lines [4]. The presented positions for the PN task and the presented numbers for the NP task were sampled roughly evenly across the given number range. No numbers were repeated. The order of the trials was randomized for each participant.

Procedure

Most participants were tested in a quiet laboratory room. Some children were tested at local venues such as a nearby children's museum. Following the procedures reported in similar studies (Booth & Siegler, 2006; Siegler & Opfer, 2003), participants were first presented with a blank number line, marked only with the endpoint values (e.g., 0 and 100 for a 6- or 7-year-old). The experimenter then explained "This is a number line. It has a 0 at this end, and 100 at this end. All of the other numbers in between go along this line." Children were then asked to mark where they thought 50 would go on the line. After responding, they were shown an identical number

[4] Six-year-olds also completed this task with a 0-20 number range; seven-year-olds completed a 0-1000 task; eight-, nine-, and ten-year-olds completed a 0-100000 task; and adults completed a 0-1000 task (see Slusser, Santiago, & Barth, 2013 for reports of these data). The smaller number range was always completed first. When switching to the larger number range, the experimenter said, for example, "Now we're going to play the game with different numbers. 0 still goes at this end, but now 100 is at the other end."

line with 50 marked in the middle. The experimenter asked if they knew why 50 went there and explained, "Because if 50 is half of 100, it goes right in the middle between 0 and 100." [5]

After this practice trial, the experimenter introduced the PN task by saying, "Now, I'm going to show you a number line with a mark on it, and you tell me what number you think goes there." For each subsequent test trial, the experimenter repeated the prompt, "What number goes here?" if needed. Immediately following the PN task the experimenter presented another practice trial and then introduced the NP task by saying, "Now I am going to ask you where a number goes on the line, and you are going to make a mark where you think it belongs." For each subsequent test trial, the experimenter repeated the prompt, "Where does (e.g.) twenty-nine go?" if needed. The numbers for each NP trial were printed on the page (see above) and said aloud by the experimenter.

Results

Analyses

Data from participants who marked over 90% of their responses within a single region comprising 10% of the number line (n=2) or produced responses that were uncorrelated with the presented numbers on either task (n=1) were excluded from the following analyses. This resulted in 96 participants: 15 6-year-olds (mean age 6;6), 13 7-year-olds (mean age 7;5), 13 8-year-olds (mean age 8;4), 14 9-year-olds (mean age 9;7), 14 10-year-olds (mean age 10;7), and 26 adults (mean age 20;11).

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[5] Providing the location of the midpoint value (as was done in the present study and in Booth & Siegler. 2006) can influence, and possibly improve, participants' estimates on subsequent test trials (Zax, Slusser, & Barth, 2017; Opfer, Thompson, & Kim, 2016). The proportion estimation framework predicts that this information may make children more likely to produce performance patterns resembling two-cycle, rather than one-cycle, models (e.g. Barth et al., 2016) while the representational shift hypothesis predicts that children may be more likely to produce linear, rather than logarithmically, spaced estimates (Opfer & Siegler, 2007). While the analysis below does not explicitly address these hypotheses, interested readers are encouraged to review the papers and articles cited above for a more detailed examination of the role of corrective feedback and explicit provision of reference points in children's number line estimates.

Patterns of estimation bias were evaluated by fitting the models of interest to each participant's individual estimates as well as to group median estimates. Estimates on the PN task were fit with the inverse unbounded model as well as the inverse one- and two-cycle versions of the proportional power model (Hollands et al., 2002). Estimates on the NP tasks were fit with the standard unbounded model as well as the standard one- and two-cycle versions of the proportional power model (see Slusser et al., 2013). We also evaluated exponential (for the PN task), logarithmic (for the NP task), and linear fits, though these were not the primary focus of the study. Note that none of these models provided the best explanation of group median data (see Appendices A and B), or for the majority of individual data. Formal model comparisons determined which model best predicted participants' performance patterns. Comparisons were made using Akaike's Information Criterion corrected for small sample sizes (AICc; see Burnham & Anderson, 2002; Burnham, Anderson, & Huyvaert, 2011). In addition to AICc comparisons, group median data were also evaluated using the Leave-One-Out Cross Validation technique (LOOCV; see Brown, 2000; see also Opfer et al., 2011 and Barth et al., 2011 for a recent discussion of LOOCV as applied to number-line data). Because AICc (and LOOCV) calculations take into account components of model complexity that are otherwise unaccounted for when calculating R^2 values, the findings reported here are based on AICc scores (and, for group data, LOOCV indices). Nevertheless, we report R^2 values below because these values, which fall within a standard range, are more easily interpreted (e.g., an R^2 of zero indicates that none of the variance is explained by the model, while an R^2 of one indicates that the model explains 100% of the variance). For a table of AICc and LOOCV values corresponding to group median estimates, see Appendices A and B.

As an atheoretical measure of general accuracy, percent absolute error (PAE) was calculated for each estimate by dividing the absolute difference between the child's estimate and the number corresponding to the presented position by the numerical range, then multiplying the quotient by 100 to express a percentage (see also Booth & Siegler, 2006, 2008; Slusser et al., 2013). The average PAE across participants is reported for each task in the figures below and generally aligns with previous reports (i.e., accuracy increases with age and experience and is relatively consistent across tasks, Siegler & Opfer, 2003).

Six- and 7-Year-Olds (0-100 Number Range)

Position-to-Number task.

Group analyses. On this task, median estimates for each age group were best characterized by the inverse one-cycle model (R^2=.968 for 6-year-olds and R^2=.991 for 7-year-olds) (Figure 3). The value of the β parameter for the 6-year-olds' median estimates was lower (β=.634) than that of the 7-year-olds' (β=.862), indicating more bias in the younger children's estimates.

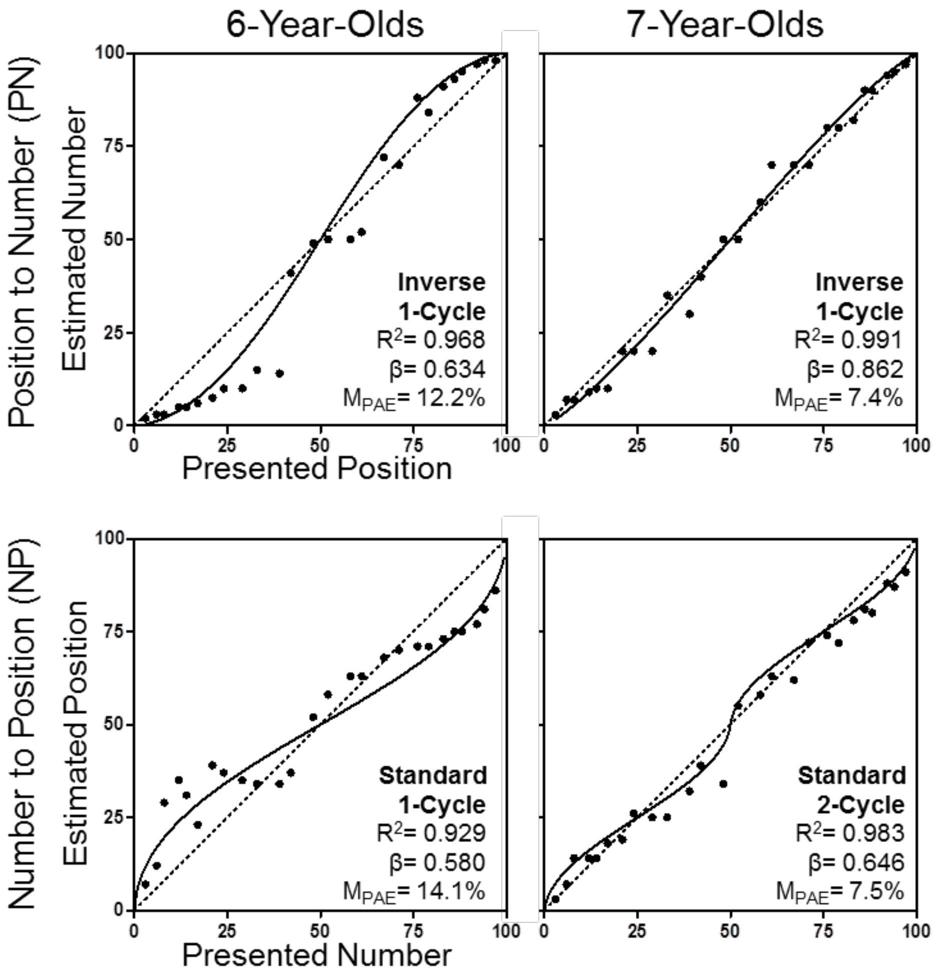
Individual analyses. Consistent with the analysis of group medians, most individual children produced estimates best characterized by the inverse one-cycle model (Table 1). Only a few children produced estimates best characterized by an inverse unbounded model or an inverse two-cycle model. These findings indicate that the majority of children formed their estimates relative to both labeled endpoints, but not an inferred midpoint.

Number-to-Position task.

Group analyses. Six-year-olds' performance on the NP task was best characterized by the standard one-cycle version of the proportional power model (R^2=.929) while 7-year-olds' performance was best characterized by the standard two-cycle version (R^2=.983) (Figure 3). This suggests that 6-year-olds used both endpoints, while 7-year-olds additionally used an inferred midpoint on this task. Furthermore, the value of the β parameter corresponding to the 6-year-olds' estimates was slightly lower (β=.580) than that of the 7-year-olds (β=.646), showing again that 6-year-olds' estimates were more biased than those of 7-year-olds.

Individual analyses. Despite the rather clear patterns generated by the median estimates of both age groups, estimation patterns of individual 6- and 7-year-olds varied (Table 1). For example, 6-year-olds' group median data were best explained by a standard one-cycle model, but the preferred models for individuals were equally distributed across all three standard models (i.e., unbounded, one-cycle, and two-cycle models). Similarly, 7-year-olds' median data were best explained by a standard two-cycle model, but roughly half of the individual 7-year-olds produced a standard one-cycle pattern and half produced a standard two-cycle pattern. In general, however, individual analyses suggest that most 6-year-olds and all 7-year-olds were able to use at least two reference points (the given endpoints) effectively on the NP task, and some in each age group were also able to use an inferred middle reference point as well.

Figure 3. Median estimates of 6- and 7-year-olds on each task. Estimated number corresponds to the marked position on the number line. The solid line represents the preferred model. The dashed line shows *y=x*.



Comparison across tasks.

On both the group and individual levels, NP estimates for 6- and 7-year-olds were best characterized by the standard versions of the proportional power models while PN estimates followed the inverse pattern, as predicted. On the NP task, estimates of the younger children (6-year-olds) followed a standard one-cycle pattern while older children (7-year-olds) seemed to produce a standard two-cycle pattern. On the PN task, however, both the 6- and 7-year-old groups consistently followed a one-cycle pattern. This suggests that children were not necessarily implementing the same strategic choice of reference points across tasks (we return to this in the Discussion section).

Nevertheless, children's β-parameter values were fairly consistent across the PN and NP tasks (Spearman rank correlation, r_s=.559, p=.002), meaning they showed a similar degree of bias on each task. For most children, these β-values were less than one on both tasks. Interestingly, however, the estimates of a few individual children resulted in β-values over one on both tasks (Table 1), in effect reversing the commonly observed trend of overestimation of smaller values on the NP task and underestimation of larger values (a trend that will be explored in some detail in the Discussion section).

Table 1. Number of participants producing estimates predicted by each model. Numbers in parentheses indicate the number of participants producing estimates yielding β-values greater than one.

		6 yo	7 yo	8 yo	9 yo	10 yo	Adults	Total
PN Task	Inverse Unbounded	3 (0)	1 (0)	1 (0)	0 (0)	1 (0)	0 (0)	**6 (0)**
	Inverse One-Cycle	11 (3)	10 (4)	9 (3)	9 (7)	8 (8)	18 (15)	**65 (40)**
	Inverse Two-Cycle	1 (1)	2 (1)	3 (2)	5 (3)	5 (2)	8 (8)	**24 (17)**
	Total	**15 (4)**	**13 (5)**	**13 (5)**	**14 (10)**	**14 (10)**	**26 (23)**	**95 (57)**
NP Task	Standard Unbounded	6 (0)	0 (0)	2 (0)	0 (0)	0 (0)	0 (0)	**6 (0)**
	Standard One-Cycle	5 (1)	7 (2)	4 (0)	7 (5)	7 (6)	9 (7)	**65 (21)**
	Standard Two-Cycle	4 (0)	6 (0)	7 (0)	7 (0)	7 (0)	17 (0)	**24 (0)**
	Total	**15 (1)**	**13 (2)**	**13 (0)**	**14 (5)**	**14 (6)**	**26 (7)**	**95 (21)**

Eight-, 9-, and 10-Year-Olds (0-1000 Number Range)

Position-to-Number task.

Group analyses. Median estimates from each age group were best characterized by the inverse one-cycle model (R^2=.980 for 8-year-olds, R^2=.995 for 9-year-olds, and R^2=.996 for 10-year-olds) (Figure 4). However, the β-value corresponding to the 8-year-olds' estimates was less than one (β=.663), while the β-values corresponding to the 9- and 10-year-olds' estimates are slightly greater than one (β=1.131 for 9-year-olds and β=1.254 for 10-year-olds).This rather unexpected result (also noted earlier in several individuals from the 6- and 7-year-old group) means that 9- and 10-year-olds, on average, are producing estimates that follow an over-then-under pattern, rather than the inverse under-then-over pattern associated with PN performance in the younger children (also see Siegler & Opfer, 2003; but see Ashcraft & Moore, 2012).

Individual analyses. The inverse one-cycle pattern best characterized individual children's estimates as well, with a majority of children in each age group producing a one-cycle pattern (Table 1). Many of these children yielded β-values greater than one; meaning their estimates, like the group-median estimates for 9- and 10-year-olds, followed an over-then-under pattern. Most of the remaining children showed two-cycle patterns, with roughly half producing estimates with corresponding β-values greater than one. Few children produced estimates best modeled by an inverse unbounded model.

Number-to-Position task.

Group analyses. On this task, group median estimates for the 8- and 9-year-old groups followed a standard two-cycle pattern (R^2=.976; β=.470 and R^2=.991; β=.722, respectively). Ten-year-olds, on the other hand, showed a standard one-cycle pattern (R^2=.983) with a β-value greater than one (β=1.274). See Figure 4.

Individual analyses. While the standard two-cycle model best characterized 8- and 9-year-olds' median estimates, on the individual level just over half of these children produced estimates best characterized by a standard two-cycle model; most others produced estimates best characterized by a standard one-cycle model (Table 1). The median estimates of the 10-year-old group were best characterized by a standard one-cycle model; individual 10-year-olds' estimates followed either the standard two-cycle or one-cycle pattern.

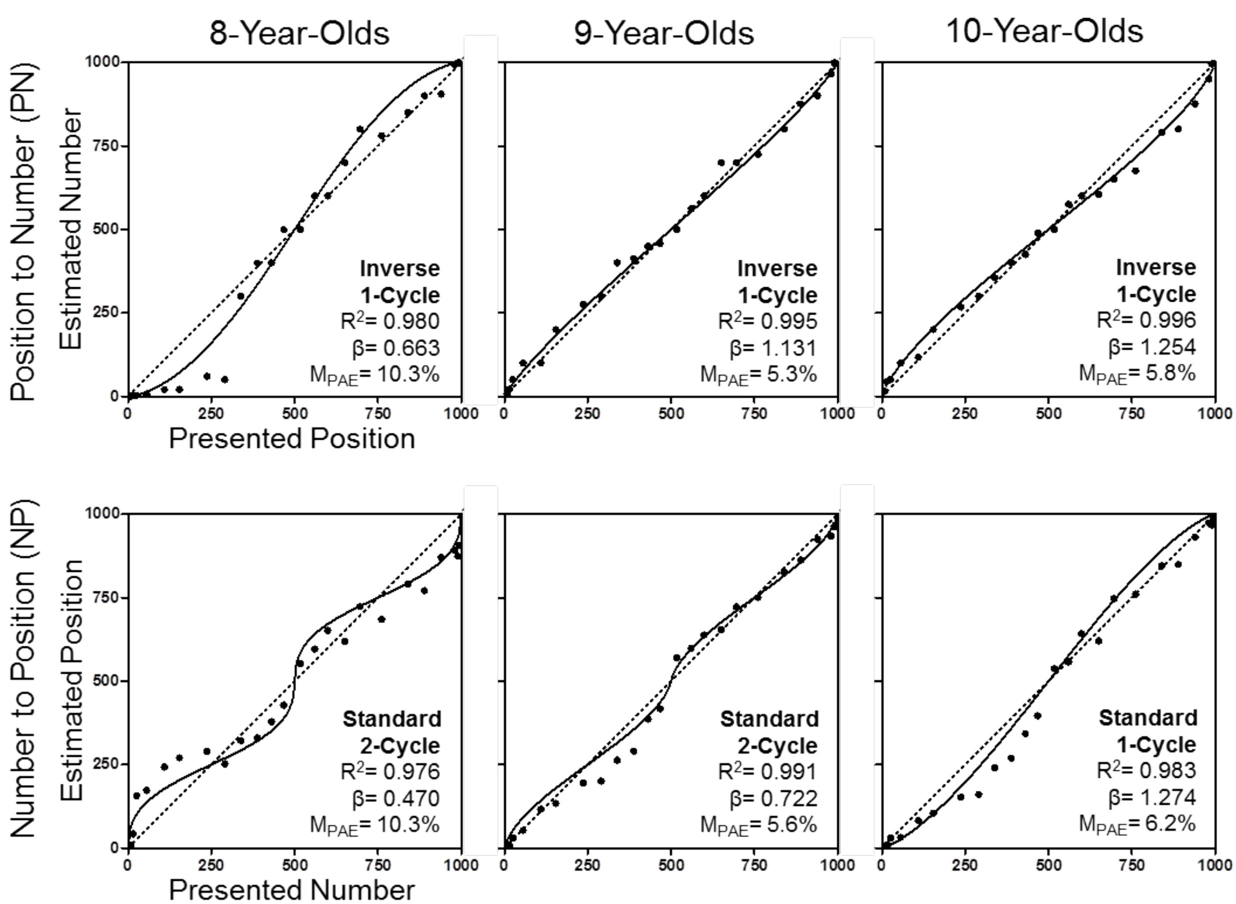
Interestingly, of the children whose estimates were best described by the one-cycle model, over half yielded β-values greater than one, meaning their estimates followed an under-then-over pattern on the NP task (and an over-then-under pattern on the PN task). In contrast, β-values corresponding to estimates best described by a two-cycle model were always less than one.

Comparison across tasks.

On both tasks, there is a distinction between the younger and older age groups: Younger (8-year-old) children tend to produce β-values less than one on both the NP and PN tasks (over-then-under on the NP task, and under-then-over on the PN task) while older (10-year-old) children tend to produce β-values greater than one on both tasks (under-then-over on the NP task, and over-then-under on the PN task). Thus, we do observe the predicted pattern of reversal from the NP to PN task, but in different directions depending on age.

Interestingly, however, within this 8-10 age group the younger children's estimates on the NP task followed a two-cycle pattern while the older children's estimates followed a one-cycle pattern. Children's estimates on the PN task, on the other hand, tended to follow a one-cycle pattern regardless of age. At the individual level, children seemed as likely to produce a one-cycle pattern as they were to produce a two-cycle pattern. Nevertheless, these 8- to 10-year-old children (like the 6- and 7-year olds) showed relatively consistent β-values across tasks, r_s=.515, p=.001. Furthermore, most produced estimates generating the same number of cycles across tasks (following either a one- or two-cycle model for both tasks) with β-values that were either consistently greater than or less than one (that is, PN patterns of performance tended to be the inverse of NP patterns, as predicted).

Figure 4. Median estimates of 8-, 9-, and 10-year-olds on each task. Estimated number corresponds to the marked position on the number line. The solid line represents the preferred model. The dashed line shows *y=x*.



Adults (0-100000 Number Range)

Position-to-Number task.

Group analyses. Group median estimates were best characterized by an inverse one-cycle model (R^2=.998) with a β-value greater than one (β=1.120) (Figure 5). Thus, like the 9- and 10-year-olds', adults' PN estimates conformed to an over-then-under pattern.

Individual analyses. The inverse one-cycle pattern best characterized a majority of the individual estimates as well (Table 1). Interestingly, a large majority of these adults produced estimates with a corresponding β-value greater than one; similar to results reported for 9- and 10-year-olds above. A few adults produced estimates that followed an inverse two-cycle pattern, all of which yielded a β-value greater than one.

Number-to-Position task.

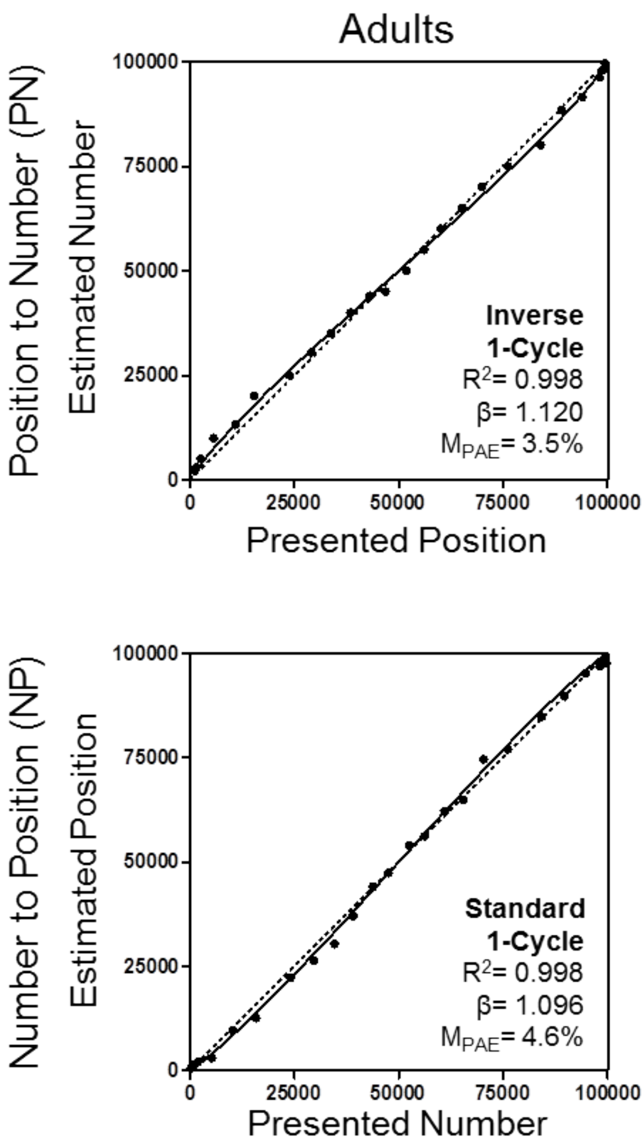
Group analyses. Group median estimates on the NP task followed a standard one-cycle pattern (R^2=.998) with a β-value greater than one (β=1.096) (Figure 5).

Individual analyses. When considered individually, adults' estimates followed either a standard one- or two-cycle model. Most of the one-cycle adults produced estimates with corresponding β-values greater than one. In contrast, none of the two-cycle adults produced estimates with corresponding β-values greater than one.

Comparison across tasks.

As with the 9- and 10-year-olds above, we see the predicted pattern of reversal from the NP to PN tasks in adults, but with β-values greater than one. While most adults produced estimates conforming to a one-cycle pattern on the PN task, individual estimates tended to follow a two-cycle pattern on the NP task. Nevertheless, many adults produced estimates that resulted in the same number of cycles across tasks. On both tasks, β-values corresponding to estimates that followed a one-cycle model were generally greater than one.

Figure 5. Median estimates of adults on each task. The solid line represents the preferred model. The dashed line shows *y*=*x*.



Discussion

This study builds on previous work showing that the psychophysical models of the proportion estimation framework provide a clear and converging explanation of numerical estimation performance on two number-line estimation tasks. We tested children and adults on a typical number-line estimation task (a number to position or NP task) and its inverse (a position to number or PN task): 6- and 7-year-olds completed a 0-100 version, 8- to 10-year-olds completed a 0-1000 version, and adults completed a 0-100000 version.

We quantitatively assessed the explanatory power of the models comprising the proportion estimation account (as well as the models comprising the logarithmic-to-linear representational shift view, see Appendices A and B) for group median estimates. We also evaluated explanations of performance at the individual level. All of the group medians and most of the individuals' results conformed to the predictions of the proportion estimation view regarding the NP and inverse PN tasks: while overall accuracy remained relatively consistent across tasks, patterns of under- or overestimation seen on one task were generally reversed for the other task. Furthermore, proportion estimation models provided a better explanation of performance than the logarithmic, exponential, and linear models at the group and individual levels – findings which do not depend upon the use of a particular model selection technique.

Importantly, patterns of developmental change on the PN task are broadly consistent with those found in previous work on the NP task, and can be explained by the developmental progression described by Slusser et al. (2013; see also Barth & Paladino, 2011; Cohen & Sarnecka, 2014; Rouder & Geary, 2014). When presented with the PN task, older children are more likely than younger children to produce two-cycle estimates (consistent with the strategic use of endpoints as well as an inferred midpoint as reference points), and with values of the model parameter β approaching one (showing less bias and improved accuracy). This mirrors the developmental trend reported for the NP task (Slusser et al, 2013). However, more one-cycle patterns are seen in the PN task as compared to the NP task, suggesting that different tasks invite different estimation strategies (Hollands & Dyre, 2000; Hollands et al., 2002). It is possible, for example, that the marked position on the PN task, or some other quality of the task, interferes with participants' ability to infer a central reference point. Consequently, participants rely only on the two labeled endpoints, thereby generating estimates that follow a one-cycle model.

Another novel finding from this study is that notable and consistent patterns of estimation bias persist on both tasks over the course of development. While estimates of older children and adults are relatively accurate on the NP and PN tasks (see Figures 4, 5), values of the exponent β do not converge around one (corresponding to little or no bias) with increasing age. Rather, younger children exhibit β-values far below one on these tasks and these values begin to exceed one (corresponding to a "flip" in the direction of estimation bias) around age nine (Figures 3, 4). The "flipped" direction of bias continues into adulthood (Figure 5). This result is consistent with two previous reports, one with adults on the NP task (Cohen & Blanc-Goldhammer, 2011), and the other with children and adults on the PN task (Ashcraft & Moore, 2012).

What is the best explanation of this phenomenon? Taken together, the present data (within subjects comparisons across NP and PN tasks) and previous findings (which address the tasks separately; Ashcraft & Moore, 2012; Cohen & Blanc-Goldhammer, 2011) suggest that the developmental "flip" in direction of bias is related to task-specific challenges inherent to bounded number-line estimation, not to fundamental changes in numerical representation or processing. This is evidenced by the fact that direction of bias changes over development in the typical bounded number-line task, but not in an unbounded version of the task (Cohen & Sarnecka, 2014) in which participants are given a length that corresponds to a single unit and must indicate the position of a target number on an unmarked, unbounded line. In fact, children (ages 4-8) perform like adults on the unbounded task, in that they too produce estimates with a pattern of positively accelerating bias (Cohen & Blanc-Goldhammer, 2011; Cohen & Sarnecka, 2014). Thus it is clear that the specific demands of the bounded number-line task, rather than a change in the way integers are represented over development, underlie this reversal in direction of bias [6]. We should therefore be unwilling to draw conclusions about the representation and processing of numerical magnitudes from data produced by typical number-line tasks.

Although findings from the bounded and unbounded tasks differ in the ways discussed above, results from both tasks show that the estimation patterns of older children and adults systematically deviate from true linearity. This finding is in contrast to an interpretation of PN data reported by Ashcraft and Moore (2012), who conducted a cross-sectional study in which PN tasks were presented to children and adults as part of a battery of tests exploring cognitive processes related to math performance. Part of the purpose of that study was to compare the logarithmic-to-linear representational shift account of children's number line estimation with the proportion judgment account. While performance on the PN task did follow the typical over/underestimation patterns that are explained by proportion estimation models (and unaccounted for by logarithmic or linear models), the authors concluded that their findings were more broadly consistent with the representational shift hypothesis. This may be due in part to the use of suboptimal model comparison techniques that did not account for differing numbers of parameters: the models were compared on the basis of R^2 values, and because the logarithmic

[6] See Cohen & Sarnecka (2014) for one specific proposal about the underlying cause of the directional change in bias observed in the bounded NP task (improvement in children's mathematical skills, specifically subtraction or division).

and linear models have more free parameters than the proportional models, they are likely to be favored by such a comparison method. It is also difficult to draw conclusions from these analyses because Ashcraft and Moore (2012) tested only the one- and two-cycle versions of the proportional model (with no unbounded version of the model). These models can only provide good fits for children who are already able to effectively use both of the endpoints in the number-line task (see Slusser et al., 2013), and it is likely that some children were not yet this proficient.

These authors also pointed out, in support of their endorsement of the log/linear shift framework, that children's R^2 values for the linear model were correlated with standardized math achievement scores, while R^2 values on the proportional models were not. However, this result does not reflect the predictive and explanatory power of the proportional framework for at least two reasons. First, the proportion estimation framework does not predict that R^2 values correspond to math achievement. This is because R^2 values for the proportional models do not relate to accurate performance in the same way the R^2 values for linear models do. R^2 values for linear fits are often used as a rough stand-in for accuracy on the task (e.g., Booth & Siegler, 2006). R^2 values for proportional models, which describe subtler variations in estimation patterns reflecting strategic and knowledge-based differences, may indicate that the model provides a good description of the data but do not directly reflect accuracy (nor should they be taken as a reflection of underlying numerical representations, see Barth et al., 2011). Consider the following comparison of two 7-year-olds from the current study: one child produced estimates on the PN task that were described very well by an inverse one-cycle model, yielding an R^2 value of 0.99. With a β-value of 0.54 and PAE of 8.35%, however, this child's estimates were far from accurate. His peer, who also produced estimates that were described very well by an inverse two-cycle model, yielding an R^2 value of 0.99, generated estimates that were relatively accurate, with a β-value of 0.84 and PAE of just 2.65%. Whereas the R^2 values for both children are nearly identical, the latter child, with his strategic use of an inferred midpoint and minimal estimation bias, would likely outperform the former on a standardized math test.

In sum, the present results support the idea that the theoretical framework of proportion estimation can quantitatively account for children's and adults' performance on both the (PN) task and the standard number-line (NP) task. While assessments relying only on logarithmic or linear functions tend to obscure important nuances detected in both the NP and PN tasks, models of proportion estimation confirm that typical (PN *and* NP) number line estimation tasks elicit estimates of numerical proportion. Accordingly, performance should not be interpreted as an indication of how children (and adults) reason about isolated numerical magnitude. In fact, these results fundamentally question the use of bounded number line tasks as an evaluation of mental representations of number, and magnitude in general, converging with prior related claims using different methods (e.g. Chesney & Matthews, 2013; Cohen & Sarnecka, 2014; Hurst et al., 2014; Rips, 2013).

While these findings challenge the hypothesis that accessing appropriate mental representations of numerical magnitude underlies developmental change in mathematical performance (e.g., Booth & Siegler, 2006), it is clear that the use of multiple reference points is integral to children's improvement on these number line tasks (see also Peeters, Degrande, Ebersbach, Verschaffel, & Luwel, 2016). Thus, increased support and instruction directed at facilitating children's reasoning about *relative* magnitude and numerosity may further support children's achievement in other mathematical domains.

Acknowledgments

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Appendix A. Model values for group median estimates on the Position to Number (PN) task.

Group	Model	Sl	Y0†	β	R^2	AICc	ΔAICc	MSE
6 Year Olds *(0-100)*	Exponential	0.048	19.259		0.784	152.645	51.982	297.987
	Linear	1.001	0.000		0.930	123.487	22.824	97.084
	Inverse Unbounded			2.212	0.647	163.034	62.371	486.482
	Inverse One-Cycle			0.634	*0.968	*100.663	---	*44.182
	Inverse Two-Cycle			0.798	0.932	120.139	19.476	93.447
7 Year Olds *(0-1000)*	Exponential	0.047	25.340		0.774	147.052	85.639	240.311
	Linear	1.010	0.000		0.986	75.552	14.139	15.362
	Inverse Unbounded			2.188	0.763	146.012	84.599	252.778
	Inverse One-Cycle			0.862	*0.991	*61.413	---	*9.764
	Inverse Two-Cycle			1.017	0.985	73.778	12.365	15.710
8 Year Olds *(0-1000)*	Exponential	0.007	294.683		0.600	266.559	74.131	55058.406
	Linear	1.000	0.000		0.958	212.483	20.055	5784.831
	Inverse Unbounded			3.458	0.574	265.682	73.254	58639.938
	Inverse One-Cycle			0.663	*0.980	*192.428	---	*2770.907
	Inverse Two-Cycle			0.964	0.958	210.056	17.628	5775.801
9 Year Olds *(0-1000)*	Exponential	0.007	340.274		0.658	255.542	104.983	34791.088
	Linear	0.959	27.559		0.995	156.028	5.469	550.430
	Inverse Unbounded			3.420	0.697	250.273	99.714	30856.963
	Inverse One-Cycle			1.131	*0.995	*150.559	---	*484.137
	Inverse Two-Cycle			1.098	0.993	160.473	9.914	731.760
10 Year Olds *(0-1000)*	Exponential	0.007	328.814		0.690	251.362	104.116	29230.116
	Linear	0.921	33.041		0.992	162.938	15.692	734.082
	Inverse Unbounded			3.478	0.690	248.950	101.704	29202.919
	Inverse One-Cycle			1.254	*0.996	*147.246	---	*421.713
	Inverse Two-Cycle			1.188	0.986	173.761	26.515	1273.010
Adults *(0-100000)*	Exponential				*does not converge*			
	Linear	0.976	1074.742		0.998	356.917	12.102	2376295.784
	Inverse Unbounded			6.091	0.471	484.384	139.569	531791063.600
	Inverse One-Cycle			1.120	*0.998	*344.815	---	*1585467.467
	Inverse Two-Cycle			1.121	0.997	362.201	17.386	3271602.527

Note: Values for each of the free parameters are reported above. Slope (Sl) and y-intercept (Y0) values are reported for models consistent with the representational shift account (i.e., the exponential and linear models). β-values are reported for models consistent with the proportion judgment framework (i.e., the inverse versions of the unbounded, one-cycle, and two-cycle models). R^2 values are reported as a standard measure of goodness-of-fit. Δ AICc refers to the difference in Akaike's Information Criterion (AICs) values compared to the model preferred by this metric. Mean squared error (MSE) is reported as the cross-validation error index for the Leave-One-Out Cross Validation (LOOCV) analysis.

* Indicates the preferred model as determined by each measure of model fit (i.e., the model yielding the highest R^2 value, the lowest AICc value, or the lowest LOOCV error index).

† Following the proposed interpretation of these models (i.e., that children's estimation patterns can be used to model their mental representations of number, e.g., Siegler & Opfer, 2003) constraints were set such that no model was allowed to project negative y-values (see also Slusser, Santiago, & Barth, 2013).

Appendix B. Model values for group median estimates on the Number to Position (NP) task.

	Model	Sl	Y0†	β	R^2	AICc	ΔAICc	MSE
6 Year Olds *(0-100)*	Logarithmic	14.773	0.000		0.754	129.615	34.729	122.887
	Linear	0.706	16.098		***0.935**	95.055	0.169	***32.526**
	Standard Unbounded			0.444	0.893	105.661	10.775	53.547
	Standard One-Cycle			0.580	0.929	***94.886**	---	35.380
	Standard Two-Cycle			0.397	0.884	107.703	12.817	57.921
7 Year Olds *(0-1000)*	Logarithmic	14.087	0.000		0.610	154.547	83.127	320.607
	Linear	0.928	0.936		0.982	74.419	2.999	14.707
	Standard Unbounded			0.441	0.779	137.380	65.960	181.367
	Standard One-Cycle			0.901	0.971	84.435	13.015	23.669
	Standard Two-Cycle			0.646	***0.983**	***71.420**	---	***14.348**
8 Year Olds *(0-1000)*	Logarithmic	91.590	0.000		0.570	254.567	71.239	33405.173
	Linear	0.827	81.788		0.969	191.444	8.116	2407.484
	Standard Unbounded			0.285	0.720	241.911	58.583	21779.060
	Standard One-Cycle			0.676	0.958	196.310	12.982	3257.299
	Standard Two-Cycle			0.470	***0.976**	***183.328**	---	***1896.492**
9 Year Olds *(0-1000)*	Logarithmic	91.051	0.000		0.481	267.861	99.709	58127.208
	Linear	0.976	0.000		0.987	180.243	12.091	1509.677
	Standard Unbounded			0.287	0.631	257.258	89.106	41281.650
	Standard One-Cycle			1.129	0.987	177.247	9.095	1472.006
	Standard Two-Cycle			0.722	***0.991**	***168.152**	---	***1007.703**
10 Year Olds *(0-1000)*	Logarithmic	89.313	0.000		0.446	271.186	86.041	66766.438
	Linear	0.971	0.000		0.978	194.255	9.110	2706.715
	Standard Unbounded			0.285	0.590	261.544	76.399	49352.230
	Standard One-Cycle			1.274	***0.983**	***185.145**	---	***2045.675**
	Standard Two-Cycle			0.768	0.979	190.184	5.039	2523.507
Adults *(0-100000)*	Logarithmic	5147.025	0.000		0.330	495.712	145.577	771777232.100
	Linear	0.997	0.000		0.997	366.088	15.953	3482204.129
	Standard Unbounded			0.165	0.465	487.944	137.809	616809646.900
	Standard One-Cycle			1.096	***0.998**	***350.135**	---	***1978883.393**
	Standard Two-Cycle			0.958	0.997	363.185	13.050	3408602.180

Note: Values for each of the free parameters are reported above. Slope (Sl) and y-intercept (Y0) values are reported for models consistent with the representational shift account (i.e., the logarithmic and linear models). β-values are reported for models consistent with the proportion judgment framework (i.e., the standard versions of the unbounded, one-cycle, and two-cycle models). R^2 values are reported as a standard measure of goodness-of-fit. Δ AICc refers to the difference in Akaike's Information Criterion (AICs) values compared to the model preferred by this metric. Mean squared error (MSE) is reported as the cross-validation error index for the Leave-One-Out Cross Validation (LOOCV) analysis.

* Indicates the preferred model as determined by each measure of model fit (i.e., the model yielding the highest R^2 value, the lowest AICc value, or the lowest LOOCV error index).

† Following the proposed interpretation of these models (i.e., that children's estimation patterns can be used to model their mental representations of number, e.g., Siegler & Opfer, 2003) constraints were set such that no model was allowed to project negative y-values (see also Slusser, Santiago, & Barth, 2013).