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Short Title: Variational Bayes Inference for HM-DCMs

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Data availability statement: The data analysis code is available on the Open Science Framework page (https://osf.io/nzgmw/?view_only=d0d2824260c1487da856eaa0d0a69150).

Acknowledgements: This work was supported by Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for JSPS Research Fellow 18J01312, 20H01720, 21H00936, and 22K13810.

Conflict of interests: The authors have no competing interests to declare.

Variational Bayes Inference for Hidden Markov Diagnostic Classification Models

Abstract

Diagnostic classification models (DCMs) can be used to track the cognitive learning states of students across multiple time points or over repeated measurements. This study developed an effective variational Bayes (VB) inference method for hidden Markov type longitudinal general DCMs. The simulations performed in this study verified the validity of the proposed algorithm for satisfactorily recovering the true parameters. Simulation and actual data analyses were conducted to compare the proposed VB and Markov chain Monte Carlo (MCMC) methods. The results revealed that the parameter estimates provided by the VB method are consistent with the MCMC method with the additional benefit of a faster estimation time. The comparative simulation also indicated differences between the two methods in terms of posterior standard deviation and coverage of 95% credible intervals. Thus, with limited computational resources and time, the proposed VB method can output comparable estimations with that of the MCMC.

Keywords: Variational Bayesian inference, diagnostic classification models, cognitive diagnostic models, hidden Markov model, longitudinal analysis, Markov chain Monte Carlo method.

Variational Bayesian Inference Posterior Approximation Algorithm for Hidden Markov Diagnostic Classification Models

1. Introduction

Diagnostic classification models (DCMs; Rupp et al., 2010; von Davier & Lee, 2019) are a powerful class of psychometric models for learning students' knowledge states – expressed as permutations of latent categorical proficiency variables called attributes (e.g., DiBello et al., 2006) – based on their responses to a set of observable tasks. Evaluating the growth or change in students' knowledge states over time is an important consideration for both teachers and students as such information can be leveraged to provide students with learning activities or interventions designed to remedy areas of deficiency or to provide teachers with formative feedback about their instructional practices. Recently, DCMs have been extended to the longitudinal case, allowing researchers to model, capture, and estimate student educational trajectories by evaluating changes in the attributes of interest across multiple measurement time points.

The development of longitudinal DCM models generally stems from two broad modeling approaches (see Table 1 of Zhan, 2020b). One approach assumes that categorical attributes are a function of higher order continuous latent traits (de la Torre & Douglas, 2004), and growth of academic proficiency is modeled with respect to these traits. The second class of longitudinal DCMs adopts a latent class formulation to directly model changes in the categorical attribute mastery patterns. From a statistical perspective, DCMs can be thought of as a restricted version of latent class models (Rupp & Templin, 2008) or mixture models (Bishop, 2006). The discrete and random nature of attributes and attribute mastery profiles imply that longitudinal DCMs can be treated as special cases of hidden Markov models (HMM; e.g., Beal, 2003; Bishop, 2006; Ghahramani & Beal, 2001) or latent transition models (LTA; Collins & Lanza, 2009; Collins & Wugalter, 1992; Martin et al., 1996). In this paper, we use the term HMM to describe models of this type over the term LTA to remain consistent with the broader statistics literature.

A unique characteristic of HMMs is its ability to model the transition of latent classes over time. An important goal of DCMs is to evaluate the mastery status of students with respect

1 to a set of well-defined discrete attributes as opposed to general academic abilities – often
2 modeled as continuous, random variables. The assumption that attributes are a function of higher
3 order proficiency variables results in the inability to directly model changes in attribute mastery
4 status, making this feature unsuitable for the development of DCMs whose purpose is to provide
5 information about the attributes themselves. Thus, we adopt the HMM approach in this study.
6 We refer to the class of models resulting from the synthesis of HMMs and DCMs as Hidden
7 Markov Diagnostic Classification Models (HM-DCMs).

8 The model development literature for longitudinal DCMs has proliferated in recent years;
9 however, parameter estimation methods for HM-DCMs have received little attention. In this
10 study, we develop a computationally efficient Bayesian estimation procedure by introducing the
11 variational Bayes (VB) inference technique (e.g., Beal, 2003; Bishop, 2006; Blei et al., 2017;
12 Grimmer, 2011; Nakajima et al., 2019; Rijmen et al. 2016). The VB method is unique in that it
13 provides a deterministic approximation of the posterior distribution. This is in contrast to other
14 well-known Bayesian estimation approaches – such as Markov chain Monte Carlo (MCMC; e.g.,
15 Brooks et al., 2011; Gelman et al., 2013) – which provide a stochastic approximation of the
16 posterior. Alternatively, the VB estimator maximizes the lower bound of the log marginal
17 likelihood instead of the marginal likelihood itself. In summary, the VB method is essentially an
18 optimization problem. This characteristic has been shown to lead to faster computational time
19 and easier convergence checks than those with conventional stochastic approximation methods.
20 VB algorithms for general HMMs have been developed (Beal, 2003; MacKay & MacKay, 1997;
21 Nakajima et al., 2019) and can be modified for their application with HM-DCMs. We also
22 extend the VB method proposed by Yamaguchi and Okada (2020a) to incorporate HM-DCMs
23 with a general DCM measurement model.

24 The VB method provides parameter estimates of complex statistical models with
25 minimal computation time. For example, the VB method was applied to the multidimensional
26 item response theory model (Cho, et al., 2021) that needs heavy computation because of multiple
27 integration of likelihood, and the computational difficulties were efficiently addressed. Similarly,

the HM-DCMs are complex and have numerous parameters requiring enormous estimation time while using MCMC method. The MCMC can provide more precise posteriors than the VB method but is computationally heavy. Therefore, to obtain Bayesian parameter estimates, MCMC method is not always the best. Our VB method can provide Bayesian like parameter estimates even when the MCMC does not work. If the situation is more complex, the MCMC method fails to estimate model parameters. In addition, the VB method can be used to explore the various measurement models in the HM-DCMs owing to its low computational cost. Information communication technology has improved the quantity of available data even in the educational setting; therefore, a computationally efficient algorithm is critical. Thus, the VB method can extend applicability of the HM-DCMs.

The remainder of this paper is organized as follows. Section 2 comprehensively reviews longitudinal DCMs and their parameter estimation methods. HM-DCM is mathematically formulated in Section 3. Section 4 presents the VB posterior of HM-DCMs and the VB estimation algorithm. Section 5 presents a parameter recovery simulation study. Section 6 demonstrates a comparison between the VB and MCMC methods. Note that the MCMC method described in Section 6 was written in JAGS language (Plummer, 2003), which is easy-to-use, because specific MCMC methods do not exist for HM-DCMs under general DCMs. Section 7 presents a small comparative study between the VB and MCMC methods using real data employed by Wang et al. (2018). Finally, Section 8 provides a discussion of the present research and recommendations for the use of the algorithm in future research.

2. Longitudinal DCMs and Parameter Estimation Methods

One approach to developing longitudinal DCMs is to assume that the latent attributes are influenced by higher order continuous latent traits. For example, Huang (2017) proposed a multilevel generalized deterministic input noisy AND-gate model (G-DINA; de la Torre, 2011) that captures changes in discrete latent attributes via a multilevel structure, where the discrete attributes are assumed to be summarized as a common continuous latent trait. Pan et al. (2020)

extended Huang's model (2017) and employed a multivariate normal distribution for the higher order traits and assessed proficiency changes under the log-linear cognitive diagnostic model (LCDM; Henson et al., 2009). Zhan, Jiao, Liao, et al. (2019) employed a similar strategy in the development of a higher-order longitudinal variant of the DINA model (Junker & Sijtsma, 2001; Macready & Dayton, 1977; Maris, 1999), and Zhan, Jiao, Man, et al. (2019) demonstrated how to estimate the longitudinal DINA model under a Bayesian setting. Lin et al. (2020) applied a growth curve approach for modeling higher order traits, and covariates were introduced to explain continuous traits. More recently, Zhan, Ma, et al. (2020) developed a higher-order latent trait DCM for hierarchical attribute structures, Zhan (2020a) introduced a Markov estimation strategy for immediate diagnostic feedback, and Zhan (2021) proposed a longitudinal probabilistic DCM that treats attributes as probabilistic entities.

A second approach for developing longitudinal DCMs is based on the HMM approach. Li et al. (2016) developed an HM-based DINA model and examined changes in the mastery status of students in a mathematics assessment that consisted of an instructional intervention component. Kaya and Leite (2017) proposed and examined the performance of the HM-DINA and HM-DINO (deterministic input noisy OR-gate; Templin & Henson, 2006) models, while Chen et al. (2018) studied and developed an estimation procedure for the HM-DINA model under a fully Bayesian framework. Madison and Bradshaw (2018a, 2018b) developed a transition diagnostic model under the LCDM for a pretest-posttest design and demonstrated how to estimate the parameters of the HM-LCDM in Mplus (Muthén & Muthén, 1998-2017) with ML using the expectation-maximization (Dempster et al., 1977) algorithm. Furthermore, Wang (2021) proposed a penalized EM algorithm for HM-DCMs that can track learning trajectories in a longitudinal setting. Wen et al. (2020) proposed an artificial neural network approach for tracking the development of cognitive skills in DCMs and further demonstrated a high accuracy of parameter recovery and classification rates of the proposed approach.

Recently, Wang, Yang, et al. (2018) modified the work of Chen et al. (2017), applied an HMM and higher order modeling approach for a mental rotation skill test, and used the examinee

1 response time to assess student learning progress. Hung and Huang (2019) provided a
2 straightforward extension of Huang (2017)'s model that incorporated the LTA approach and a
3 higher-order formulation to model repeated measurements under the G-DINA model.

4 One important aspect of HMMs, which is often overlooked, is that they are singular
5 models (e.g., Nakajima et al., 2019; Watanabe, 2018); that is, the posterior distribution cannot be
6 approximated with a normal distribution and the usual results from normal asymptotic theory do
7 not hold. This means that ML estimation methods are not appropriate to determine the
8 parametric variations in HMMs despite being used in previous studies (e.g., Madison &
9 Bradshaw 2018a, 2018b). To be more precise, the Fisher information matrix of a singular model
10 has at least one eigenvalue with a value of zero (Nakajima et al., 2019). As a result, the
11 approximated standard errors with the Fisher information matrix cannot be employed.

12 Bayesian estimation methods have been recommended as an alternative to ML-based
13 techniques for their generalizability (Nakajima et al., 2019; Watanabe, 2018). However,
14 performing probabilistic posterior inference using MCMC is computationally intensive, and
15 additionally, the convergence checks are difficult to evaluate (e.g., Bishop, 2006). The number of
16 HM-DCM model parameters becomes exceeding large as the number of attributes, students,
17 items, and/or measurement occasions increases. Under such conditions, MCMC estimation
18 procedures become computationally infeasible and prohibitively expensive in terms of time
19 considerations. Moreover, current MCMC methods for HM-DCMs are limited to restricted DCM
20 measurement models (e.g., DINA, DINO) rather than a broader class of general DCMs (e.g.,
21 LCDM, G-DINA, general diagnostic model (GDM); von Davier, 2008).

22 Furthermore, previous simulation-based research has proved the superiority of the VB
23 method compared to the ML approach; for example, a numerical comparison between the two by
24 Beal (2003) revealed that the estimation errors of VB were smaller in an HMM, in addition to the
25 smaller mean square errors in linear regression reported by Tzikas et al. (2008). Moreover, a
26 comparison between the VB inference algorithm and the EM algorithm was also conducted for
27 the DINA model (Yamaguchi, 2020b), whose results indicated that the two algorithms were

equally fast and that posterior SDs were reasonable while the SEs under the ML method were extremely large in several situations.

The VB method has recently gained traction in the field of psychometrics. For example, Rijmen et al. (2016) derived a VB algorithm for item response theory models. Jeon et al. (2017a) developed a maximization-maximization algorithm, a VB-based procedure, for the generalized linear mixed model. Humphreys and Titterington (2003) and White and Murphy (2014) derived a VB inference algorithm for latent class models. Furthermore, Yamaguchi and Okada (2020a) introduced a mixture model formulation of the DCMs and derived a VB estimation algorithm for general DCMs that extended the DINA model-based algorithm (Yamaguchi & Okada, 2020b). Yamaguchi and Okada (2020b) indicated that the VB estimation method provided similar parameter estimates as MCMC but at a much faster estimation time.

3. Formulation of the HM-DCM

The basic formulation of the measurement model of general DCMs is the restricted latent class formulation (e.g., Yamaguchi & Okada, 2020a, 2020b; Yamaguchi & Templin, 2022). First, let X_{ijt} represent a random variable denoting the item response provided by respondent i ($i = 1, \dots, I$) to item j ($j = 1, \dots, J_t$) at time t ($t = 1, \dots, T$) with realization x_{ijt} that takes on a value of 1 for a correct response and 0 otherwise. The latent classes in DCMs are called attribute mastery patterns and are expressed as a vector $\alpha_l = [\alpha_{l1}, \dots, \alpha_{lK}]^T$, ($l = 1, \dots, L$) that is defined as a permutation of K latent binary variables α_{lk} , which acquire a value of 1 if the l th pattern contains a mastered k th attribute and 0 otherwise. The superscript T represents the transpose. The total number of latent classes is $L = 2^K$ and an $L \times K$ attribute mastery pattern matrix is written as α .

An important assumption of DCMs is the equivalence constraints applied on the correct item response parameters, which indicates that some attribute mastery patterns exhibit the same correct item response probability. The Q-matrix (e.g., Tatsuoka, 1983), which represents the connection between items and latent attributes, denotes this constraint and plays an essential role

in the DCM formulation. The Q-matrix is expressed as a $J \times K$ matrix Q with binary elements. The j th row of Q is $\mathbf{q}_j = [q_{j1}, \dots, q_{jK}]$ with element $q_{jk} = 1$ if item j requires attribute k and $q_{jk} = 0$ otherwise. The vector $\mathbf{q}_j$ indicates the set of attributes the item can distinguish. For example, if $K = 4$ attributes are assumed in a test and the first and second attributes are measured by item j then $\mathbf{q}_j$ is $[1, 1, 0, 0]$. In this example, there are $2^4 = 16$ possible attribute mastery patterns, but item j only provides information about the first two attributes. Therefore, this item can distinguish among four patterns, $[0, 0, *, *]$, $[0, 1, *, *]$, $[1, 0, *, *]$, and $[1, 1, *, *]$, where the symbol $*$ indicates the value of the element that can be ignored without any loss of information. These reduced patterns, or item specific patterns, represent groupings of the original attribute mastery patterns. The pattern $[0, 0, *, *]$, for example, implies that the patterns $[0, 0, 0, 0]$, $[0, 0, 0, 1]$, $[0, 0, 1, 0]$, and $[0, 0, 1, 1]$ have the same item response probability. The other reduced patterns define their own groupings in the same manner. This connection is represented as a restriction matrix, named the G-matrix (Yamaguchi & Okada, 2020a).

Let a reduced attribute mastery pattern be denoted by α_{jh}^* with h ($h = 1, \dots, H_j = 2^{\sum_k q_{jk}}$) as its indicator. Then, the element of the G-matrix for the j th item, g_{jhl} , becomes 1 if α_{jh}^* is the same as α_l except for the $*$ elements, which are coded as 0. The G-matrix is defined for each item in the same manner (see Table 1, Yamaguchi and Okada, 2020a). Putting this G-matrix element in Equation 1, the modified item response function of the general DCM at time t becomes

$$P(X_{ijt} = x_{ijt} | \mathbf{z}_{it}, \boldsymbol{\theta}_{jt}) = \prod_l \prod_h \left\{ [\theta_{jht}^{x_{ijt}} (1 - \theta_{jht})^{1-x_{ijt}}]^{g_{jhtl}} \right\}^{z_{itl}}, \quad (1)$$

where θ_{jth} represents the correct item response probability of the j th item if a respondent has the h th reduced attribute mastery pattern at time t . Note that the value of θ_{jht} is an element of $\boldsymbol{\theta}_{jt}$, which is a parameter vector of length $2^{\sum_k q_{jk}}$ and is smaller than L , and z_{itl} is the l th element of the vector $\mathbf{z}_{it}$ representing the latent class indicator variable of respondent i at time t . Only one element of $\mathbf{z}_{it}$ is 1, and the others are 0, called as one-of- L representation. Note that the number of items can vary across time points but for the simplicity of exposition and

notation, we assume that the numbers of items over all the measurement occasions are the same (i.e., $J = J_1 = J_2 = \dots = J_T$). Additionally, we assume a different Q-matrix for each measurement time point, so we set subscript t on the element of Q-matrix: q_{jkt} . Moreover, elements of a G-matrix generated from a Q-matrix should have a time subscript t if the elements of Q-matrix vary across measurement occasions represented as g_{jhtl} .

An initial attribute mastery pattern for student i is denoted as $\mathbf{z}_{i1}$, which is assumed to follow a categorical distribution with parameter $\boldsymbol{\pi} = [\pi_1, \dots, \pi_L]^\top$, where $\pi_l \geq 0$ and $\sum \pi_l = 1$, and it is expressed as

$$P(\mathbf{z}_{i1} | \boldsymbol{\pi}) = \prod_l \pi_l^{z_{i1l}}. \quad (2)$$

An essential component of the HMM is the transition probability $\tau_{ll'}$, which is an element of the transition matrix $\mathbf{T}$:

$$\mathbf{T} = \begin{bmatrix} \tau_{11} & \dots & \tau_{1l'} & \dots & \tau_{1L} \\ \vdots & \ddots & & & \vdots \\ \tau_{l1} & & \tau_{ll'} & & \tau_{lL} \\ \vdots & & & \ddots & \vdots \\ \tau_{L1} & \dots & \tau_{Ll'} & \dots & \tau_{LL} \end{bmatrix}, \quad (3)$$

where $\tau_{ll'} \geq 0, \forall l, l'$ and $\sum_{l'} \tau_{ll'} = 1, \forall l$. The transition probability $\tau_{ll'}$ denotes the student transition probability from attribute mastery pattern l to l' . Note that we assume this transition probability is the same for all students. Moreover, we only take into account adjoining time points (first-order Markov model); however, note that it is possible to consider more time points simultaneously, allowing one to attain more information in the model. Using this matrix representation, the transition of attribute mastery pattern is represented as follows:

$$P(\mathbf{z}_{it} | \mathbf{z}_{i(t-1)}, \mathbf{T}) = \prod_l \prod_{l'} \tau_{ll'}^{z_{i(t-1)l} z_{itl'}}. \quad (4)$$

A detailed complete data likelihood and prior setting are given in Supplemental Material A.

4. Variational Bayes Algorithm for the HM-DCM

4.1 Assumptions and Variational Posterior Distributions

The variational posteriors of the HM-DCM under the following factorization assumption are derived:

$$P(\mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T} | \mathbf{X}) \approx q(\mathbf{Z})q(\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}) = \left(\prod_i q(\mathbf{z}_i) \right) q(\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}), \quad (5)$$

where $\mathbf{Z}$, $\boldsymbol{\Theta}$, and $\mathbf{X}$ represent the sets of all $\mathbf{z}_{it}$, θ_{jht} , and x_{ijt} , respectively. The subscript of variational posterior distribution is dropped for simplicity, and we distinguish them in their arguments.

First, variational distribution of the initial mixing parameter $\boldsymbol{\pi}$ is a Dirichlet distribution, given as follows:

$$q(\boldsymbol{\pi}) = \frac{1}{\text{Beta}(\boldsymbol{\delta}^*)} \prod_l \pi_l^{\delta_l^* - 1}, \quad (6)$$

where $\delta_l^* = \sum_i \mathbb{E}_{q(\mathbf{z}_i)}[z_{i1l}] + \delta_l^0$. The variational posterior $q(\theta_{jht})$ is a Beta distribution:

$$q(\theta_{jht}) = \frac{1}{\text{Beta}(a_{jht}^*, b_{jht}^*)} \theta_{jht}^{a_{jht}^* - 1} (1 - \theta_{jht})^{b_{jht}^* - 1}, \quad (7)$$

and its parameters are

$$\begin{cases} a_{jht}^* = \sum_i \sum_l \mathbb{E}_{q(\mathbf{z}_i)}[z_{itl}] g_{jhtl} x_{ijt} + a_{jht}^0, \\ b_{jht}^* = \sum_i \sum_l \mathbb{E}_{q(\mathbf{z}_i)}[z_{itl}] g_{jhtl} (1 - x_{ijt}) + b_{jht}^0. \end{cases} \quad (8)$$

Finally, the variational posterior $q(\boldsymbol{\tau}_l)$ is a Dirichlet distribution again:

$$q(\boldsymbol{\tau}_l) = \frac{1}{\text{Beta}(\boldsymbol{\omega}_l^*)} \prod_{l'} \tau_{ll'}^{\omega_{ll'}^* - 1}, \quad (9)$$

where the parameter is $\omega_{ll'}^* = \sum_{t=2} \sum_i \mathbb{E}_{q(\mathbf{z}_i)}[z_{i(t-1)l} z_{itl'}] + \omega_{ll'}^0$.

These variational posteriors are well-known distributions, and their expectation and variance have a closed form. The mean and variance of the variational posteriors are derived using their posterior parameters as follows:

$$\begin{cases} \mathbb{E}_{q(\boldsymbol{\pi})}[\pi_l] = \frac{\delta_l^*}{\sum_l \delta_l^*}, \\ \mathbb{V}_{q(\boldsymbol{\pi})}[\pi_l] = \frac{\delta_l^* (\sum_l \delta_l^* - \delta_l^*)}{(\sum_l \delta_l^*)^2 (1 + \sum_l \delta_l^*)}, \end{cases} \quad (10)$$

$$\begin{cases} \mathbb{E}_{q(\tau_l)}[\tau_{ll'}] = \frac{\omega_{ll'}^*}{\sum_{l'} \omega_{ll'}^*}, \\ \mathbb{V}_{q(\tau_l)}[\tau_{ll'}] = \frac{\omega_{ll'}^* (\sum_{l'} \omega_{ll'}^* - \omega_{ll'}^*)}{(\sum_{l'} \omega_{ll'}^*)^2 (1 + \sum_{l'} \omega_{ll'}^*)}, \end{cases} \quad (11)$$

1 and

$$\begin{cases} \mathbb{E}_{q(\theta_{jht})}[\theta_{jht}] = \frac{a_{jht}^*}{a_{jht}^* + b_{jht}^*}, \\ \mathbb{V}_{q(\theta_{jht})}[\theta_{jht}] = \frac{a_{jht}^* b_{jht}^*}{(a_{jht}^* + b_{jht}^*)^2 (1 + a_{jht}^* + b_{jht}^*)}. \end{cases} \quad (12)$$

2 Derivation of these variational posterior are given in Supplemental Material B. An attribute
3 mastery pattern of the t th time point is estimated as $l = \underset{l}{\operatorname{argmax}} \mathbb{E}[\mathbf{z}_{it}]$.

4 **4.2 Forward-backward Algorithm and Variational EM Algorithm**

5 To update these variational posteriors, expectations of latent attribute mastery pattern,
6 $\mathbb{E}_{q(\mathbf{z}_i)}[\mathbf{z}_{it}]$ and $\mathbb{E}_{q(\mathbf{z}_i)}[\mathbf{z}_{it} \mathbf{z}_{i(t-1)}^\top]$, need to be calculated and not the parametric distribution of
7 $q(\mathbf{z}_i)$ as follows:

$$\begin{cases} q(\mathbf{z}_{it}) = \sum_{\mathbf{z}_{i \setminus t}} q(\mathbf{z}_i), \\ q(\mathbf{z}_{it}, \mathbf{z}_{it-1}) = \sum_{\mathbf{z}_{i \setminus \{t, t-1\}}} q(\mathbf{z}_i), \end{cases} \quad (13)$$

8 where summations were taken over $\mathbf{z}_{i \setminus t}$ and $\mathbf{z}_{i \setminus \{t, t-1\}}$, which are vectors whose t th and $(t -$
9 $1)$ th elements are removed from $\mathbf{z}_i$. When the number of time points is large, calculating the

10 summation in Equation (13) is computationally difficult. Instead, calculating two expectations

11 $\mathbb{E}_{q(\mathbf{z}_{it})}[\mathbf{z}_{itl}]$ and $\mathbb{E}_{q(\mathbf{z}_i)}[\mathbf{z}_{itl'} \mathbf{z}_{i(t-1)l}]$ is enough to develop a variational EM algorithm. We

12 employ a well-known forward-backward algorithm (e.g., Bishop, 2006; Rabiner, 1989), also

13 known as the Baum-Welch algorithm (Baum, 1972), based on the Beal (2003) and Suyama and

14 Sugiyama (2017), to calculate these expectations, as provided in Supplemental Material C.

15 Supplemental Material D describes the calculation of the evidence lower bound (ELBO) which is

16 another name of lower bound log marginal likelihood in the context of variational inference.

The VB algorithm for the HM-DCM is summarized as follows. First, we set the hyperparameters $\mathbf{A}^0, \mathbf{B}^0, \boldsymbol{\delta}^0$, and $\boldsymbol{\Omega}^0$ and initialize $\mathbf{A}^*, \mathbf{B}^*, \boldsymbol{\delta}^*$, and $\boldsymbol{\Omega}^*$. Next, we implement the VE (variational expectation)-step by calculating $\mathbb{E}_{q(\mathbf{z}_{it})}[\mathbf{z}_{itl}]$ and $\mathbb{E}_{q(\mathbf{z}_i)}[\mathbf{z}_{itl}'\mathbf{z}_{i(t-1)l}], \forall i, t, l$. Third, $\mathbf{A}^*, \mathbf{B}^*, \boldsymbol{\delta}^*$, and $\boldsymbol{\Omega}^*$ are updated, which is the VM (variational maximization)-step. If the change of ELBO is smaller than the prespecified threshold, the algorithm is terminated and the estimated mean and variance of variational posteriors are returned, otherwise the algorithm returns to the VE-step.

5. Simulation Study 1: Parameter Recovery

5.1. Simulation Settings

A simulation study was conducted to examine the parameter recovery performance of the proposed VB algorithm. This section focuses on evaluating the effectiveness of the VB method in recovering model parameters. Then, the comparative simulation between the VB and MCMC methods is described in simulation study 2. We also compare the performance of the VB algorithm to that of standard MCMC procedures for the HM-DINA model (Wang et al., 2018) in the real data analysis section.

In this study, we manipulated three factors, namely the (1) sample size, either small (500) or large (2000); (2) Q-matrix, consisting of either four or six attributes; and (3) item quality conditions representing the high ($p_j^{\min} = 0.1$ and $p_j^{\max} = 0.9$) and low-quality items ($p_j^{\min} = 0.2$ and $p_j^{\max} = 0.8$), where the definitions of $p_j^{\min}$ and $p_j^{\max}$ will be shown later. Thus, a total of $2 \times 2 \times 2 = 8$ simulation conditions were evaluated. The two Q-matrices investigated are shown in Table S1 of Supplemental Material E. In the simulation, we assumed that the Q-matrix is the same across all time points (measurement occasions). The four attribute Q-matrix shown on the left side of Table S1 consists of 30 items. Under this Q-matrix specification, items one through eight measure one attribute, items nine through 20 measure two attributes, items 21 through 28 measure three attributes, and items 29 and 30 measure four attributes. The six attribute Q-matrix shown on the right side of Table S1 consists of 44 items,

where items one through 18 measure one attribute, items 19 through 33 measure two attributes, items 35 through 39 measure three attributes, and items 40 through 44 measure four attributes.

True item parameters were set to satisfy the monotonicity conditions (e.g., Henson et al., 2009). Let h'_a be an element of a subscript set of h satisfying $\sum_k \alpha_{h'jk}^* = a$ for $a =$

$0, \dots, \sum_k q_{jk}$. Then, we represent $p_j^{\max} = \theta_{jh'_{\sum_k q_{jk}} t}$ for all time t . The $p_j^{\max}$ parameter

represents the item response probability for item j corresponding to the attribute pattern where all required attributes have been mastered. Similarly, the $p_j^{\min}$ parameter for item j was represented as $p_j^{\min} = \theta_{jh'_0 t}$. The $p_j^{\min}$ parameter represents the correct item response

probability for item j corresponding to the pattern where any of attribute required by the item

are not mastered. The item parameters were set as $\theta_{jh'_a t} = p_j^{\min} + a \frac{p_j^{\max} - p_j^{\min}}{\sum_k q_{jk}}$. Note that the

item parameters and Q-matrix configuration were identical across time points. The total number of time points was fixed to $T = 5$.

The true transition matrix $\mathbf{T}$ was also fixed for each Q-matrix condition. In particular, we set $P(\alpha_{k(t+1)} = 1 | \alpha_{kt} = 0) = 3/10$, $P(\alpha_{k(t+1)} = 0 | \alpha_{kt} = 1) = 1/10$, and $P(\alpha_{k(t+1)} = 0 | \alpha_{kt} = 0) = P(\alpha_{k(t+1)} = 1 | \alpha_{kt} = 1) = 4/10$, for $t = 1, \dots, T - 1$, where the first probability represents a change from non-mastery status to mastery status with respect to the k th attribute at the subsequent measurement occasion; the second probability represents the condition where an examinee becomes a non-master of (or forgets) attribute k after having mastered the attribute at the previous measurement occasion; and the final probability represents the condition of no change in mastery or non-mastery status with respect to attribute k at the subsequent measurement occasion. Note that the sum of the four probabilities is not one, and the attribute pattern-wise transition probabilities were defined by the normalized versions of the products of attribute-wise transition probabilities. This transition configuration represents the case where attribute mastery status across measurement occasions tends to be stable, whereas mastering an attribute is more probable than forgetting an attribute.

The tetrachoric correlations among attributes were fixed to 0.4 and we assumed a multivariate normal distribution to represent tetrachoric correlations among attributes based on Chiu and Douglas (2013). More precisely, for every student, a continuous latent attribute vector $\tilde{\alpha} = [\tilde{\alpha}_1, \dots, \tilde{\alpha}_K]^\top$ was generated from a multivariate normal distribution with mean vector $\mathbf{0}$ and correlation matrix Σ_{cor} , where all the off-diagonal elements were fixed to 0.4. If the condition $\tilde{\alpha}_k \geq \Phi^{-1}\left(\frac{k+0.5}{K+1}\right)$ was satisfied, then α_k was set to 1, otherwise $\alpha_k = 0$. We utilized this setting as high correlations among attributes are known to generate highly biased attribute mastery patterns. Thus, this simulation setting ensures a diversity of attribute mastery and transition patterns.

Assuming the above settings, the data generating procedure was as follows. First, initial attribute mastery patterns were generated. Then, the first item responses were generated based on the initial attribute mastery patterns, assumed Q-matrix, and item parameters. The attribute mastery patterns at the second measurement occasion were calculated from the true transition matrix and the initial attribute mastery patterns. These item response and transition procedures were repeated for all the remaining time points.

The hyperparameter setting for the mixing parameter δ^0 was set to $\mathbf{1}_L$, a vector of length L consisting entirely of ones. A^0 and B^0 were set as follows. The hyperparameter $a_{jh'_{at}}^0$ was set to $1.0001 + a/(\sum_k q_{jk} - 1)$, and constant 10^{-4} was selected for computational stability. Similarly, hyperparameter $b_{jh'_{at}}^0$ was set to $1.0001 + (\sum_k q_{jk} - a)/(\sum_k q_{jk} - 1)$. This setting ensures that correct item response probabilities monotonically increase as the number of mastered attributes increases. In addition, a correct item response probability of all mastering patterns was greater than 0.5 and all non-mastering pattern was less than 0.5 in this prior setting. Hyperparameters Ω^0 for transition probabilities were all set to one. The initial values of $\mathbb{E}_{q(z_i)}[z_{it}]$ and $\mathbb{E}_{q(z_i)}[z_{it}z_{i(t-1)}^\top]$ were set to $1/L$ and $1/(L \times L)$, respectively. The simulation was replicated 200 times for every condition. An estimation program was written in R 4.0.3 for a 64-bit Windows 10 PC (R Core Team, 2020). The maximum number of iterations was set to 500, and the algorithm was terminated if the absolute change in the ELBO

was smaller than 10^{-4} . The simulation code is available in the following open science framework (OSF) page:
https://osf.io/nzgmv/?view_only=d0d2824260c1487da856eaa0d0a69150.

5.2. Results

The biases and RMSE results for the fifth time point are shown in Figures 1 and 2, respectively. The biases were approximately zero in almost all conditions except for the low-quality items and small samples. The low-quality item, small sample, and four attribute condition indicated slightly positive biases. However, they were generally small in magnitude (less than 0.05). Similarly, the biases in the low-quality items, small sample, and six attribute condition indicated slightly negative biases but most of them were greater than 0.05. These results indicate that parameter recovery was accurate, even in the final measurement occasion. The RMSE results shown in Figure 2 reveal that the estimates were relatively small in magnitude, with an average value of less than 0.15 across all the simulation conditions. The large sample size condition indicated that the RMSEs were less than 0.10. The small sample size conditions sometimes generated relatively large RMSEs; however, if the quality of items was high, the RMSEs values tended to be less than 0.10. Despite the worst performance observed for a small sample size and low-quality item, these conditions were modeled by the algorithm with sufficient accuracy, with the highest RMSE value of 0.15. The biases and RMSE results from the first to fourth time points can be found in Supplemental Material F.

The bias and RMSE results for the transition probabilities are shown in Figures 3 and 4, respectively. The biases were concentrated around 0, so the parameters were recovered very well. In addition, most of the RMSE values were less than 0.05 across all the conditions, suggesting a negligible variation in parameter estimation. An accurate recovery of transition probabilities was exhibited for the four-attribute and small sample size conditions, owing to the range of RMSE values lying between 0.05 and 0.08.

Figure 5 depicts the results for the CCR_k and CCR_{All} classification rates for each condition and time point, where CCR_k and CCR_{All} represent k th attribute and attribute pattern

correct classification rates. The difference of time points is not shown for all conditions. Moreover, no explicit differences were observed for different number of attributes. The high-quality items exhibited high CCRs, that is, greater than 0.95, and the CCR_{All} values were greater than 0.90. In addition, even for low-quality items, the CCR for each attribute was greater than 0.90, provided the sample size was sufficiently large. For low quality and large sample size conditions, the CCR_{All} rates were between 0.70 and 0.75. Note that we have omitted the results obtained for the π parameter because they do not contribute in attaining the correct classification results.

6. Simulation 2: Comparison Between VB and MCMC Methods

6.1. *Simulation Settings*

The objective of the second simulation is to compare VB and MCMC methods. We particularly focus on estimation time, bias, RMSE, posterior SD, coverage of 95% CI, and CCRs. The 95% CIs of VB were based on variational posteriors. The variational posterior of the initial mixing and transition probability parameters were described as a Dirichlet distribution, and we marginalized it to calculate 95% CI for each parameter. Owing to the nature of Dirichlet distributions, for example, variational posterior of π_l is a Beta distribution: $\text{Beta}(\delta_l^*, \sum_{l' \neq l} \delta_{l'}^*)$, and its 95% CI was based on equal-tailed interval. The same calculation was applied for the transition parameters.

In our preliminary exploration, the MCMC estimation, which was written in JAGS language version 4.3.0, was slow to convergence in the first simulation settings. Therefore, we employed relatively simple conditions. First, we modified the Q-matrix setting. The Q-matrix employed in the second simulation assumed three attributes and consisted of two sets of the following sub-Q-matrix: $[\mathbb{I}_K, [1,1,0]^T, [1,0,1]^T, [0,1,1]^T]^T$, where $\mathbb{I}_K$ is $K \times K$ identity matrix. Therefore, the Q-matrix had 12 items. The total number of time points was decreased to $T = 4$. The manipulated factors were the (1) sample size, either small (100) or large (1,000), and (2) item quality conditions representing the high ($p_j^{\min} = 0.1$ and $p_j^{\max} = 0.9$) and low-quality items

($p_j^{\min} = 0.2$ and $p_j^{\max} = 0.8$). Thus, four simulation conditions were evaluated. Data generation and VB estimation settings were the same as in the first simulation.

MCMC was written in JAGS language and measurement model parts were LCDM following Zhan et al. (2019). The priors of the LCDM were also the same as those reported by Zhan et al. (2019, Table 6), including a Normal distribution with mean and variance equal to -1.096 and 4, respectively, expressed as $N(-1.096, 4)$, for intercepts, a truncated Normal distribution $N^+(0, 4)$ for positive main effects, and $N(0, 4)$ for interaction effects. The other priors were the same as in the previous simulation. The number of MCMC chains were three, the iteration number was 10,000, burn-in period was 5,000, and thinning interval was 10. We assumed Rubin's $\hat{R}$ (Gelman & Rubin, 1992) for MCMC convergence criterion, wherein $\hat{R} \leq 1.1$. The estimation was conducted with R language on the 64 bit version of Windows 10 Pro OS with 3.60 GHz Intel Xeon CPU and 32 GB memory. The simulation was repeated 100 times. Simulation code is available at the previously shown OSF web page.

6.2. Results

First, almost all $\hat{R}$ s were less than 1.1, so MCMC iterations converged. Only two parameters in the 1,000 sample size and low-quality items condition did not converge; however, these did not significantly affect the results. Table 1 indicates the average estimation time (s) of the MCMC and VB methods. The average estimation times of MCMC were significantly affected by the sample size. The MCMC of smaller sample sizes was approximately 400 s, but larger sample sizes took more than 8,000 s. The VB method was implemented by both sample size and item-quality but its average estimation times were less than 5 s. The VB method was at least 340 times and sometimes several thousand times faster than the MCMC method in this simulation.

Table 2 shows the average bias result of each model parameter set with the MCMC and VB methods. The initial mixing parameter π and transition probability τ were correctly estimated by the two methods. The MCMC indicated slightly larger biases for correct item response probability parameter θ in small sample size conditions than the VB method.

Particularly, the 100 samples and low-quality items condition indicated the largest bias of the MCMC and VB method. The biases were sufficiently small in 1,000 sample size settings.

Table 3 shows the results of RMSE of the MCMC and VB methods. Similar to the bias result, the 100 samples and low-quality items condition showed largest discrepancy between the MCMC and VB methods. The VB exhibited larger RMSEs for all parameter sets than the MCMC, indicating that the parameter estimates of the VB method in the condition were unstable in comparison with those of the MCMC. However, the other conditions indicated almost same RMSEs between the two methods.

Table 4 presents posterior SDs of the MCMC and VB methods. The posterior SDs of correct item response probability were similar between the two methods under the three conditions, except for the 100 sample and low-items condition. However, the SD of the VB method of initial mixing and transitions probability parameters were generally underestimated. The underestimation phenomenon was relatively mitigated in the case of the high-quality items.

Table 5 lists the 95% CI coverage results of the MCMC and VB methods. The 95% CIs of the MCMC method approached 0.95, which is close to the theoretically expected value. The VB method, however, produced generally worse results. The 95% CI of the VB method for correct item response probability parameter in the high-quality items conditions was 0.932 or 0.921, and these values were better than the other parameters. The 95% CI coverages of the VB method for initial mixing and transition probability parameter in the high-quality item conditions ranged from 0.898 to 0.913, which deviates from 0.95. The other values of the 95% of the VB method were less than 0.90, indicating significant underestimation. This was because of shorter estimation of 95% CI of the VB method, and this underestimation phenomenon must be recognized.

Finally, Figure 6 depicts CCR results. Both MCMC and VB methods indicated almost same CCR_k s and CCR_{All} for all time points and in all conditions. Therefore, the MCMC and VB methods provided identical results for attribute mastery recovery.

In summary, the VB and MCMC methods resulted in almost unbiased parameter recovery and similar RMSEs. The posterior SDs of the VB method were generally smaller than those of the MCMC method. This difference is also clearly shown in the coverage of 95% CI results. More precisely, the coverages of 95% CI with the VB methods were generally worse than those of the MCMC method, whose coverages approached the theoretically expected values. These results indicate that the VB has some benefits and disadvantages in terms of parameter estimation.

7. Application to Empirical Data

7.1. *Analysis of a spatial reasoning test*

We analyze a spatial reasoning test analyzed by Chen et al. (2017) and Wang et al. (2018) to demonstrate the use of the algorithm and to compare the performance of the VB algorithm to MCMC-based estimation. The spatial reasoning test consists of 50 items that are divided into five blocks with each block containing 10 items. Each block was followed by a learning intervention that provided feedback on the performance of the previous block; thus, each block in the spatial reasoning test corresponds to a single measurement occasion (time point) for a total of five time points.

To ensure a balanced block design, each of the 351 participants were randomly assigned to one of the five test versions, wherein each version varied with respect to block arrangement (see Table 1 of Wang et al. (2018)). The assessment measured the following four attributes: (1) 90° x -axis rotation, (2) 90° y -axis rotation, (3) 180° x -axis rotation, and (4) 180° y -axis rotation. The Q-matrix for the spatial reasoning assessment is provided in Supplemental Material G. A detailed description of the experimental procedure, an example of experimental material, and interventions can be found in Wang et al. (2018). The data and MCMC estimation code are available in the `hmc dm` R package (Zhang et al., 2020).

The current version of the `hmc dm` package (Version 1.1.1) does not support estimation of general DCMs; therefore, for comparability between the MCMC-based estimation routine

implemented in the package and the VB estimation algorithm, the G-matrix was modified to reflect the DINA model. In particular, we set $g_{j1l} = 1 - \prod_k \alpha_{lk}^{q_{jk}}$ and $g_{j2l} = \prod_k \alpha_{lk}^{q_{jk}}$. Under this setting, the guessing and slip parameters correspond to the θ parameters with $\theta_{j1} = guess_j$ and $\theta_{j2} = 1 - slip_j$, respectively. Furthermore, we modified the VB algorithm to accommodate the non-decreasing attribute constraint so that the assumption that mastered attributes are not forgotten is accounted for (Chen et al., 2017). Therefore, the number of mastered attributes under the non-decreasing attribute constraint setting either stayed the same or increased across time points.

The maximum number of iterations in the VB algorithm was set to 500 and the convergence criteria was set to 10^{-4} . The initial values and prior settings of the VB algorithm were same as those of the simulation study described in the previous sections. For the MCMC-based estimation routine, a total of four chains were specified, each with 40,000 MCMC samples, the first 20,000 of which were discarded as burn-in. All other settings utilized the default specifications of the `hmcDM` package. The convergence criteria was assessed via Gelman-Rubin's $\hat{R}$, wherein a value of $\hat{R} \leq 1.1$ was used as the evidence of convergence in the parameters. Convergence checks were carried out using the `CODA` package (Plummer et al., 2006). Data analysis was conducted on a laptop with 16 GB memory and Intel Core i7 1.90 GHz CPU. Code from the data analysis can be found in the previously shown OSF page.

7.2. Results

The VB algorithm converged within the specified number of iterations. The $\hat{R}$ values of all parameters were less than 1.1, providing evidence of proper convergence. The estimation times of the VB and MCMC algorithms were 42.286 s and 1454.312 s, respectively. Therefore, parameter estimation with the VB algorithm was 34.393 times faster than the estimation via MCMC. We note that the VB algorithm was written in the base R language, whereas the MCMC algorithm was written in C++, a computationally faster language. Despite this disadvantage, the VB algorithm was found to be computationally more efficient than the MCMC algorithm.

We then compared the parameter estimates obtained from the two estimation procedures. The left panels of Figure 7 display the $guess_j$ and $1 - slip_j$ parameter estimates obtained from the VB and MCMC procedures. As can be seen from these two panels, the correlations between parameter estimates from the two estimation procedures were greater than 0.99, indicating strong estimation consistency. The posterior SDs of the $guess_j$ and $1 - slip_j$ estimates obtained from the two methods are shown in the right panels of Figure 7. In general, the SDs obtained from the VB algorithm were slightly smaller than those obtained from the MCMC algorithm. These results are consistent with previous findings by Yamaguchi and Okada (2020b).

The comparative results of the mixing parameter π are shown in Table 6. The greatest difference in estimation (>0.04) was observed for the [0100] pattern. However, the differences in the values of the other patterns were less than 0.025. The correlation between estimates was 0.993. In addition, the SDs of the VB algorithm were found to be smaller than those obtained from the MCMC algorithm. Taken together, these results provide strong support for the quality of parameter estimation produced by the VB algorithm.

Transition probability comparisons between the two estimation procedures are provided in Figure 8. Once again, the parameter estimates produced by the two algorithms were highly consistent with a correlation of 0.969 (left panel). In addition, like the estimation results of the mixing and item parameters, the posterior SDs under VB estimation were generally smaller than those produced under MCMC (right panel). The difference is slightly large, so this result provided a caution when interpreting the posterior SDs of the transition parameters with the VB method.

Agreement with respect to the individual attributes and mastery profiles between the two estimation methods at each time point are given in Table 7. The mastery agreement results obtained for the 90° x -axis, 90° y -axis rotation, and 180° y -axis attributes were all greater than 0.95 at each time point. Values for the 180° x -axis attribute ranged between 0.926 (third time point) and 0.951 (first time point). Agreement in attribute profiles at each time point is shown on

the rightmost column of Table 7; the values range from 0.877 (third time point) to 0.914 (fifth time point). Thus, taken together, these results indicate that both the VB and MCMC algorithms provide similar estimates; however, as noted earlier, the VB algorithm was found to be computationally more efficient than the MCMC algorithm.

8. Conclusion and Discussion

This study proposes an effective variational Bayesian estimation method for hidden Markov longitudinal DCMs under a general DCM measurement model. Based on the comparison of the results obtained under the VB and MCMC algorithms from the simulation study and real data analysis, we conclude that the proposed method constitutes a valid estimation procedure, particularly in terms of point estimation.

Owing to its formulation from a general family of DCMs, the algorithm developed in this study allows for the specification and estimation of specific, or restricted, DCMs (e.g., DINA, DINO, etc.). The utility of this formulation was demonstrated in the applied analysis as the hidden Markov DINA model was readily obtained from the general model by appropriately modifying the G-matrix. Other models, including the LCDM, GDM, G-DINA, and their restricted forms can also be modeled with minimal modifications to the code.

Because our study demonstrates the utility of the proposed method across a set of limited conditions, future studies can focus on assessing parameter recoveries in a wider range of conditions using our simulation and estimation code, which have been made publicly available for researchers and practitioners who wish to extend our simulation study. Another limitation of this study is that our method assumes that the Q-matrix and number of attributes are known across all time points. Q-matrix learning from item response data is one important research topic in DCMs, and various methods have been proposed in recent years (e.g., Gu et al., 2018; C. W. Liu et al., 2020; J. Liu et al., 2012, 2013; Yinghan Chen et al., 2018, 2021; Yinyin Chen et al., 2020; Yunxiao Chen et al., 2015; Culpepper, 2019b, 2019a; Xu & Shang, 2018). It may be useful to use such Q-matrix estimation methods in the longitudinal DCM context. However, we believe

that this assumption is reasonable because while assessing the growth of students, the target attribute set was already assumed, and the Q-matrix was clarified based on previous test items assessments. Without such prior research of the relationships between the items and attributes, we cannot evaluate the growth or change of attributes mastery over repeated assessments. Therefore, the assumption of a known Q-matrix is natural and reasonable in operational settings.

In addition, the VB method does not impose monotonicity constraint on item parameters. This is because ordering constraints among parameters is difficult to take into account; we believe this is a fruitful area of future research. In our simulations, specifying the appropriate priors to satisfy the monotonicity condition enabled good parameter recovery. Therefore, manipulation of prior distributions may be an alternative strategy for ensuring strict monotonicity constraints.

In relation to monotonicity constraints, identifiability is an important topic on the DCM studies. Liu et al. (2023) established strict and generic identifiability of HM-DCMs, including estimation of Q-matrix. From theorem 3 presented by Liu et al. (2023), the model parameters are strictly identifiable if $\pi_l > 0, l = 1, \dots, 2^K$, the rank of transition matrix $\mathbf{T}$ is 2^K , and the sub-Q-matrix corresponding to the first K items is the identity matrix for max time points $T > 3$. For $T = 2$, in addition to the conditions for $T > 3$, for arbitrary pair of attribute mastery patterns, there exists at least one item with $\mathbf{q}$ -vector that generates different item response vector probabilities. Omitting the rank condition of transition matrix $\mathbf{T}$ from strict identifiability condition, the above-mentioned conditions ensure generic identifiability of model parameters (Liu et al., 2023, Theorem 4). Furthermore, monotonicity constraints prevent label switching of attribute mastery and non-mastery. Our formulation did not require monotonicity constraints, and it is possible to attribute mastery/non-mastery label permutation. However, priors were specified to distinguish correct item response parameters, and the simulation results showed appropriate parameter recovery so that the prior setting can work to prevent identifiability issues. In future, theoretical property of the priors in the HM-DCMs should be clarified.

1 Selection of the estimation method must be considered for the HM-DCMs. From our
2 two simulations and a real data analysis example, we conclude that the VB method works faster
3 than the MCMC method, although point estimates of the VB method were as good as those of
4 the MCMC method. The coverage result of the VB was worse than that of the MCMC method
5 because of underestimation of posterior SD. Therefore, both the methods have pros and cons.
6 The VB method can work when the MCMC method takes extraordinarily long estimation time.
7 The MCMC method may provide more accurate estimates than the VB method with strong
8 machine power and enough time for estimation. The estimation method should be selected
9 accordingly.

10 An estimation method has specific strengths and weaknesses, and its selection should
11 be based on the purpose of the data analysis. For example, usual MCMC techniques need more
12 computational power than the VB inference algorithm; however, they have the unique property
13 of guarantying asymptotic convergence to the proper posterior distribution under mild
14 conditions. If a data analyst has access to powerful computational resources and time and
15 requires precise estimation, it would be wise to perform the estimation with MCMC-based
16 strategies. However, if an analyst is restricted to limited computational resources and time, the
17 VB method may be preferable over MCMC estimation. The real data analysis demonstrated that
18 point estimates between the two estimation procedures could be very similar in some cases,
19 lending support for the VB method when computational efficiency is desired.

20 Regarding the comparison between the VB- and ML-based methods, because of the
21 conceptual differences between Bayesian-based and ML-based estimation, comparison between
22 the two is impractical. Theoretically, the computational burden of the VB algorithm is the same
23 as that of the EM algorithm. Yamaguchi & Okada (2020b) compared the computational speeds
24 of the VB and EM algorithms in the DINA model and found no difference. Additionally, the VB
25 method can readily compute posterior SDs; however, calculating the standard errors of the ML
26 estimates requires additional calculation. This is because the HMMs are singular models, which

1 implies that the Fisher information matrix can become rank deficient, and its inverse cannot be
2 computed in this situation.

3 Our VB method was derived from the minimum independent assumption, based on
4 independence of latent attribute indicators from model parameters. Moreover, we did not assume
5 distribution form of variational posteriors. Even under these assumptions, our minimum
6 assumption provides well-known distribution form of VB posteriors. Therefore, the derived VB
7 posteriors are the most natural approximation. In the other VB methods such as predetermining
8 the distribution family of posteriors, we can assume another specific distribution form such as
9 normal distributions, but it may be far from true posteriors. If independence between latent
10 indicator variables and model parameters is ignored, more accurate variational posteriors could
11 be obtained. However, in such a situation, the distribution form of the VB posterior cannot be
12 expressed in a well-known form and VB algorithm becomes complex. In addition, the posteriors
13 variance may be difficult to evaluate.

14 The variational estimation methods have several drawbacks as mentioned earlier. The
15 importance weighted variational inference method (Burda et al., 2015), which was applied for
16 the multidimensional item response theory models (Liu, et al., 2022), can provide tighter bound
17 of log marginal likelihood than mean field approximation. Optimizing the tighter ELBO over
18 model parameters can overcome the disadvantage of mean field approximation, such as smaller
19 estimation of posterior variance, suggesting that importance weighted variational inference is a
20 promising method. However, it has several shortcomings. First, its parameter estimation
21 algorithm may become more complex than the usual mean field approximation because
22 optimization of importance weighted ELBO is based on the gradient method. Furthermore, the
23 parameter update step is based on the graduate method with numerical differentiation, and it may
24 become unstable. In addition, importance weighted variational inference needs a random
25 sampling step to evaluate tighter ELBO, so its computational cost may become large. Detailed
26 theoretical discussion is provided by Liu et al. (2022, pp. 5-6). A comparison of our variational
27 method and importance weighted variational inference should be conducted in future.

In addition, simulation 2 indicated underestimation of posterior SD and the coverage of 95% CI of the VB method. Bootstrap procedure in conjunction with the VB method (Dang and Maestrini, 2022) can be used to address this problem. This procedure randomly selects bootstrap sample and applies the VB method to obtain multiple VB estimates. The bootstrap credible interval is calculated from the VB estimates with better convergence probability than the 95% CI with variational posterior. The VB estimation incurs lower computational cost than the MCMC, so the bootstrap method is a realistic option to fix the underestimation problem of the VB method.

Finally, analyzing the actual data revealed that for a small sample size, it is difficult to capture the transition of student performance, and some transition patterns can become sparse or empty; that is, in addition to their variability, the estimates close to the boundary of parameter space cannot be estimated appropriately. Therefore, further analyses are required to warrant the assessment of the differences between the VB and MCMC methods in the context of longitudinal DCMs.

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- 17

Table 1.

Estimation time (s) of the Markov chain Monte Carlo (MCMC) and variational Bayes (VB) methods in simulation 2

Sample size	Item quality	MCMC		VB		MCMC/VB
		Mean	(SD)	Mean	(SD)	
100	High	435.265	(14.733)	0.684	(0.159)	636.534
100	Low	435.779	(15.073)	1.274	(0.313)	342.108
1,000	High	8281.627	(273.653)	2.456	(0.175)	3371.727
1,000	Low	8220.844	(259.587)	4.597	(0.743)	1788.270

Table 2.

Bias of the Markov chain Monte Carlo (MCMC) and variational Bayes (VB) methods in simulation 2

Sample size	Item quality	Correct item response probability: θ		Initial mixing parameter: π		Transition probability: τ	
		MCMC	VB	MCMC	VB	MCMC	VB
100	High	-.009	.002	.000	.000	.000	.000
100	Low	-.020	.010	.000	.000	.000	.000
1,000	High	.000	.001	.000	.000	.000	.000
1,000	Low	-.001	.001	.000	.000	.000	.000

1

Table 3.

Root mean square error (RMSE) of the Markov chain Monte Carlo (MCMC) and variational Bayes (VB) methods in simulation 2

Sample size	Item quality	Correct item response probability: θ		Initial mixing parameter: π		Transition probability: τ	
		MCMC	VB	MCMC	VB	MCMC	VB
100	High	.069	.071	.028	.029	.042	.045
100	Low	.084	.100	.042	.054	.048	.067
1,000	High	.026	.026	.010	.010	.018	.018
1,000	Low	.035	.036	.015	.015	.026	.027

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3

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Table 4.

Posterior SD of the Markov chain Monte Carlo (MCMC) and variational Bayes (VB) methods in simulation 2

Sample size	Item quality	Correct item response probability: θ		Initial mixing parameter: π		Transition probability: τ	
		MCMC	VB	MCMC	VB	MCMC	VB
100	High	.077	.071	.034	.028	.055	.046
100	Low	.096	.078	.056	.028	.075	.046
1,000	High	.027	.024	.010	.009	.019	.015
1,000	Low	.037	.028	.015	.009	.029	.015

2

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Table 5.

95% coverage ratio of the Markov chain Monte Carlo (MCMC) and variational Bayes (VB) methods in simulation 2

Sample size	Item quality	Correct item response probability: θ		Initial mixing parameter: π		Transition probability: τ	
		MCMC	VB	MCMC	VB	MCMC	VB
100	High	.946	.932	.951	.906	.960	.911
100	Low	.946	.851	.935	.659	.961	.778
1,000	High	.951	.921	.948	.913	.959	.898
1,000	Low	.952	.867	.950	.720	.969	.723

Table 6.

Estimation results of mixing the parameters between variational inference and the Gibbs sampling method in the spatial reasoning test

Attribute mastery pattern	VB			MCMC		
	Mean	[	SD]	Mean	[	SD]
0000	.340	[	.025]	.315	[	.044]
1000	.006	[	.004]	.027	[	.024]
0100	.004	[	.003]	.046	[	.043]
1100	.064	[	.013]	.056	[	.023]
0010	.005	[	.004]	.013	[	.012]
1010	.044	[	.011]	.023	[	.017]
0110	.026	[	.008]	.017	[	.012]
1110	.039	[	.010]	.034	[	.021]
0001	.005	[	.004]	.016	[	.014]
1001	.005	[	.004]	.009	[	.008]
0101	.024	[	.008]	.021	[	.016]
1101	.009	[	.005]	.015	[	.013]
0011	.009	[	.005]	.013	[	.012]
1011	.024	[	.008]	.014	[	.012]
0111	.006	[	.004]	.008	[	.008]
1111	.388	[	.025]	.372	[	.034]

1

Table 7.

Estimation results of attribute mastery agreement between variational Bayes inference and the Gibbs sampling method in the spatial reasoning test

Time point	Attribute				All
	90° x-axis	90° y-axis	180° x-axis	180° y-axis	
1	.951	.974	.951	.977	.891
2	.974	.966	.943	.986	.903
3	.957	.963	.926	.983	.877
4	.957	.969	.931	.986	.886
5	.960	.983	.943	.991	.914

1

2 Figure 1.

3 *Histograms of biases of correct item response probabilities at the fifth time point in simulation study*

1

2 Figure 2.

Histograms of RMSEs of correct item response probabilities at the fifth time point in simulation study

1 Figure 3.

2 *Histograms of biases of transition parameters in simulation study*

1

Figure 4.

Histograms of RMSEs of transition parameters in simulation study

1

2 Figure 5.

Plots of the correct classification ratio (CCR) obtained for each attribute and whole pattern for each condition in simulation study

Figure 6.

1 *Plots of the correct classification ratio (CCR) obtained for each attribute and whole pattern for each condition with the Markov chain Monte Carlo and variational Bayesian methods in simulation study 2*

Figure 7.

Scatter plot of posterior mean and SD of guess and slip parameters with variational Bayesian inference (VB) and Markov chain Monte Carlo (MCMC) in in the spatial reasoning test

Figure 8.
Scatter plot of posterior mean and SD of transition probability with variational Bayesian inference (VB) and Markov chain Monte Carlo (MCMC) in in the spatial reasoning test

Supplemental Material A. 1. Complete Data Likelihood of the Hidden Markov Diagnosis

The item response function of the general DCM at time t is given as

$$P(X_{ijt} = x_{ijt} | \mathbf{z}_{it}, \boldsymbol{\theta}_{jt}) = \prod_l \prod_h \left\{ [\theta_{jht}^{x_{ijt}} (1 - \theta_{jht})^{1-x_{ijt}}]^{g_{jhtl}} \right\}^{z_{itl}}. \quad (1)$$

Under usual assumptions of conditional independence given for an attribute mastery pattern (the local independence assumption), the joint probability of $\mathbf{X}_{it} = [X_{i1t}, \dots, X_{ijt}]^\top$ is computed as follows:

$$\begin{aligned} P(\mathbf{X}_{it} = \mathbf{x}_{it} | \mathbf{z}_{it}, \boldsymbol{\theta}_t) &= \prod_j P(X_{ijt} = x_{ijt} | \mathbf{z}_{it}, \boldsymbol{\theta}_{jt}), \\ &= \prod_j \prod_l \prod_h \left\{ [\theta_{jht}^{x_{ijt}} (1 - \theta_{jht})^{1-x_{ijt}}]^{g_{jhtl}} \right\}^{z_{itl}}, \end{aligned} \quad (2)$$

where $\boldsymbol{\theta}_t$ is a set of item response probability parameter vectors $\{\boldsymbol{\theta}_{1t}, \dots, \boldsymbol{\theta}_{jt}\}$, and $\mathbf{x}_{it}$ is a realization of $\mathbf{X}_{it}$. Independence among time points given latent class and exchangeability of students leads to a complete data likelihood:

$$\begin{aligned} P(\mathbf{X} | \mathbf{Z}, \boldsymbol{\theta}) &= \prod_i \prod_t P(\mathbf{X}_{it} = \mathbf{x}_{it} | \mathbf{z}_{it}, \boldsymbol{\theta}_t) \\ &= \prod_i \prod_t \prod_j \prod_l \prod_h \left\{ [\theta_{jht}^{x_{ijt}} (1 - \theta_{jht})^{1-x_{ijt}}]^{g_{jhtl}} \right\}^{z_{itl}}, \end{aligned} \quad (3)$$

where $\mathbf{X}$, $\mathbf{Z}$, and $\boldsymbol{\theta}$ are sets of all $\mathbf{x}_{it}$, $\mathbf{z}_{it}$, $\boldsymbol{\theta}_t$, respectively. The relationship between the different time points of $\mathbf{z}_{it}$ is modeled in the next section.

Prior distributions of the model parameters are introduced to derive the VB algorithm.

The prior requirement for the evaluation of the correct item response probability parameter θ_{jht} is a beta distribution composed of non-negative hyperparameters a_{jht}^0 and b_{jht}^0 ,

$$P(\theta_{jht} | a_{jht}^0, b_{jht}^0) = \frac{1}{\text{Beta}(a_{jht}^0, b_{jht}^0)} \theta_{jht}^{a_{jht}^0-1} (1 - \theta_{jht})^{b_{jht}^0-1}, \quad (4)$$

where $\text{Beta}(a_{jht}^0, b_{jht}^0)$ is the beta function defined as $\text{Beta}(a, b) = \int_0^1 w^{(a-1)} (1 - w)^{(b-1)} dw$.

Under this prior specification, the joint probability of $\boldsymbol{\theta}$ under assumption of independence is given as follows:

$$\begin{aligned}
 P(\boldsymbol{\Theta}|\mathbf{A}^0, \mathbf{B}^0) &= \prod_j \prod_h \prod_t P(\theta_{jht} | a_{jht}^0, b_{jht}^0), \\
 &= \prod_j \prod_h \prod_t \frac{1}{\text{Beta}(a_{jht}^0, b_{jht}^0)} \theta_{jht}^{a_{jht}^0-1} (1 - \theta_{jht})^{b_{jht}^0-1}.
 \end{aligned} \tag{5}$$

1 The distribution of initial attribute mastery pattern $\mathbf{z}_{it=1}$, is given as

$$P(\mathbf{z}_{it=1} | \boldsymbol{\pi}) = \prod_l \pi_l^{z_{it=1}l}. \tag{6}$$

2 Under exchangeability of students, the joint probability of the initial attribute mastery indicator
3 matrix $\mathbf{Z}_1 = [\mathbf{z}_{11}, \dots, \mathbf{z}_{I1}]^\top$ is

$$P(\mathbf{Z}_1 | \boldsymbol{\pi}) = \prod_i \prod_l \pi_l^{z_{i1}l}. \tag{7}$$

4 Following standard practice, the prior for the estimation of $\boldsymbol{\pi}$ is obtaining a Dirichlet
5 distribution with parameter $\boldsymbol{\delta}^0 = [\delta_1^0, \dots, \delta_L^0]^\top$ satisfying $\delta_l^0 \geq 0, \forall l$:

$$P(\boldsymbol{\pi} | \boldsymbol{\delta}^0) = \frac{1}{\text{Beta}(\boldsymbol{\delta}^0)} \prod_l \pi_l^{\delta_l^0-1}, \tag{8}$$

6 where $\text{Beta}(\boldsymbol{\delta}^0)$ is a real-valued beta function defined as $\frac{\prod_l \Gamma(\delta_l^0)}{\Gamma(\sum_l \delta_l^0)}$, and $\Gamma(\cdot)$ is a gamma
7 function.

8 The transition model is defined as

$$P(\mathbf{z}_{it} | \mathbf{z}_{i(t-1)}, \mathbf{T}) = \prod_l \prod_{l'} \tau_{ll'}^{z_{i(t-1)}l z_{it}l'}. \tag{9}$$

9 The students are exchangeable and the conditional distribution of the attribute mastery pattern
10 index matrix $\mathbf{Z}_t$ is

$$P(\mathbf{Z}_t | \mathbf{Z}_{t-1}, \mathbf{T}) = \prod_i \prod_l \prod_{l'} \tau_{ll'}^{z_{i(t-1)}l z_{it}l'}. \tag{10}$$

11 For time $t = 2$ to T , the conditional distribution of the set of attribute mastery pattern index
12 matrices $\{\mathbf{Z}_t\}_{t=2}^T$ is

$$P(\{\mathbf{Z}_t\}_{t=2}^T | \{\mathbf{Z}_t\}_{t=1}^{T-1}, \mathbf{T}) = \prod_{t=2} \prod_i \prod_l \prod_{l'} \tau_{ll'}^{z_{i(t-1)}l z_{it}l'}. \tag{11}$$

13 The prior distribution for $\boldsymbol{\tau}_l = [\tau_{l1}, \dots, \tau_{lL}]^\top$ is a Dirichlet distribution with non-negative hyper
14 parameter $\boldsymbol{\omega}_l^0 = [\omega_1^0, \dots, \omega_L^0]^\top$:

$$P(\boldsymbol{\tau}_l | \boldsymbol{\omega}_l^0) = \frac{1}{\text{Beta}(\boldsymbol{\omega}_l^0)} \prod_{l'} \tau_{ll'}^{\omega_{ll'}^0-1}. \tag{12}$$

15 Subsequently, the joint distribution of $\mathbf{T}$ is given as

$$P(\mathbf{T}|\mathbf{\Omega}^0) = \prod_l \left(\frac{1}{\text{Beta}(\mathbf{\omega}_l^0)} \prod_{l'} \tau_{ll'}^{\omega_{ll'}^0 - 1} \right), \quad (13)$$

1 where $\mathbf{\Omega}^0$ is a set of hyper parameters $\{\mathbf{\omega}_1^0, \dots, \mathbf{\omega}_L^0\}$.

2 Finally, by combining Equations (3), (5), (7), (8), (11), and (13) and applying Bayes'
 3 theorem, the joint posterior of $\mathbf{Z}, \mathbf{\Theta}$, and $\mathbf{\pi}$ in the HM-DCM is decomposed as follows:

$$\begin{aligned} & P(\mathbf{Z}, \mathbf{\Theta}, \mathbf{\pi}, \mathbf{T} | \mathbf{X}, \mathbf{A}^0, \mathbf{B}^0, \mathbf{\delta}^0, \mathbf{\Omega}^0) \\ &= P(\mathbf{X} | \mathbf{Z}, \mathbf{\Theta}) P(\mathbf{\Theta} | \mathbf{A}^0, \mathbf{B}^0) P(\mathbf{Z} | \mathbf{\pi}, \mathbf{T}) P(\mathbf{\pi} | \mathbf{\delta}^0) P(\mathbf{T} | \mathbf{\Omega}^0), \\ &= P(\mathbf{\Theta} | \mathbf{A}^0, \mathbf{B}^0) P(\mathbf{T} | \mathbf{\Omega}^0) \{P(\mathbf{X}_{t=1} | \mathbf{Z}_{t=1}, \mathbf{\Theta}_{t=1}) P(\mathbf{Z}_{t=1} | \mathbf{\pi}) P(\mathbf{\pi} | \mathbf{\delta}^0)\} \\ &\quad \times \prod_{t=2}^T P(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{\Theta}_t) P(\mathbf{Z}_t | \mathbf{Z}_{t-1}, \mathbf{T}). \end{aligned} \quad (14)$$

4 From Equation (14), the log joint posterior is calculated as follows:

$$\begin{aligned} & \log P(\mathbf{Z}, \mathbf{\Theta}, \mathbf{\pi}, \mathbf{T} | \mathbf{X}, \mathbf{A}^0, \mathbf{B}^0, \mathbf{\delta}^0, \mathbf{\Omega}^0) \\ &= \sum_t \sum_j \sum_h \{ (a_{jht}^0 - 1) \log \theta_{jht} + (b_{jht}^0 - 1) \log (1 - \theta_{jht}) \} \\ &+ \sum_l \sum_{l'} (\omega_{ll'}^0 - 1) \log \tau_{ll'} + \sum_l (\delta_l^0 - 1) \log \pi_l \\ &+ \sum_i \sum_l z_{i1l} \log \pi_l + \sum_{t=2} \sum_i \sum_l \sum_{l'} z_{i(t-1)l} z_{itl'} \log \tau_{ll'} \\ &+ \sum_t \sum_i \sum_j \sum_h \sum_l \{ z_{itl} g_{jhl} [x_{ijt} \log \theta_{jht} + (1 \\ &- x_{ijt}) \log (1 - \theta_{jht})] \} + \text{constant}. \end{aligned} \quad (15)$$

Supplemental Material B. Derivation of Variational Posteriors

A detailed conceptual overview and a comprehensive theoretical treatment of the VB inference methodology is available in the reports of Yamaguchi and Okada (2020a, 2020b), Bishop (Bishop, 2006), and Nakajima et al. (2019). VB inference is formulated as an optimization problem, wherein the goal is to find the most appropriate distribution that is closest to the true posterior distribution. Let $q(\boldsymbol{\vartheta})$ be an unknown parametric distribution, called the variational posterior, and let $P(\boldsymbol{\vartheta}|\mathbf{Y})$ denote the true posterior of $\boldsymbol{\vartheta}$ given a real-valued random vector $\mathbf{Y}$. The Kullback-Leibler (KL) divergence is defined as

$$\text{KL}(q(\boldsymbol{\vartheta})||P(\boldsymbol{\vartheta}|\mathbf{Y})) = \int q(\boldsymbol{\vartheta}) \log \frac{q(\boldsymbol{\vartheta})}{P(\boldsymbol{\vartheta}|\mathbf{Y})} d\boldsymbol{\vartheta}. \quad (16)$$

Given this definition, an optimization problem is formulated as

$$\hat{q} = \text{argmin} \text{KL}(q(\boldsymbol{\vartheta})||P(\boldsymbol{\vartheta}|\mathbf{Y})). \quad (17)$$

Because it is difficult to directly obtain the value of the posterior $P(\boldsymbol{\vartheta}|\mathbf{Y})$, we work on the joint distribution of data and parameters, $P(\mathbf{Y}, \boldsymbol{\vartheta}) = P(\mathbf{Y}|\boldsymbol{\vartheta})P(\boldsymbol{\vartheta})$, where $P(\mathbf{Y}|\boldsymbol{\vartheta})$ is the model likelihood function and $P(\boldsymbol{\vartheta})$ is a prespecified prior distribution.

We have to restrict the space of the approximate densities $q(\boldsymbol{\vartheta})$ by employing the mean-field variational family of approximate densities. This family of approximate densities assumes that the variational posteriors can be factorized into N independent variational densities; that is,

$$q(\boldsymbol{\vartheta}) = \prod_{n=1}^N q_n(\vartheta_n), \quad (18)$$

Where the distribution $q(\boldsymbol{\vartheta})$ is decomposed into N independent distributions $q_1(\vartheta_1), \dots, q_N(\vartheta_N)$, which adopt exclusive subsets $\boldsymbol{\vartheta} = \{\vartheta_1, \dots, \vartheta_n, \dots, \vartheta_N\}$, $n = 1, \dots, N$. Then, the VB problem can be formulated as

$$\hat{q} = \text{argmin} \text{KL}(q(\boldsymbol{\vartheta})||P(\mathbf{Y}, \boldsymbol{\vartheta})) \text{ subject to } q(\boldsymbol{\vartheta}) = \prod_{n=1}^N q_n(\vartheta_n). \quad (19)$$

Under this independence constraint, the optimal $q_n(\vartheta_n)$ that minimizes $\text{KL}(q(\boldsymbol{\vartheta})||P(\mathbf{Y}, \boldsymbol{\vartheta}))$ can be shown to have the following form:

$$q_n(\vartheta_n) \propto \exp(\mathbb{E}_{n' \neq n}[\log P(\mathbf{Y}, \boldsymbol{\vartheta})]), \quad (20)$$

where $\mathbb{E}_{n' \neq n}[\mathbf{x}] = \int \mathbf{x} \prod_{n' \neq n}^N q_{n'}(\vartheta_{n'}) d\vartheta_{n'}$. As the optimal distributions are inter-dependent, an iterative algorithm is needed to obtain the approximated distributions. Equation (20) is employed to derive the optimal variational posterior that satisfies the conditional conjugacy, that is, the conditional version of usual conjugacy in Bayesian inference (formal explanation is found in Nakajima et al. 2019, p.42) in the current HM-DCM setting.

The variational posteriors of the HM-DCM under the following factorization assumption of the posterior are derived:

$$P(\mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T} | \mathbf{X}) \approx q(\mathbf{Z})q(\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}) = \left(\prod_i q(\mathbf{z}_i) \right) q(\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}). \quad (21)$$

The subscript of variational posterior distribution is dropped for simplicity, and we distinguish the variational posteriors in their arguments. The HM-DCM assumes independence of parameters, and then, the variational posterior $q(\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T})$ is decomposed into a set of fine-grained independent distributions:

$$q(\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}) = \left(\prod_t \prod_j \prod_h q(\theta_{jht}) \right) q(\boldsymbol{\pi}) \left(\prod_l q(\boldsymbol{\tau}_l) \right). \quad (22)$$

The derivation procedure is the same as that detailed in the Appendix A of Yamaguchi and Okada (2020a). First, variational distribution of initial mixing parameter $\boldsymbol{\pi}$ is defined as

$$\begin{aligned}
 q(\boldsymbol{\pi}) &\propto \exp(\mathbb{E}_{q(\mathbf{Z})q(\boldsymbol{\Theta}, \mathbf{T})}[\log P(\mathbf{Z}_{t=1} | \boldsymbol{\pi}) + \log P(\boldsymbol{\pi} | \boldsymbol{\delta}^0)]), \\
 &\propto \exp\left(\sum_l \left\{ \sum_i \mathbb{E}_{q(\mathbf{z}_i)}[z_{it=1l}] + \delta_l^0 - 1 \right\} \log \pi_l\right).
 \end{aligned} \quad (23)$$

Therefore, $q(\boldsymbol{\pi})$ is a Dirichlet distribution:

$$q(\boldsymbol{\pi}) = \frac{1}{\text{Beta}(\boldsymbol{\delta}^*)} \prod_l \pi_l^{\delta_l^* - 1}, \quad (24)$$

where $\delta_l^* = \sum_i \mathbb{E}_{q(\mathbf{z}_i)}[z_{i1l}] + \delta_l^0$. A similar process is used derive $q(\boldsymbol{\pi})$, where the variational posterior $q(\theta_{jht})$ is

$$q(\theta_{jht}) \propto \exp(\mathbb{E}_{q(\mathbf{Z})q(\boldsymbol{\pi}, \mathbf{T})}[\log P(\mathbf{X} | \boldsymbol{\Theta}, \mathbf{Z}) + \log P(\boldsymbol{\Theta} | \mathbf{A}^0, \mathbf{B}^0)]), \quad (25)$$

$$\propto \exp \left(\left\{ \sum_i \sum_l \mathbb{E}_{q(\mathbf{z}_i)}[z_{itl}] g_{jhl} x_{ijt} + a_{jht}^0 - 1 \right\} \log \theta_{jht} + \left\{ \sum_i \sum_l \mathbb{E}_{q(\mathbf{z}_i)}[z_{itl}] g_{jhl} (1 - x_{ijt}) + b_{jht}^0 - 1 \right\} \log(1 - \theta_{jht}) \right).$$

1 Then, $q(\theta_{jht})$ is a Beta distribution:

$$q(\theta_{jht}) = \frac{1}{\text{Beta}(a_{jht}^*, b_{jht}^*)} \theta_{jht}^{a_{jht}^* - 1} (1 - \theta_{jht})^{b_{jht}^* - 1}, \quad (26)$$

2 and its parameters are

$$\begin{cases} a_{jht}^* = \sum_i \sum_l \mathbb{E}_{q(\mathbf{z}_i)}[z_{itl}] g_{jhl} x_{ijt} + a_{jht}^0, \\ b_{jht}^* = \sum_i \sum_l \mathbb{E}_{q(\mathbf{z}_i)}[z_{itl}] g_{jhl} (1 - x_{ijt}) + b_{jht}^0. \end{cases} \quad (27)$$

3 Finally, the variational posterior $q(\mathbf{T})$ can be decomposed as $q(\mathbf{T}) = \prod_l q(\boldsymbol{\tau}_l)$. Therefore,

4 focusing on the $q(\boldsymbol{\tau}_l)$:

$$\begin{aligned} q(\boldsymbol{\tau}_l) &\propto \exp \left(\mathbb{E}_{q(\mathbf{z})q(\boldsymbol{\pi}, \boldsymbol{\theta})} \left[\log P(\boldsymbol{\tau}_l | \boldsymbol{\omega}_l^0) + \sum_{t=2} \log P(\mathbf{Z}_t | \mathbf{Z}_{t-1}, \boldsymbol{\tau}_l) \right] \right), \\ &\propto \exp \left(\sum_{l'} \left\{ \sum_{t=2} \sum_i \mathbb{E}_{q(\mathbf{z}_i)}[z_{i(t-1)l} z_{itl'}] + \omega_{ll'}^0 - 1 \right\} \log \tau_{ll'} \right). \end{aligned} \quad (28)$$

5 This implies that $q(\boldsymbol{\tau}_l)$ is a Dirichlet distribution again:

$$q(\boldsymbol{\tau}_l) = \frac{1}{\text{Beta}(\boldsymbol{\omega}_l^*)} \prod_{l'} \tau_{ll'}^{\omega_{ll'}^* - 1}, \quad (29)$$

6 where the parameter is $\omega_{ll'}^* = \sum_{t=2} \sum_i \mathbb{E}_{q(\mathbf{z}_i)}[z_{i(t-1)l} z_{itl'}] + \omega_{ll'}^0$.

7 These variational posteriors are well-known distributions, and their expectation and
8 variance have a closed form. The mean and variance of the variational posteriors are derived
9 using their posterior parameters:

$$\begin{cases} \mathbb{E}_{q(\boldsymbol{\pi})}[\pi_l] = \frac{\delta_l^*}{\sum_l \delta_l^*}, \\ \mathbb{V}_{q(\boldsymbol{\pi})}[\pi_l] = \frac{\delta_l^* (\sum_l \delta_l^* - \delta_l^*)}{(\sum_l \delta_l^*)^2 (1 + \sum_l \delta_l^*)}, \end{cases} \quad (30)$$

$$\begin{cases} \mathbb{E}_{q(\tau_l)}[\tau_{ll'}] = \frac{\omega_{ll'}^*}{\sum_{l'} \omega_{ll'}^*}, \\ \mathbb{V}_{q(\tau_l)}[\tau_{ll'}] = \frac{\omega_{ll'}^* (\sum_{l'} \omega_{ll'}^* - \omega_{ll'}^*)}{(\sum_{l'} \omega_{ll'}^*)^2 (1 + \sum_{l'} \omega_{ll'}^*)}, \end{cases} \quad (31)$$

1 and

$$\begin{cases} \mathbb{E}_{q(\theta_{jht})}[\theta_{jht}] = \frac{a_{jht}^*}{a_{jht}^* + b_{jht}^*}, \\ \mathbb{V}_{q(\theta_{jht})}[\theta_{jht}] = \frac{a_{jht}^* b_{jht}^*}{(a_{jht}^* + b_{jht}^*)^2 (1 + a_{jht}^* + b_{jht}^*)}. \end{cases} \quad (32)$$

2

3

Supplemental Material C. Forward-backward Algorithm

Expectations of $q(\mathbf{z}_i)$, that are, $\mathbb{E}_{q(\mathbf{z}_i)}[\mathbf{z}_{it}]$ and $\mathbb{E}_{q(\mathbf{z}_i)}[\mathbf{z}_{it}\mathbf{z}_{i(t-1)}^\top]$ are calculated using:

$$\begin{cases} q(\mathbf{z}_{it}) = \sum_{\mathbf{z}_{i \setminus t}} q(\mathbf{z}_i), \\ q(\mathbf{z}_{it}, \mathbf{z}_{i(t-1)}) = \sum_{\mathbf{z}_{i \setminus \{t, t-1\}}} q(\mathbf{z}_i). \end{cases} \quad (33)$$

The computational cost of these expectations becomes large with increasing time points. We employ the well-known forward-backward algorithm, which is computationally efficient, to calculate these expectations. The forward-backward algorithm can be seen a special case of the sum-product algorithm, which is explained in Bishop (2006, Chapter 8). The sum-product algorithm uses dependency among variables to exchange the order of summation and product operation, and this can reduce the computational cost. Consequently, the algorithm needs recursive calculation. Particularly, the forward-backward algorithm uses two recursively calculated terms from time 1 or time T . Intuitively, these terms accumulate information from time 1 or T to time t to calculate expectation of variational posterior at time t . To derive forward-backward algorithm for the HM-DCMs, we need several new terms and decomposition of $q(\mathbf{z}_i)$.

Let us assume the following decomposition of $q(\mathbf{z}_i)$:

$$q(\mathbf{z}_i) \propto \tilde{\mathbf{P}}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)}) \tilde{\mathbf{P}}(\mathbf{z}_{it} | \mathbf{z}_{i(t-1)}) \tilde{\mathbf{P}}(\mathbf{x}_{it} | \mathbf{z}_{it}) \tilde{\mathbf{P}}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T} | \mathbf{z}_{it}), \quad (34)$$

where

$$\tilde{\mathbf{P}}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)}) = \exp\{\mathbb{E}_{q(\boldsymbol{\theta}, \boldsymbol{\pi}, \mathbf{T})}[\log P(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)} | \boldsymbol{\theta}_{1:(t-1)}, \boldsymbol{\pi}, \mathbf{T})]\}, \quad (35)$$

$$\tilde{\mathbf{P}}(\mathbf{z}_{it} | \mathbf{z}_{i(t-1)}) = \exp\{\mathbb{E}_{q(\mathbf{T})}[\log P(\mathbf{z}_{it} | \mathbf{z}_{i(t-1)}, \mathbf{T})]\}, \quad (36)$$

$$\tilde{\mathbf{P}}(\mathbf{x}_{it} | \mathbf{z}_{it}) = \exp\{\mathbb{E}_{q(\boldsymbol{\theta})}[\log P(\mathbf{x}_{it} | \mathbf{z}_{it}, \boldsymbol{\theta}_t)]\}, \quad (37)$$

$$\tilde{\mathbf{P}}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T} | \mathbf{z}_{it}) = \exp\{\mathbb{E}_{q(\boldsymbol{\theta}, \mathbf{T})}[\log P(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T} | \mathbf{z}_{it}, \boldsymbol{\theta}_{(t+1):T}, \mathbf{T})]\}. \quad (38)$$

Here, the function $\tilde{\mathbf{P}}(\cdot)$ is not a usual probability function because it is not normalized, which is indicated by the tilde mark. The calculation of expectations in Equations (35)–(38) is not so difficult but rather cumbersome; see Supplemental Material A. From here, subscript with colon

1 “.” means that the set has a sequential element from the time point of left of the colon to the right
 2 time point. For example, $\mathbf{X}_{i,1:(t-1)}$ is a set of observed random variables of J item responses of
 3 individual i with the first time point to the $t - 1$ time point. Likewise, $\mathbf{Z}_{i,(t+1):T}$ is a set of
 4 latent attribute mastery pattern indicator of student i from the $(t + 1)$ th to the final T th time
 5 point. The others are interpreted in the same manner.

6 Substituting Equations (35)–(38) into Equation (34), that is, $q(\mathbf{z}_i) \propto$
 7 $\tilde{\mathbf{P}}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)})\tilde{\mathbf{P}}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)})\tilde{\mathbf{P}}(\mathbf{x}_{it}|\mathbf{z}_{it})\tilde{\mathbf{P}}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T}|\mathbf{z}_{it}), q(\mathbf{z}_{it})$ can be simply
 8 represented:

$$\begin{aligned} q(\mathbf{z}_{it}) &\propto \sum_{\mathbf{z}_{i \setminus t}} \tilde{\mathbf{P}}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)})\tilde{\mathbf{P}}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)})\tilde{\mathbf{P}}(\mathbf{x}_{it}|\mathbf{z}_{it})\tilde{\mathbf{P}}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T}|\mathbf{z}_{it}), \\ &= \tilde{\mathbf{P}}(\mathbf{x}_{it}|\mathbf{z}_{it}) \left\{ \sum_{\mathbf{z}_{i,1:(t-1)}} \tilde{\mathbf{P}}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)})\tilde{\mathbf{P}}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)}) \right\} \left\{ \sum_{\mathbf{z}_{i,(t+1):T}} \tilde{\mathbf{P}}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T}|\mathbf{z}_{it}) \right\}, \\ &= f(\mathbf{z}_{it})b(\mathbf{z}_{it}), \end{aligned} \quad (39)$$

9 where

$$f(\mathbf{z}_{it}) = \tilde{\mathbf{P}}(\mathbf{x}_{it}|\mathbf{z}_{it}) \sum_{\mathbf{z}_{i,1:(t-1)}} \tilde{\mathbf{P}}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)})\tilde{\mathbf{P}}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)}), \quad (40)$$

$$b(\mathbf{z}_{it}) = \sum_{\mathbf{z}_{i,(t+1):T}} \tilde{\mathbf{P}}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T}|\mathbf{z}_{it}). \quad (41)$$

10 $f(\mathbf{z}_{it})$ and $b(\mathbf{z}_{it})$ are calculated recursively, and these functions are more useful to reduce the
 11 computational burden than adopting the naïve approach because Equations (40) and (41) are
 12 represented as

$$f(\mathbf{z}_{it}) = \tilde{\mathbf{P}}(\mathbf{x}_{it}|\mathbf{z}_{it}) \sum_{\mathbf{z}_{i(t-1)}} \tilde{\mathbf{P}}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)})f(\mathbf{z}_{i(t-1)}), \quad (42)$$

$$b(\mathbf{z}_{it}) = \sum_{\mathbf{z}_{i(t+1)}} \tilde{\mathbf{P}}(\mathbf{x}_{i(t+1)}|\mathbf{z}_{i(t+1)})\tilde{\mathbf{P}}(\mathbf{z}_{i(t+1)}|\mathbf{z}_{it})b(\mathbf{z}_{i(t+1)}). \quad (43)$$

13 These equations indicate that $f(\mathbf{z}_{it})$ is recursively calculated using the previous value of
 14 $f(\mathbf{z}_{i(t-1)})$, and $b(\mathbf{z}_{it})$ is employing $b(\mathbf{z}_{i(t+1)})$. More precisely, first, $f(\mathbf{z}_{i1}) =$

1 $\tilde{P}(\mathbf{x}_{i1}|\mathbf{z}_{i1})\tilde{P}(\mathbf{z}_{i1})$ is calculated, where $\tilde{P}(\mathbf{z}_{i1}) = \exp\{\mathbb{E}_{q(\boldsymbol{\pi})}[\log P(\mathbf{z}_{i1}|\boldsymbol{\pi})]\}$, and recursively
 2 applies Equation (42). Similarly, start with $b(\mathbf{z}_{iT}) = 1$ and update it with Equation (43). When
 3 we calculate $f(\mathbf{z}_{it})$ and $b(\mathbf{z}_{it})$, they should be normalized each time to avoid underflow.
 4 Particularly, the normalizing constant for $f(\mathbf{z}_{it})$, which is $\gamma_t(\mathbf{x}_{it}) = \sum_l f(\mathbf{z}_{itl})$, is important to
 5 calculate the normalizing constant for $q(\mathbf{z}_i)$.

6 After calculating $f(\mathbf{z}_{it})$ and $b(\mathbf{z}_{it})$ for all time points, the distribution $q(\mathbf{z}_{it})$ is
 7 calculated using $f(\mathbf{z}_{itl} = 1)$ and $b(\mathbf{z}_{itl} = 1)$ for all l and normalized subsequently, and its
 8 expectation is represented as

$$\mathbb{E}_{q(\mathbf{z}_{it})}[\mathbf{z}_{itl}] = \frac{f(\mathbf{z}_{itl} = 1)b(\mathbf{z}_{itl} = 1)}{\sum_l f(\mathbf{z}_{itl} = 1)b(\mathbf{z}_{itl} = 1)}. \quad (44)$$

9 Here, $f(\mathbf{z}_{itl} = 1) = \tilde{P}(\mathbf{x}_{it}|\mathbf{z}_{itl} = 1)\sum_{\mathbf{z}_{i(t-1)}} \tilde{P}(\mathbf{z}_{itl} = 1|\mathbf{z}_{i(t-1)})f(\mathbf{z}_{i(t-1)})$, and it is a product of
 10 unnormalized item response vector probability of respondent i with l th attribute mastery
 11 pattern at time t and $\sum_{\mathbf{z}_{i(t-1)}} \tilde{P}(\mathbf{z}_{itl} = 1|\mathbf{z}_{i(t-1)})f(\mathbf{z}_{i(t-1)})$, which is the sum of unweighted
 12 probability of $\mathbf{z}_{itl} = 1$, given previous time point attribute mastery pattern vector $\mathbf{z}_{i(t-1)}$ with
 13 weight $f(\mathbf{z}_{i(t-1)})$. Thus, $f(\mathbf{z}_{itl} = 1)$ represents the combination of current item response
 14 values and the previous attribute mastery pattern.

15 Similarly, $q(\mathbf{z}_{it}, \mathbf{z}_{i(t-1)})$ can be calculated as

$$\begin{aligned}
 & q(\mathbf{z}_{it}, \mathbf{z}_{i(t-1)}) \\
 & \propto \sum_{\mathbf{z}_{i \setminus \{t, t-1\}}} \tilde{P}(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)})\tilde{P}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)})\tilde{P}(\mathbf{x}_{it}|\mathbf{z}_{it})\tilde{P}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T}|\mathbf{z}_{it}), \\
 & = \tilde{P}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)})\tilde{P}(\mathbf{x}_{it}|\mathbf{z}_{it}) \left\{ \sum_{\mathbf{z}_{i,1:(t-2)}} \tilde{P}(\mathbf{X}_{i,1:(t-2)}, \mathbf{Z}_{i,1:(t-2)})\tilde{P}(\mathbf{z}_{i(t-1)}|\mathbf{z}_{i(t-2)})\tilde{P}(\mathbf{x}_{i(t-1)}|\mathbf{z}_{i(t-1)}) \right\} \\
 & \times \left\{ \sum_{\mathbf{z}_{i,(t+1):T}} \tilde{P}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T}|\mathbf{z}_{it}) \right\}, \\
 & = \tilde{P}(\mathbf{z}_{it}|\mathbf{z}_{i(t-1)})\tilde{P}(\mathbf{x}_{it}|\mathbf{z}_{it})f(\mathbf{z}_{i(t-1)})b(\mathbf{z}_{it}).
 \end{aligned}$$

- 1 For all l and l' , calculate $\tilde{P}(z_{itl'} = 1 | z_{i(t-1)l} = 1) \tilde{P}(x_{it} | z_{itl'} = 1) f(z_{i(t-1)l} = 1) b(z_{itl'} = 1)$,
 2 normalize it to obtain the distribution $q(z_{itl'}, z_{i(t-1)l})$, and its expectation $\mathbb{E}_{q(z_i)}[z_{itl'} z_{i(t-1)l}]$ is
 3 represented as

$$\begin{aligned} & \mathbb{E}_{q(z_i)}[z_{itl'} z_{i(t-1)l}] \\ &= \frac{\tilde{P}(z_{itl'} = 1 | z_{i(t-1)l} = 1) \tilde{P}(x_{it} | z_{itl'} = 1) f(z_{i(t-1)l} = 1) b(z_{itl'} = 1)}{\sum_l \sum_{l'} \tilde{P}(z_{itl'} = 1 | z_{i(t-1)l} = 1) \tilde{P}(x_{it} | z_{itl'} = 1) f(z_{i(t-1)l} = 1) b(z_{itl'} = 1)}. \end{aligned} \quad (46)$$

- 4 First, using the result of the data generating process at time $t = 2$, the joint distribution
 5 is decomposed as follows:

$$P(x_{i1}, z_{i1} | \Theta_1, \pi) = P(x_{i1} | z_{i1}, \Theta_1) P(z_{i1} | \pi), \quad (47)$$

- 6 and from time $t = 3$ to T ,

$$\begin{aligned} & P(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)} | \Theta_{1:(t-1)}) \\ &= P(x_{i1} | z_{i1}, \Theta_1) P(z_{i1} | \pi) \prod_{t'=2}^{t-1} P(x_{it'} | z_{it'}, \Theta_{t'}) P(z_{it'} | z_{i(t'-1)}, \mathbf{T}). \end{aligned} \quad (48)$$

- 7 Then, the exponential function in Equation (35) can be expressed as

$$\begin{aligned} & \mathbb{E}_{q(\Theta, \pi, \mathbf{T})} [\log P(\mathbf{X}_{i,1:(t-1)}, \mathbf{Z}_{i,1:(t-1)} | \Theta_{1:(t-1)}, \pi, \mathbf{T})] \\ &= \mathbb{E}_{q(\Theta, \pi, \mathbf{T})} \left[\sum_l z_{i1l} \log \pi_l + \sum_{t'=2}^{t-1} \sum_l \sum_{l'} z_{i(t-1)l} z_{itl'} \log \tau_{ll'} \right. \\ & \quad + \sum_{t=1}^{t-1} \sum_j \sum_h \sum_l z_{it'l} g_{jhl} \{x_{ijt'} \log \theta_{jht'} \\ & \quad \left. + (1 - x_{ijt'}) \log(1 - \theta_{jht'})\} + \sum_{t'=2}^{t-1} \sum_l \sum_{l'} z_{i(t-1)l} z_{itl'} \log \tau_{ll'} \right], \end{aligned} \quad (49)$$

$$\begin{aligned}
 &= \sum_l z_{i1l} \mathbb{E}_{q(\boldsymbol{\pi})} [\log \pi_l] + \sum_{t'=2}^{t-1} \sum_l \sum_{l'} z_{i(t-1)l} z_{itl} \mathbb{E}_{q(\boldsymbol{\tau}_l)} [\log \tau_{ll'}] \\
 &\quad + \sum_{t'=1}^{t-1} \sum_j \sum_h \sum_l z_{it'l} g_{jhlt'} \left\{ x_{ijt'} \mathbb{E}_{q(\boldsymbol{\theta}_{jht})} [\log \theta_{jht'}] \right. \\
 &\quad \left. + (1 - x_{ijt'}) \mathbb{E}_{q(\boldsymbol{\theta}_{jht})} [\log(1 - \theta_{jht'})] \right\}.
 \end{aligned}$$

1 The list of expectations for Equation (49) are as follows:

$$\mathbb{E}_{q(\boldsymbol{\pi})} [\log \pi_l] = \psi(\delta_l^*) - \psi\left(\sum_l \delta_l^*\right), \forall l, \quad (50)$$

$$\mathbb{E}_{q(\boldsymbol{\tau}_l)} [\log \tau_{ll'}] = \psi(\omega_{ll'}^*) - \psi\left(\sum_{l'} \omega_{ll'}^*\right), \forall l, l', \quad (51)$$

$$\mathbb{E}_{q(\boldsymbol{\theta}_{jht})} [\log \theta_{jht}] = \psi(a_{jht}^*) - \psi(a_{jht}^* + b_{jht}^*), \forall j, h, t, \quad (52)$$

$$\mathbb{E}_{q(\boldsymbol{\theta}_{jht})} [\log(1 - \theta_{jht})] = \psi(b_{jht}^*) - \psi(a_{jht}^* + b_{jht}^*), \forall j, h, t, \quad (53)$$

2 where $\psi(x)$ is the gamma function defined for the value $x > 0$ and is expressed as

$$\frac{d}{dx} \log \Gamma(x) = \int_0^\infty y^{(x-1)} \exp(-y) dy. \quad (54)$$

4 Similarly, the expectation part of Equation (36) is given as

$$\mathbb{E}_{q(\mathbf{T})} [\log P(\mathbf{z}_{it} | \mathbf{z}_{i(t-1)}, \mathbf{T})] = \sum_l \sum_{i'} z_{i(t-1)l} z_{itl} \mathbb{E}_{q(\mathbf{T})} [\log \tau_{ll'}], \quad (54)$$

5 and that of Equation (37)—that is $\tilde{P}(\mathbf{x}_{it} | \mathbf{z}_{it})$ —can be written as

$$\begin{aligned}
 &\mathbb{E}_{q(\boldsymbol{\theta})} [\log P(\mathbf{x}_{it} | \mathbf{z}_{it}, \boldsymbol{\theta}_t)] \\
 &= \sum_j \sum_h \sum_l z_{itl} g_{jhlt} \left\{ x_{ijt} \mathbb{E}_{q(\boldsymbol{\theta}_{jht})} [\log \theta_{jht}] + (1 - x_{ijt}) \mathbb{E}_{q(\boldsymbol{\theta}_{jht})} [\log(1 - \theta_{jht})] \right\}, \quad (55)
 \end{aligned}$$

6 and finally, the term in Equation (38), that is, $\tilde{P}(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T} | \mathbf{z}_{it})$, is calculated as

$$\begin{aligned}
 &\mathbb{E}_{q(\boldsymbol{\theta}, \mathbf{T})} [\log P(\mathbf{X}_{i,(t+1):T}, \mathbf{Z}_{i,(t+1):T} | \mathbf{z}_{it}, \boldsymbol{\theta}_{(t+1):T}, \mathbf{T})] \\
 &= \sum_{t'=t+1}^T \sum_l \sum_{l'} z_{i(t-1)l} z_{itl} \mathbb{E}_{q(\boldsymbol{\tau}_l)} [\log \tau_{ll'}] \quad (56)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t'=t+1}^T \sum_j \sum_h \sum_l z_{it'l} g_{jhlt'} \left\{ x_{ijt'} \mathbb{E}_{q(\theta_{jht})} [\log \theta_{jht'}] \right. \\
 & \quad \left. + (1 - x_{ijt'}) \mathbb{E}_{q(\theta_{jht})} [\log (1 - \theta_{jht'})] \right\}.
 \end{aligned}$$

1 Using the above elements, we can calculate Equations (35)–(38).

2

Supplemental Material D. Evidence lower bound (ELBO)

Finally, the evidence lower bound (ELBO), which is the negatively signed $\text{KL}(q(\mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}) || P(\mathbf{X}, \mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T} | \mathbf{A}^0, \mathbf{B}^0, \boldsymbol{\delta}^0, \boldsymbol{\Omega}^0))$, can be derived using the definition of evidence of lower bound (Beal, 2003) as follows:

$$\begin{aligned}
 \log P(\mathbf{X} | \mathbf{A}^0, \mathbf{B}^0, \boldsymbol{\delta}^0, \boldsymbol{\Omega}^0) &\geq ELBO, \\
 &= -\text{KL}(q(\mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}) || P(\mathbf{X}, \mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T} | \mathbf{A}^0, \mathbf{B}^0, \boldsymbol{\delta}^0, \boldsymbol{\Omega}^0)), \\
 &= \sum_{\mathbf{Z}} \iiint q(\mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T}) \log \frac{P(\mathbf{X}, \mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T} | \mathbf{A}^0, \mathbf{B}^0, \boldsymbol{\delta}^0, \boldsymbol{\Omega}^0)}{q(\mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{T})} d\boldsymbol{\Theta} d\boldsymbol{\pi} d\mathbf{T}, \\
 &= \int q(\boldsymbol{\Theta}) \log \frac{P(\boldsymbol{\Theta} | \mathbf{A}^0, \mathbf{B}^0)}{q(\boldsymbol{\Theta})} d\boldsymbol{\Theta} + \int q(\boldsymbol{\pi}) \log \frac{P(\boldsymbol{\pi} | \boldsymbol{\delta}^0)}{q(\boldsymbol{\pi})} d\boldsymbol{\pi} \\
 &\quad + \int q(\mathbf{T}) \log \frac{P(\mathbf{T} | \boldsymbol{\Omega}^0)}{q(\mathbf{T})} d\mathbf{T} + \sum_i \log \zeta(\mathbf{x}_{i1:T}).
 \end{aligned} \tag{57}$$

Details of each term in Equation (57) are mentioned in the Supplemental Material B. The final term $\zeta(\mathbf{x}_{i1:T})$ is the normalizing constant for $q(\mathbf{z}_i)$ and is calculated using the normalizing constant $\gamma_t(\mathbf{x}_{it})$ as follows (Beal, 2003; p.95, Equation 3.74):

$$\prod_{t=1}^T \gamma_t(\mathbf{x}_{it}) = \zeta(\mathbf{x}_{i1:T}). \tag{58}$$

Therefore, $\sum_t \log \gamma_t(\mathbf{x}_{it}) = \log \zeta(\mathbf{x}_{i1:T})$.

The first term is

$$\begin{aligned}
 &\int q(\boldsymbol{\Theta}) \log \frac{P(\boldsymbol{\Theta} | \mathbf{A}^0, \mathbf{B}^0)}{q(\boldsymbol{\Theta})} d\boldsymbol{\Theta} \\
 &= \mathbb{E}_{q(\boldsymbol{\Theta})} \left[\sum_t \sum_j \sum_h \left\{ \log \frac{\beta(a_{jht}^*, b_{jht}^*)}{\beta(a_{jht}^0, b_{jht}^0)} + (a_{jht}^0 - a_{jht}^*) \log \theta_{jht} \right. \right. \\
 &\quad \left. \left. + (b_{jht}^0 - b_{jht}^*) \log(1 - \theta_{jht}) \right\} \right],
 \end{aligned} \tag{59}$$

$$\begin{aligned}
 &= \sum_t \sum_j \sum_h \left\{ \log \frac{\beta(a_{jht}^*, b_{jht}^*)}{\beta(a_{jht}^0, b_{jht}^0)} + (a_{jht}^0 - a_{jht}^*) [\psi(a_{jht}^*) - \psi(a_{jht}^* + b_{jht}^*)] \right. \\
 &\quad \left. + (b_{jht}^0 - b_{jht}^*) [\psi(b_{jht}^*) - \psi(a_{jht}^* + b_{jht}^*)] \right\}.
 \end{aligned}$$

1 The second term is

$$\begin{aligned}
 &\int q(\boldsymbol{\pi}) \log \frac{P(\boldsymbol{\pi}|\boldsymbol{\delta}^0)}{q(\boldsymbol{\pi})} d\boldsymbol{\pi} = \mathbb{E}_{q(\boldsymbol{\pi})} \left[\log \frac{\text{Beta}(\boldsymbol{\delta}^*)}{\text{Beta}(\boldsymbol{\delta}^0)} + \sum_l \{\delta_l^0 - \delta_l^*\} \log \pi_l \right] \\
 &= \log \frac{\text{Beta}(\boldsymbol{\delta}^*)}{\text{Beta}(\boldsymbol{\delta}^0)} + \sum_l \left[(\delta_l^0 - \delta_l^*) \left\{ \psi(\delta_l^*) - \psi \left(\sum_{l'} \delta_{l'}^* \right) \right\} \right].
 \end{aligned} \tag{60}$$

2 Finally, the third term is given by

$$\begin{aligned}
 &\int q(\mathbf{T}) \log \frac{P(\mathbf{T}|\boldsymbol{\Omega}^0)}{q(\mathbf{T})} d\mathbf{T} \\
 &= \mathbb{E}_{q(\mathbf{T})} \left[\sum_l \left\{ \log \frac{\text{Beta}(\boldsymbol{\omega}_l^*)}{\text{Beta}(\boldsymbol{\omega}_l^0)} + \sum_{l'} [(\omega_{ll'}^0 - \omega_{ll'}^*) \log \tau_{ll'}] \right\} \right] \\
 &= \sum_l \left\{ \log \frac{\text{Beta}(\boldsymbol{\omega}_l^*)}{\text{Beta}(\boldsymbol{\omega}_l^0)} + \sum_{l'} \left[(\omega_{ll'}^0 - \omega_{ll'}^*) \left(\psi(\omega_{ll'}^*) - \psi \left(\sum_{l''} \omega_{ll''}^* \right) \right) \right] \right\}.
 \end{aligned} \tag{61}$$

1 **Supplemental Material E. Q-matrix employed in obtaining the simulation results**

Table S1.

Four attribute Q-matrix (left) and six attribute Q-matrix (right) employed in the simulation study

Item	Attribute				Sum
	1	2	3	4	
1	1	0	0	0	1
2	0	1	0	0	1
3	0	0	1	0	1
4	0	0	0	1	1
5	1	0	0	0	1
6	0	1	0	0	1
7	0	0	1	0	1
8	0	0	0	1	1
9	1	1	0	0	2
10	1	0	1	0	2
11	1	0	0	1	2
12	0	1	1	0	2
13	0	1	0	1	2
14	0	0	1	1	2
15	1	1	0	0	2
16	1	0	1	0	2
17	1	0	0	1	2
18	0	1	1	0	2
19	0	1	0	1	2
20	0	0	1	1	2
21	1	1	1	0	3
22	1	1	0	1	3
23	1	0	1	1	3
24	0	1	1	1	3
25	1	1	1	0	3
26	1	1	0	1	3
27	1	0	1	1	3
28	0	1	1	1	3
29	1	1	1	1	4
30	1	1	1	1	4
Sum	16	16	16	16	

Item	Attribute						Sum
	1	2	3	4	5	6	
1	1	0	0	0	0	0	1
2	0	1	0	0	0	0	1
3	0	0	1	0	0	0	1
4	0	0	0	1	0	0	1
5	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1
7	1	0	0	0	0	0	1
8	0	1	0	0	0	0	1
9	0	0	1	0	0	0	1
10	0	0	0	1	0	0	1
11	0	0	0	0	1	0	1
12	0	0	0	0	0	1	1
13	1	0	0	0	0	0	1
14	0	1	0	0	0	0	1
15	0	0	1	0	0	0	1
16	0	0	0	1	0	0	1
17	0	0	0	0	1	0	1
18	0	0	0	0	0	1	1
19	1	1	0	0	0	0	2
20	1	0	1	0	0	0	2
21	1	0	0	1	0	0	2
22	1	0	0	0	1	0	2
23	1	0	0	0	0	1	2
24	0	1	1	0	0	0	2
25	0	1	0	1	0	0	2
26	0	1	0	0	1	0	2
27	0	1	0	0	0	1	2
28	0	0	1	1	0	0	2
29	0	0	1	0	1	0	2
30	0	0	1	0	0	1	2
31	0	0	0	1	1	0	2
32	0	0	0	1	0	1	2
33	0	0	0	0	1	1	2
34	1	1	1	0	0	0	3
35	0	1	1	1	0	0	3
36	0	0	1	1	1	0	3
37	0	0	0	1	1	1	3
38	1	0	1	0	1	0	3
39	0	1	0	1	0	1	3
40	1	1	1	1	0	0	4
41	0	1	1	1	1	0	4
42	0	0	1	1	1	1	4
43	1	1	0	1	1	0	4
44	0	1	1	0	1	1	4
Sum	12	15	16	16	15	12	

1 **Supplemental Material F. Mean bias and RMSEs of correct item response probabilities at**
 2 **the first to fourth time points**

3 Figure S1

Histograms of biases of correct item response probabilities at the first time point in simulation study

1 Figure S2

Histograms of RMSEs of correct item response probabilities at the first time point in simulation study

1 Figure S3

2 *Histograms of biases of correct item response probabilities at the second time point*

1

2 Figure S4

3 *Histograms of RMSEs of correct item response probabilities at the second time point*

1

2 Figure S5

3 *Histograms of biases of correct item response probabilities at the third time point*

1

2 Figure S6

3 *Histograms of RMSEs of correct item response probabilities at the third time point*

1

2 Figure S7

3 *Histograms of biases of correct item response probabilities at the fourth time point*

1

2 Figure S8

3 *Histograms of RMSEs of correct item response probabilities at the fourth time point*

Supplemental Material G. Q-matrix employed in the Real Data Example

Table S2

Q-matrix of the Spatial Reasoning Test employed by Wang et al. (2018)

Item	Attribute				Item	Attribute			
	90° x-axis	90° y-axis	180° x-axis	180° y-axis		90° x-axis	90° y-axis	180° x-axis	180° y-axis
1	1	0	0	0	26	0	1	1	0
2	0	1	0	0	27	0	0	0	1
3	0	0	0	1	28	1	0	0	0
4	0	0	1	0	29	0	1	0	0
5	1	1	0	0	30	1	0	0	1
6	0	1	1	0	31	0	1	0	0
7	1	1	0	0	32	1	0	0	0
8	1	0	0	1	33	0	0	1	0
9	0	0	1	0	34	0	0	0	1
10	0	1	1	0	35	1	1	0	0
11	0	0	1	0	36	0	1	1	0
12	0	1	0	0	37	1	0	0	1
13	0	0	0	1	38	0	1	0	0
14	1	0	0	0	39	0	0	0	1
15	0	1	1	0	40	1	1	0	0
16	1	1	0	0	41	0	0	0	1
17	0	1	0	0	42	0	0	1	0
18	0	0	1	0	43	1	0	0	0
19	0	1	1	0	44	0	1	0	0
20	1	0	0	1	45	0	0	0	1
21	0	1	0	0	46	1	1	0	0
22	0	0	0	1	47	0	1	1	0
23	0	0	1	0	48	1	1	0	0
24	1	1	0	0	49	0	1	0	0
25	1	0	0	0	50	1	0	0	1