Running head: PRODUCTS OF VARIABLES IN STRUCTURAL EQUATION MODELING (XRAM)

Products of Variables in Structural Equation Models: Multiplicative Reticular Action
Models

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Abstract

A general method is introduced in which variables that are products of other variables in the context of a structural equation model (SEM) can be decomposed into the sources of variance due to the multiplicands. The result is a new category of SEM which we call a Multiplicative Reticular Action Model (XRAM). XRAM can include interactions between latent variables, multilevel random coefficients, latent variable moderators, and novel constructs such as factors of paths and twin genetic decomposition of multilevel random coefficients. The method relies on an assumption that all variance sources in a model can be decomposed into linear combinations of independent normal standardized variables. Although the distribution of a variable that is an outcome of multiplication between other variables is not normal, the assumption is that it can be decomposed into sources that are normal if one takes into account the non-normality induced by the multiplication. The method is applied to an example to show how in a special case it is equivalent to known unbiased and efficient estimators in the statistical literature. Two simulations are presented that demonstrate the precision of the approximation and implement the method to estimate parameters in a multilevel autoregressive framework.

Products of Variables in Structural Equation Models: Multiplicative Reticular Action Models

Introduction

The expected covariance matrix of a Structural Equation Model (SEM) which is composed of a linear combination of variables has long been known (Wright, 1921) as well as its equivalence to a method of path coefficients (Wright, 1934). This equivalence was formalized into what is called the Reticular Action Model (RAM) by McArdle and McDonald (McArdle & McDonald, 1984). While linear combinations of variables are powerful and useful, nonlinear models have become increasingly important as mixed effects models, moderation models, and nonlinear dynamical systems models have become necessary in the current data-rich research environment.

When a variable in an SEM is the outcome of the product of two variables and at least one of the product terms is a manifest variable, there are a variety of methods that can be used to estimate coefficients in the SEM (e.g., Mehta & Neale, 2005; Neale, 1998). But when both of the product terms are latent, the problem becomes more difficult. Methods have been proposed that require nonlinear constraints on measurement models (the product indicant technique, Kenny & Judd, 1984), approximating non-normal distributions using mixture distributions (the latent moderated structure technique, Klein & Moosbrugger, 2000; Moosbrugger, Schermelleh-Engel, & Klein, 1997), multiple group approaches and estimating latent scores so that they can be dealt with as manifest variables (Schumacker, 2002), and variations on unconstrained approaches (unconstrained and quasi-maximum likelihood techniques, Marsh, Wen, & Hau, 2004; Marsh et al., 2007). In the current article we take a different approach and present a method for the decomposition of the variance and covariance of products of variables that was first

1 proposed and elaborated by Wall and Amemiya (Wall & Amemiya, 2001, 2003). The
 2 approach presented here makes the assumption that the non-normality of a variable
 3 constructed as the product of other variables can be completely accounted for if a model
 4 can account for all sources of variance that are included in the product and that those
 5 sources of variance are multivariate normally distributed.

We begin our argument from the standpoint of SEM path diagrams in order to help
 make the logic more tractable. In a linear combination, estimated coefficients are
 constants that are multiplied by variables and then all of these products are summed
 together and assigned to an outcome variable. For instance, a simple bivariate regression
 using mean centered multivariate normal variables can be written as,

$$z_i = b_1x_i + b_2y_i + e_i \quad (1)$$

6 where x_i and y_i are the predictor variables and e_i is the residual at each row index i , and
 7 b_1 , b_2 , and 1 are constants. If the variance of x , y , and e are V_x , V_y , and V_e , respectively
 8 and the covariance between x and y is C_{xy} , a RAM path diagram isomorphic to this
 9 bivariate regression can be drawn as in Figure 1-a. Compare Equation 1 with the path
 10 diagram and note that two things are happening when the three one-headed arrows attach
 11 to the box representing the manifest variable z . In the equation, b_1x_i , b_2y_i , and e_i are first
 12 summed. Second, the result of that sum is assigned to the variable z_i . In the equation,
 13 summation is represented by one symbol, “+”, and assignment is represented by another
 14 symbol, “=”. The path diagram welds together these two symbols from the equation.
 15 This is not a problem as long as *only* linear combinations are used. In that case,
 16 summation and assignment are always used together. However, if one wishes to use an
 17 n-ary operator other than summation, the assignment operator and the n-ary operator
 18 must be disambiguated in the path diagram.

19 In Figure 1-b, a new n-ary operator path diagram node is proposed: the summation

operator is represented by a plus sign within a circle. If this summation operator is treated just like a latent variable with zero unique variance, then the algorithmic application of the RAM path tracing rules (Boker, McArdle, & Neale, 2002) will correctly produce the components of covariance of the expected covariance matrix of any SEM model that is composed solely of linear combinations of variables. Note that in Figure 1-c, the summation operator can be mixed together with standard addition-and-assignment while the standard RAM tracing rules will continue to produce the correct components of covariance.

Now that the assignment operator can be distinguished from the summation operator, it is possible to propose a new n-ary operation between variables: multiplication. Consider the simplest case, a product of a constant and two predictor variables,

$$z_i = b_1 x_i y_i + e_i , \quad (2)$$

where the variables x and y are multivariate normal with mean zero and variances V_x and V_y respectively, and the residual e is independent normally distributed with mean zero and variance V_e .

A path model for Equation 2 is displayed in Figure 2-a, where the n-ary operator multiplication is represented by an asterisk within a circle. While the summation operator could be treated as if it were a latent variable with zero variance, the multiplication operator cannot, as will be demonstrated below. Thus, we will need to specify that an n-ary operator is a fourth type of node in a path diagram: i) squares represent manifest variables; ii) circles represent latent variables; iii) triangles represent constants for means and intercepts; and iv) n-ary operators are represented by a small circle surrounding the selected operator symbol. This distinction is necessary for two reasons that will be described in more detail later in the article: different n-ary operators have different identity operations and also trigger different path coefficient rules.

Diagramming the problem in this way has numerous advantages. Since multiplication is commutative, neither the vector $\mathbf{x}$ nor $\mathbf{y}$ has a privileged place in the multiplication, just as one would expect algebraically. This is not explicit in some diagrammatic representations for moderation where a moderating variable is singled out. This might be perceived as just a philosophical difference. However, placing both multiplicands of a commutative binary operation on equal visual footing emphasizes that the SEM network of variables can be perturbed at either upstream variable with an equivalent downstream effect.

Variance of the Product of Two Variables

Given two vectors $\mathbf{x}$ and $\mathbf{y}$, each with N elements, Goodman (1960) first derived the variances of the product of two variables and showed that these were unbiased estimators. If $\Delta\mathbf{x} = \mathbf{x} - \mu(\mathbf{x})$ and $\Delta\mathbf{y} = \mathbf{y} - \mu(\mathbf{y})$, the exact variance is given by

$$\begin{aligned} \text{var}(\mathbf{xy}) = & \text{cov}(\Delta\mathbf{x}^2, \Delta\mathbf{y}^2) + \text{var}(\mathbf{x})\text{var}(\mathbf{y}) + \mu(\mathbf{x})^2\text{var}(\mathbf{y}) + \mu(\mathbf{y})^2\text{var}(\mathbf{x}) + \\ & (\mu(\mathbf{x})\mu(\mathbf{y}))^2 + 2\mu(\mathbf{x})\mu(\mathbf{y})\text{cov}(\mathbf{x}, \mathbf{y}) + 2\mu(\mathbf{x})\text{cov}(\Delta\mathbf{x}, \Delta\mathbf{y}^2) + \\ & 2\mu(\mathbf{y})\text{cov}(\Delta\mathbf{x}^2, \Delta\mathbf{y}) - \text{cov}(\mathbf{x}, \mathbf{y})^2 . \end{aligned} \quad (3)$$

Bohrnstedt and colleagues (1969) extended this result to the bivariate normal case and showed that the variance of the product, $\text{var}(\mathbf{xy})$, can be estimated as (Bohrnstedt & Marwell, 1978, Equation 15)

$$\begin{aligned} \text{var}(\mathbf{xy}) = & \mu(\mathbf{x})^2\text{var}(\mathbf{y}) + \mu(\mathbf{y})^2\text{var}(\mathbf{x}) + 2\mu(\mathbf{x})\mu(\mathbf{y})\text{cov}(\mathbf{x}, \mathbf{y}) + \\ & \text{var}(\mathbf{x})\text{var}(\mathbf{y}) + \text{cov}(\mathbf{x}, \mathbf{y})^2 . \end{aligned} \quad (4)$$

Bohrnstedt and Marwell (1978) also derived reliability for this estimator when the observed product variable included a normally distributed error term.

Now consider the regression equation

$$z_i = b_1 x_i y_i + e_i \quad (5)$$

1 where $\mathbf{x}$ and $\mathbf{y}$ are mean centered, multivariate normal, and the vector of residuals, $\mathbf{e}$, is
 2 mean centered, normally distributed, and independent of the predictor variables $\mathbf{x}$ and $\mathbf{y}$.
 3 This may be drawn as a path diagram as shown in Figure 2-a. A new set of tracing rules
 4 will be required when this product symbol is used. However, note that the total variance
 5 at the product symbol can be estimated from Equation 6. The downstream contribution
 6 of this variance to $\mathbf{z}$ will be scaled by b_1^2 . We can thus linearly decompose the variance of $\mathbf{z}$
 7 into terms due to the contribution of the error term $\mathbf{y}$ and the product of $\mathbf{x}$ and $\mathbf{y}$ such
 8 that

$$\begin{aligned}
 \text{var}(\mathbf{z}) = & b_1^2[\mu(\mathbf{x})^2\text{var}(\mathbf{y}) + \mu(\mathbf{y})^2\text{var}(\mathbf{x}) + 2\mu(\mathbf{x})\mu(\mathbf{y})\text{cov}(\mathbf{x}, \mathbf{y}) + \\
 & \text{var}(\mathbf{x})\text{var}(\mathbf{y}) + \text{cov}(\mathbf{x}, \mathbf{y})^2] + \text{var}(\mathbf{e}) .
 \end{aligned} \tag{6}$$

9 Since $\mu(\mathbf{x}) = \mu(\mathbf{y}) = 0$, Equation 6 simplifies to

$$\text{var}(\mathbf{z}) = b_1^2[\text{var}(\mathbf{x})\text{var}(\mathbf{y}) + \text{cov}(\mathbf{x}, \mathbf{y})^2] + \text{var}(\mathbf{e}) . \tag{7}$$

10 We next derive a general estimator for the variance decomposition of products of
 11 variables and apply it to the example problem in Equation 7 in order to demonstrate how
 12 the known estimator for the variance of the product of two variables is a special case
 13 result of the general estimator. This method relies on the model in the equivalent path
 14 diagram shown in Figure 2-b.

15 **Expectations by Method of Moments for Products of** 16 **Variables**

17 Suppose we are given an SEM as a path diagram in which all nodes with incoming
 18 edges are additionally labeled as sum or product of their incoming edges. For the time
 19 being, we will assume that the graph is acyclic (recursive). It may be that the results can
 20 be extended to cases with cycles as well, provided that for all node values, a infinite series
 21 calculated along each cycle converges.

We are interested in all moments of the vector of all observed variables. The following describes how those can be computed analytically assuming a fixed set of parameters. The main idea as first proposed by Wall and Amemiya (Wall & Amemiya, 2001, 2003) is to represent the variance sources of the SEM as independent standard-normally distributed sources and then represent all variables as polynomial over these sources. Then, all moments become expectations of polynomials, which in turn are sums of monomials. In this way, the computation of the moments is reduced to computing the expectation of a monomial of independently standard-normally distributed variables—a problem that has a known computational solution.

At the highest level, the algorithm proceeds in three steps,

1. All variables in the SEM are represented by a linear combination of some independently normally distributed variables $w_1, \dots, w_n$ with known variances such that the covariance matrix of all variables is the symmetrical matrix $\mathbf{S}$ from the RAM matrix formulation.
2. Progressing top-down in the asymmetrical graph of the path diagram, polynomial representations of all variables in the $w_1, \dots, w_n$ are computed.
3. Polynomial representations of all requested moments are computed and evaluated into numbers.

Suppose that we have an SEM represented in standard RAM notation such that the, the model-expected covariance matrix of the observed variables, $\mathbf{C}_{xx}$ is calculated as

$$\mathbf{C}_{xx} = \mathbf{F}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{S}((\mathbf{I} - \mathbf{A})^{-1})^T\mathbf{F}^T \quad (8)$$

where for all variables, both latent and manifest, $\mathbf{A}$ is the matrix of regression coefficients, $\mathbf{S}$ is the matrix of variances and covariances, and $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{F}$ filters out the latent variables so that $\mathbf{C}_{xx}$ contains only the model-expected covariance matrix of the observed variables.

In order to transform the model into a model with only independent variables, we will operate on $\mathbf{S}$, the matrix of model variances and covariances both latent and manifest. We first compute the Eigenvalue decomposition of $\mathbf{S}$,

$$\mathbf{S} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \quad (9)$$

Let $W = w_1, \dots, w_n$ be independently normally distributed variables with zero mean and variances given by the diagonal entries of $\mathbf{D}$, the Eigenvalues of $\mathbf{S}$, where n is the number of total variables in the SEM. Then $\mathbf{Q}W$ is an n -dimensional random variable with zero mean and covariance

$$\mathbb{V}(\mathbf{Q}W) = \mathbb{E}(\mathbf{Q}\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{Q}^T) = \mathbf{S} \quad (10)$$

- 1 So each variable can be expressed by a linear combination of the W variables with the
- 2 corresponding row of Q as weights, plus a constant term that gives the mean of that
- 3 variable.

Using the product and sum nodes, every variable in the model (both observed and latent) x_i can now be represented as a polynomial f_i in $w_1, \dots, w_n$. If there are m variables in total, the $k = (k_1, \dots, k_m)$ -th moment of the joint distribution of the vector $X = (x_1, \dots, x_m)$ is

$$M_k(X) = \mathbb{E}\left(\prod_{i=1}^m f_i^{k_i}\right) \quad (11)$$

where the expectation is taken with respect to the roots w_i . In particular, $M_k(X)$ is the expected value of a polynomial in w_i , where the coefficients are combination of the regression weights in the SEM. Let this polynomial be $g = \prod_{i=1}^m f_i^{k_i}$. This polynomial is a sum of monomials in w_i . Since the expectation is linear, we can separate the computation of the expectation to the single monomials, and thus reduce our problem to computing the expectation of a monomial of independently normally distributed variables, i.e., an expectation of the form

$$\mathbb{E}(w_1^{e_1} \dots w_n^{e_n}) \quad (12)$$

which is given by the product of the expectations of the single variables with their exponents,

$$\mathbb{E}(w_1^{e_1} \cdots w_n^{e_n}) = \prod_{i=1}^n \mathbb{E}(w_i^{e_i}) \quad (13)$$

1 This last equation is not completely trivial, and so for completeness sake, we give a
2 quick proof:

Theorem 1.

$$\mathbb{E}(W_1^{e_1} \cdots W_n^{e_n}) = \prod_{i=1}^n \mathbb{E}(W_i^{e_i}) \quad (14)$$

Proof.

$$\begin{aligned} \mathbb{E}(W_1^{e_1} \cdots W_n^{e_n}) &= \int_{-\infty}^{\infty} w_1^{e_1} \cdots w_n^{e_n} pdf(w_1) \cdots pdf(w_n) dw_1 \cdots dw_n \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} w_1^{e_1} pdf(w_1) dw \right) w_2^{e_2} \cdots w_n^{e_n} pdf(w_2) \cdots pdf(w_n) dw_2 \cdots dw_n \\ &= \mathbb{E}(W_1^{e_1}) \int_{-\infty}^{\infty} w_2^{e_2} \cdots w_n^{e_n} pdf(w_2) \cdots pdf(w_n) dw_2 \cdots dw_n \\ &\vdots \\ &= \prod_{i=1}^n \mathbb{E}(W_i^{e_i}) \end{aligned}$$

3 □

Thus, we are left with computing the higher-order moments of independent standard-normally distributed variables with zero mean and unit variance,

$$\mathbb{E}(W^e), \quad (15)$$

where e is an integer. These moments are known (e.g., Papoulis & Pillai, 2002) to be

$$\mathbb{E}(W^e) = \begin{cases} 0 & \text{if } e \text{ is odd} \\ \mathbb{V}(W)^{e/2} \prod_{i=1}^{\frac{e-2}{2}} (2i+1) & \text{if } e \text{ is even} \end{cases} \quad (16)$$

1 *Example 1: Bivariate Product of Variables Regression*

2 As an example, we will transform the SEM model in Figure 2-a into the equivalent
 3 SEM model shown in Figure 2-b such that all sources of variance and covariances are
 4 independent, normally distributed variables with mean zero and unit variance. The first
 5 step is to remove all covariances between variables by replacing them with unit variance
 6 sources. Thus, we add the latent variable w_2 and replace the covariance path with the
 7 value C_{xy} in Figure 2-a with regression paths from w_2 to x and y with values $C_{xy}^{\frac{1}{2}}$. Thus,
 8 the covariance between x and y can be calculated as $C_{xy}^{\frac{1}{2}} \cdot 1 \cdot C_{xy}^{\frac{1}{2}} = C_{xy}$. However, now the
 9 variances for x and y become residual variances that must be reduced by the total effect of
 10 w_2 which is $C_{xy}^{\frac{1}{2}} \cdot 1 \cdot C_{xy}^{\frac{1}{2}} = C_{xy}$. So, the residual variance of x is $V_x - C_{xy}$ and the residual
 11 variance of y is $V_y - C_{xy}$.

12 We can now replace the variance terms in Figure 2-a with the independent normally
 13 distributed unit variance variables w_1 , w_3 , and w_4 and regression paths to x , y , and e
 14 respectively. The regression weights for these variables become the square root of the
 15 residual variances for x , y , and e as shown in Figure 2-b. The i th variable in the path
 16 diagram in Figure 2-b can now be represented as a polynomial f_i of what we will refer to
 17 as *root nodes*, i.e., the independent, normally distributed variables $w_1, \dots, w_n$ with zero
 18 mean and unit variance. This transformation of an SEM model into polynomials of root
 19 nodes can be applied to any SEM that can be represented as a RAM model, including
 20 models that have n-ary operators such as introduced here. We will call models that
 21 conform to RAM conventions along with addition and product n-ary operators
 22 Multiplicative RAM (XRAM) models.

23 Returning to the example in Figure 2-b, we can compute the moments, including all
 24 joint moments of any variable or pair of variables. Thus, we can decompose the variance
 25 of the product variable $\mathbf{z} = b_1 \mathbf{x} \mathbf{y} + \mathbf{e}$ into components of variance and covariance of the

1 other variables in the SEM as follows:

$$\mathbb{V}(z) = \mathbb{E}(zz) \quad (17)$$

$$= \mathbb{E}((b_1xy + e)(b_1xy + e)) \quad (18)$$

$$= \mathbb{E}(b_1^2x^2y^2 + 2b_1xye + e^2) . \quad (19)$$

2 We next transform the variables x , y , and e into their monomial equivalents,

$$x = w_1(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2(\frac{1}{2}C_{xy})^{\frac{1}{2}} \quad (20)$$

$$y = w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2(\frac{1}{2}C_{xy})^{\frac{1}{2}} \quad (21)$$

$$e = w_4(V_e)^{\frac{1}{2}} . \quad (22)$$

3 Next we substitute Equations 20, 21 and 22 into Equation 19 and then expand, collect,

1 and cancel terms to find that

$$\begin{aligned} \mathbb{V}(z) = & \mathbb{E}(b_1^2(w_1(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})^2(w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})^2 + \\ & 2b_1(w_1(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})(w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})(w_4V_e^{\frac{1}{2}}) + \\ & (w_4V_e^{\frac{1}{2}})^2) \end{aligned} \quad (23)$$

$$\begin{aligned} = & \mathbb{E}(b_1^2(w_1^2(V_x - \frac{1}{2}C_{xy}) + w_2^2\frac{1}{2}C_{xy} + 2w_1w_2(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}) \\ & (w_3^2(V_y - \frac{1}{2}C_{xy}) + w_2^2\frac{1}{2}C_{xy} + 2w_2w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}) + \\ & 2b_1w_4V_e^{\frac{1}{2}}(w_1(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})(w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}}) + \\ & w_4^2V_e) \end{aligned} \quad (24)$$

$$\begin{aligned} = & \mathbb{E}(b_1^2(w_1^2w_3^2(V_x - \frac{1}{2}C_{xy})(V_y - \frac{1}{2}C_{xy}) + w_1^2w_2^2(V_x - \frac{1}{2}C_{xy})\frac{1}{2}C_{xy} + \\ & 2w_1^2w_2w_3(V_x - \frac{1}{2}C_{xy})(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}} + \\ & w_2^2w_3^2\frac{1}{2}C_{xy}(V_y - \frac{1}{2}C_{xy}) + w_4^2\frac{1}{2}C_{xy}^2 + 2w_2^3w_3\frac{1}{2}C_{xy}(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}} + \\ & 2w_1w_2w_3^2(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}(V_y - \frac{1}{2}C_{xy}) + 2w_1w_2^3(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}\frac{1}{2}C_{xy} + \\ & 4w_1w_2^2w_3(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}} + \\ & 2b_1w_4V_e^{\frac{1}{2}}(w_1(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})(w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}}) + \\ & w_4^2V_e) \end{aligned} \quad (25)$$

2 Referring to Equation 16, the expectation of the root nodes, w_i are $\mathbb{E}(w_i) = 0$, $\mathbb{E}(w_i^2) = 1$,

3 $\mathbb{E}(w_i^3) = 0$, and $\mathbb{E}(w_i^4) = 3$. Keeping in mind that the expected value of a constant is that

4 constant, we can substitute these expected central moments and simplify and collect terms

1 and cancel so that

$$\begin{aligned}
 \mathbb{V}(z) = & b_1^2((1)(1)(V_x - \frac{1}{2}C_{xy})(V_y - \frac{1}{2}C_{xy}) + (1)(1)(V_x - \frac{1}{2}C_{xy})\frac{1}{2}C_{xy} + \\
 & 2(1)(0)(0)(V_x - \frac{1}{2}C_{xy})(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}} + \\
 & (1)(1)\frac{1}{2}C_{xy}(V_y - \frac{1}{2}C_{xy}) + (3)\frac{1}{2}C_{xy}^2 + 2(0)(0)\frac{1}{2}C_{xy}(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}} + \\
 & 2(0)(0)(1)(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}(V_y - \frac{1}{2}C_{xy}) + 2(0)(0)(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}\frac{1}{2}C_{xy} + \\
 & 4(0)(1)(0)(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}}(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}}\frac{1}{2}C_{xy}^{\frac{1}{2}} + \\
 & 2b_1(0)V_e^{\frac{1}{2}}(w_1(V_x - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})(w_3(V_y - \frac{1}{2}C_{xy})^{\frac{1}{2}} + w_2\frac{1}{2}C_{xy}^{\frac{1}{2}})) + \\
 & (1)V_e
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 \mathbb{V}(z) = & b_1^2((V_x - \frac{1}{2}C_{xy})(V_y - \frac{1}{2}C_{xy}) + (V_x - \frac{1}{2}C_{xy})\frac{1}{2}C_{xy} + \\
 & \frac{1}{2}C_{xy}(V_y - \frac{1}{2}C_{xy}) + 3\frac{1}{2}C_{xy}^2) + V_e
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 \mathbb{V}(z) = & b_1^2(V_x V_y - \frac{1}{2}C_{xy}V_y - \frac{1}{2}C_{xy}V_x + \frac{1}{2}C_{xy}^2 + \frac{1}{2}C_{xy}V_x - \frac{1}{2}C_{xy}^2 + \\
 & \frac{1}{2}C_{xy}V_y - \frac{1}{2}C_{xy}^2 + 3\frac{1}{2}C_{xy}^2) + V_e
 \end{aligned} \tag{28}$$

$$\mathbb{V}(z) = b_1^2(V_x V_y + C_{xy}^2) + V_e, \tag{29}$$

2 which is the result in Equation 7 derived from the results of Goodman (Goodman, 1960)
 3 and Bohrnstedt and colleagues (Bohrnstedt & Marwell, 1978).

4 *Example 2: Autoregression with Latent Random Coefficient*

5 Since the method of moments variance/covariance decomposition of products of
 6 variables does not rely on the existence of variable scores, latent variables may be used as
 7 multiplicands without needing to estimate latent variable scores. This means that existing
 8 models may be cast into new forms and novel model structures may be fit. Consider
 9 for example model in Figure 3-a where the autoregression parameters vary between
 10 participants following a normal distribution with mean zero and variance 1. Thus, this is
 11 equivalent to a multilevel model grouped by participant.

1 In this example, $\mathbf{S}$ is already diagonal with dimension $n = 5$. Let us assume that
 2 $v_b = 1.0$, $v_{x1} = 1.0$, and that the residual variances, v_e are constrained to be equal so that
 3 $v_{e2} = v_{e3} = v_{e4} = 0.2$. We want to compute the model expected mean and variance of the
 4 last observed variable, $x4$. For the polynomial representations, we have

$$b = v_b + \mu b \quad (30)$$

$$w1 = v_{x1} \quad (31)$$

$$w2 = b \cdot w1 + v_{e2} = (v_b + \mu b)v_{x1} + v_{e2} \quad (32)$$

$$w3 = b \cdot w2 + v_{e3} = (v_b + \mu b)^2 v_{x1} + (v_b + \mu b)v_{e2} + v_{e3} \quad (33)$$

$$w4 = b \cdot w3 + v_{e4} = (v_b + \mu b)^3 v_{x1} + (v_b + \mu b)^2 v_{e2} + (v_b + \mu b)v_{e3} + v_{e4} \quad (34)$$

5 Since each summand in $w4$ has at least one variable to the power of 1, we see that

$$6 \mathbb{E}(w4) = 0.$$

7 For the second moment, we obtain

$$M_2(x4) = \mathbb{E}(x4^2) \quad (35)$$

$$= \mathbb{E}((v_b^6 + 15v_b^4 + 15v_b^2 + 1)v_{x1}^2 + (v_b^4 + 6v_b^2 + 1)v_{e2}^2 + \quad (36)$$

$$(v_b^2 + 1)v_{e3}^2 + v_{e4}^2) \quad (37)$$

$$= (15 + 15 \cdot 3 + 15 \cdot 1 + 1) \cdot 1 + (3 + 6 \cdot 1 + 1) \cdot 1 + (1 + 1) \cdot 1 + 1 \quad (38)$$

$$= 76 + 10 + 2 + 1 = 89 \quad (39)$$

8 In order to obtain the variance while taking the first and second moments into account,

$$9 \text{ we calculate } \mathbb{V}(x4) = M_2(x4) - \mathbb{E}(w4)^2 = 89 - 0 = 89.$$

10 While the example multilevel model in Figure 3-a can be fit using a multilevel SEM
 11 model, the form of Figure 3-a is novel. Normally, one identifies latent variables with a
 12 measurement model composed of manifest or latent variable indicators. The indicators for
 13 the latent variable b in Figure 3-a are essentially the autoregressive coefficient paths. Thus
 14 in products of variables models, we may begin to factor analyze the paths in a model as

well as the variables in the model. In our opinion, this represents a paradigm shift in how theories may be able to be instantiated in SEMs.

For instance, consider the model in Figure 3-b. This model is a combination of a latent intercept and slope model and a latent autoregression model. The latent intercept and slope essentially detrend the raw data simultaneously with fitting the autoregressive coefficient. And all three: intercept, slope, and autoregressive coefficient are at an individual level. Thus, we can now ask, “Is there an association between a person’s intercept and their stability?” The covariance path between the intercept, I , and autoregression coefficient, b in Figure 3-b allows that test.

Simulations

Simulation One: Comparing Estimators of Product Variance

A short simulation was run in order to estimate the adequacy of the estimator of the variance of a product $\mathbb{V}(z)$ of two normally distributed multiplicands, x and y , using Equation 29 and compare it to the estimator derived by Bohrnstedt (Bohrnstedt & Marwell, 1978, their Equation 14) which requires scores for x and y . There were 5 simulation conditions, each with 4 levels, for a total of 1,024 total condition cells. Variance and covariance conditions were chosen so as to ensure a positive definite predictor covariance matrix. The five conditions were: i) $\mathbb{V}(x) = \{2, 3, 4, 5\}$; ii) $\mathbb{V}(y) = \{3, 4, 5, 6\}$; iii) $\mathbb{C}(x, y) = \{-1, -\frac{1}{3}, \frac{1}{3}, 1\}$; iv) $\mu(x) = \{-2, -\frac{2}{3}, \frac{2}{3}, 2\}$; iv) $\mu(y) = \{-2, -\frac{2}{3}, \frac{2}{3}, 2\}$. For each combination of conditions, a data set with 10,000 observations for the predictors x and y was generated using the `rmvnorm()` function from the `mtvnorm` R package (Mi, Miwa, & Hothorn, 2009). The two simulated variables x and y were then multiplied and the variance of the resulting product of was calculated as the reference true product variance for that cell. Next, the two estimators were applied and the resulting estimated variance for that cell was stored.

1 As a result of running this simulation we noted a typo in (Bohrnstedt & Marwell,
 2 1978, their Equation 14), which omits the product of the two variances, $\text{var}(\mathbf{x})\text{var}(\mathbf{y})$. The
 3 corrected equation is

$$\begin{aligned} \text{var}(\mathbf{xy}) = & \mu(\mathbf{x})^2\text{var}(\mathbf{y}) + \mu(\mathbf{y})^2\text{var}(\mathbf{x}) + \text{cov}((\mathbf{x} - \mu(\mathbf{x}))^2, (\mathbf{y} - \mu(\mathbf{y}))^2) + \\ & 2\mu(\mathbf{y})\text{cov}((\mathbf{x} - \mu(\mathbf{x}))^2, (\mathbf{y} - \mu(\mathbf{y}))) + 2\mu(\mathbf{x})\text{cov}((\mathbf{x} - \mu(\mathbf{x})), (\mathbf{y} - \mu(\mathbf{y}))^2) + \\ & 2\mu(\mathbf{x})\mu(\mathbf{y})\text{cov}(\mathbf{x}, \mathbf{y}) + \text{var}(\mathbf{x})\text{var}(\mathbf{y}) + \text{cov}(\mathbf{x}, \mathbf{y})^2. \end{aligned} \quad (40)$$

4 Since all of the conditions had negligible bias and very low relative error, the results
 5 are presented aggregated over all conditions. Five statistics (correlation, signed error, root
 6 mean squared error, relative error, and relative RMSE) are presented in Table 1 where
 7 $\mathbb{V}(S)$ is the variance of the simulated product and $\mathbb{V}(E)$ is the estimated variance using
 8 one of the two estimators.

9 The raw variance estimates from Equation 29 versus the simulated product variance
 10 values are plotted in Figure 4-a and the estimated variance values from the more
 11 complicated estimator from Equation 40 are plotted in Figure 4-b. Both estimators are
 12 unbiased and have very low relative RMSE.

13 *Simulation Two: Autoregression with Latent Random Coefficient*

14 The second simulation demonstrates the use of the algorithm for use in estimating
 15 SEM models that include products of variables. The algorithm presented in this article
 16 was implemented in the Ω nyx software (Oertzen, Brandmaier, & Tsang, 2015) and a
 17 simulation run in which the true model and estimated model were of the form shown in
 18 Figure 3-a. Following the naming convention in Figure 3-a, parameters were set to the
 19 following: Variance of the first observation, $Vx_1 = 1.0$; Mean and variance of the
 20 autoregression coefficient, $\mu b = 0.2$ and $Vb = 0.1$; Residuals constrained to be equal for
 21 downstream variables x_i , $Ve = .25$. Using these parameters, 10,000 data sets with

$N = 1,000$ in each were simulated and the model was fit to each data set. The results are presented in Table 2.

As can be seen in Table 2, the absolute and relative error are very small, indicating that bias is negligible for all parameters. In addition, the RMSE and relative RMSE are also both reasonably small, indicating that estimation precision is good for all parameters. The estimated parameters that include product terms, μb and Vb , are not recovered as precisely as the original variance term and error term, but the precision of the product terms is still well within acceptable bounds. Figure 5 plots the relative error for all 10,000 trials of the simulation, indicating the distribution of the relative errors are reasonably close to normally distributed. The density of these distributions demonstrates the stability of the estimation procedure. Estimation of each data set with $N = 1,000$ required on average 1.8 seconds to run on a 2017 Macbook Pro running Mac OS High Sierra. The standard deviation of the estimation time was 0.17 seconds, again demonstrating the stability of the estimation algorithm when applied to this problem.

Discussion

The covariance decomposition of products of variables is a general method that will provide unbiased estimates of parameters when predictor variables are normally distributed. This is an advance over previous methods because it allows estimation of theoretically interesting models that have previously been problematic. Three novel products of variables models are diagrammed in Figure 6.

Figure 6-a is a second order latent differential equation model that allows the twin decomposition of the individual-level frequency and damping parameters into additive genetic, common environmental, and unique (ACE) components. The upper half of this model follows the form of the standard ACE twin model where twin pairs are either monozygotic or dizygotic and the additive genetic correlation is set to either 1.0 or 0.5

1 respectively. In this way, one can test behavior genetic theories about regulation processes
 2 that are modeled by second order differential equations.

3 Figure 6-b displays a three latent variable model where one loading for the factor X
 4 and one loading for the factor Z are moderated by the factor Y . This model might be
 5 applied when ones theory includes context-dependent or situation-specific definitions of
 6 the latent variables X and Z . For instance, X might be a latent variable “Quality of Life”
 7 and Z might be “Mobility”. If Y is the construct “Environmental Affordances” then one
 8 might find that the relationship between Quality of Life and its indicators might change
 9 depending on the Environmental Affordances available to an individual.

10 Figure 6-c repeats the random coefficients autoregressive model discussed above, but
 11 frees two more parameters, c and d , which allow a population average shape of change in
 12 the autoregression coefficients to emerge. This type of model may be particularly useful in
 13 studies where an intervention is given between occasion 2 and 3. In this case, a hypothesis
 14 might predict that stabilization increases (or decreases) after the intervention.

15 *A Caution about Identity Functions*

16 In RAM, removing a path and setting a path to zero are identical operation.
 17 However, this is not the case in XRAM. Note that the identity function for addition is to
 18 add zero whereas the identity function for multiplication is multiply by one. If there is no
 19 desired effect between predictor and outcome in a normal SEM, a regression path from the
 20 predictor to the outcome is set to zero. However, if a regression path pointing from one
 21 multiplicand to a multiplication operator symbol is set to zero in an XRAM diagram, then
 22 the other multiplicand is multiplied by zero. This may not be the intention of the modeler.

To make this distinction explicit, consider the difference between the bivariate
 additive regression displayed in Figure 1-c and the bivariate product SEM displayed in
 Figure 1-a. Making all multiplications explicit with dot symbols, the additive regression

model can be written as

$$z_i = b_1 \cdot x_i + b_2 \cdot y_i + 1 \cdot e_i . \quad (41)$$

Now if we remove the path from x to the plus operator we find that

$$z_i = 0 \cdot x_i + b_2 \cdot y_i + 1 \cdot e_i \quad (42)$$

and thus the effect of y remains. However, now consider the bivariate product model in Figure 1-a. Again, making all paths explicit, this model can be written as

$$z_i = b_1 \cdot 1 \cdot x_i \cdot 1 \cdot y_i + 1 \cdot e_i . \quad (43)$$

If we remove the path from x to the multiplication operator we find that

$$z_i = b_1 \cdot 0 \cdot x_i \cdot 1 \cdot y_i + 1 \cdot e_i , \quad (44)$$

1 and thus neither x nor y now has an effect on z .

2 *A Caution about Identification*

3 The reader may wonder why we did not use a classic interaction model with direct
4 effects as an example model to illustrate the procedure. Figure 7 illustrates why this
5 model is unidentified using the current method. The reason why it is unidentified as a
6 structural model predicting covariances is apparent by counting seven free parameters
7 while there are only six degrees of freedom in a 3×3 covariance matrix. One must watch
8 for local underidentification in models that include products of variables. The overall
9 model may be globally identified as determined by counting parameters and degrees of
10 freedom in the covariance matrix, but there still be areas of local underidentification. This
11 is the same kind of problem that occurs when one neglects to scale a latent variable by
12 fixing its variance or a loading.

13 It may be more difficult to see why identification works when one premultiplies two
14 manifest variables and creates a third variable for an interaction model. In this case there

is a 4×4 covariance matrix and so there are not negative degrees of freedom. But why is the 4×4 covariance matrix of full rank? This is due to the contributions of higher order moments introduced by the multiplication formed at the individual data rows when the premultiplication is performed. This model is identified because a product of two normal distributions is not itself normal. We expect that we can leverage individual data row moments in order to identify the model in a manner similar to that proposed by Wall and Amemiya (Wall & Amemiya, 2003) and thus provide identification for a wider category of models when full information maximum likelihood is used as a fit function for raw data.

Assumptions and Limitations

The algorithm in this article makes the strong assumption that the variables in an XRAM allow the transformation of the model into an equivalent model in which all variance sources are represented as linear combinations of independent normally distributed standardized variables. This is a relaxation of the usual SEM assumption of multivariate normality. The outcome of a product of variables will not be normally distributed. However, the XRAM assumption is that the non-normality of a product outcome variable can be completely accounted by the non-normality generated by multiplying multivariate normal variance sources. Thus, all variables that are not outcome variables must be multivariate normal. To be explicit, this means that all residuals must be normally distributed, including the residuals of variables that are outcomes of products of variables.

This article presents a novel addition to the already complex infrastructure of structural equation modeling. We acknowledge that there are many special cases that need to be tested before products of variables models can be recommended in those cases. There are cases where assumptions made by normal SEM may not be sufficient to be able to use the XRAM method.

For instance, since the XRAM relies on normality of predictor variables, ordinal predictors cannot be directly used as a multiplicand. If one has ordinal manifest variables that one wishes to use in a product, we recommend a method such as OpenMx definition variables (Neale, 1998; Neale et al., 2016). In principal, the method for estimating a normally distributed latent variable from ordinal indicators as implemented in OpenMx should be able to be used to create a latent multiplicand, but we have not yet fully tested this case.

Perhaps the biggest limitation at this point is that we have not considered all of the possibilities for how nesting will work when products of variables are used. For this reason, we cannot yet recommend the use of the likelihood ratio test as a method to test for the significance of the contribution of a product. This limitation needs to be addressed in future work. When XRAM assumptions are met, our intuition is that maximum likelihood based fit statistics will be distributed in the same way as in normal SEM models, since all variance sources will be multivariate normal. However, this remains to be proven. In order for XRAM to be widely useful, fit statistics also need to be evaluated in real world conditions—when models are misspecified and assumptions of multivariate normality of variance sources are only weakly met.

18 Conclusions

Products of variables can be included in structural equation models and model expectations can be calculated using the methods introduced here. This means that products of any combination of manifest and latent variables are now possible to implement in standard SEM software in an automated and easy-to-specify way. Three important and commonly used models that will benefit from XRAM are: models with interactions between latent variables, latent moderator models, and multivariate multilevel models. It seems evident that a large number of novel categories of models will now be

1 possible. Three of the most interesting include: i) What might be called a *factor of paths*
2 model where the coefficients in an SEM are themselves modeled as an outcome of a latent
3 variable; ii) Twin models that decompose individual-level path coefficients into additive
4 genetic, common environment, and unique variance components; iii) Moderated loadings
5 models where the measurement model for a latent variable is moderated by a latent
6 environmental context variable. We believe that the algorithm presented here represents a
7 paradigm shift for representing theories within the context of structural equation models.

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Table 1

Comparisons of simulated variance of a product and two estimators of the product variance.

	Equation 29	Equation 40
$\text{cor}(\mathbb{V}(S), \mathbb{V}(E))$	0.9991	0.9993
$\text{mean}(\mathbb{V}(S) - \mathbb{V}(E))$	-0.0293	-0.0106
$\text{sqrt}(\text{mean}((\mathbb{V}(S) - \mathbb{V}(E))^2))$	0.6464	0.5513
$\text{mean}((\mathbb{V}(S) - \mathbb{V}(E))/\mathbb{V}(S))$	-0.0009	-0.0003
$\text{sqrt}(\text{mean}((\mathbb{V}(S) - \mathbb{V}(E))^2/\mathbb{V}(S)))$	0.0179	0.0144

Table 2

Mean error, RMSE, relative error, and relative RMSE for Simulation Two.

	$Vx1$	μb	Vb	Ve
Simulated Value	1.0000	0.2000	0.1000	0.2500
$\text{mean}(Est)$	1.0009	0.1997	0.0991	0.2497
$\text{sd}(Est)$	0.0455	0.0200	0.0120	0.0081
$\text{mean}(Sim - Est)$	0.0009	-0.0003	-0.0008	-0.0003
$\text{sqrt}(\text{mean}((Sim - Est)^2))$	0.0455	0.0200	0.0120	0.0081
$\text{mean}((Sim - Est)/Sim)$	0.0009	-0.0013	-0.0089	-0.0014
$\text{sqrt}(\text{mean}((Sim - Est)^2)/Sim)$	0.0455	0.0999	0.1201	0.0326

Figure Captions

Figure 1. Three equivalent path diagrams of the bivariate regression in Equation 1. (a) Standard RAM path diagram of a bivariate regression. (b) Equivalent path diagram where the n-ary operator addition, represented by $\oplus$, is completely disambiguated from the assignment operator. (c) Equivalent path diagram when the residual term is allowed to separately add and assign its contribution to z .

Figure 2. Path diagrams of a product of two variables. The asterisk surrounded by a circle represents the product of the variables connected to it by incoming arrows. (a) Mean-centered variables with a single product term. (b) Equivalent path diagram with variances sources isolated to be independent normally distributed variables with mean zero and variance one.

Figure 3. Path diagrams of autoregression models with random coefficients. (a) Pre-detrended autoregression where participants' autoregression coefficient is extracted as a latent variable. (b) Elaborated autoregression where participants' raw scores are simultaneously detrended by a latent intercept and slope model and the detrended scores fit by an autoregression. In this second model, each participant has their own intercept, slope, and autoregression coefficient. In addition there may be a nonzero covariance between the intercept and autoregression parameter: for instance, individuals who start at a higher score may be more (or less) stable than those that start a lower score.

Figure 4. Simulated variance values versus estimated values. (a) Estimated values from Equation 29 which requires scores for the multiplicands. (b) Estimated values from Equation 40 which can be calculated using only variances and covariances of the multiplicands and thus is applicable to latent variables.

Figure 5. Percent relative error for each of the parameters in the latent autoregression simulation with uniform distributed starting values and 10,000 replications.

Figure 6. Path diagrams of two novel products of variables models that were not previously possible to specify. (a) Second order latent differential equations model with twin decomposition of the variance of the frequency and damping parameters into additive genetic, common environment, and unique components. (b) Latent variable moderation of loadings of a measurement model. (c) Random coefficients autoregression model with overall mean autoregression and variance of the individual specific components. In this case, the autoregression loadings are also factor analyzed to find a population average time varying autoregression coefficient across the four occasions.

Figure 7. An interaction model with direct effects has more parameters than statistics and thus is unidentified.

a **b** **c**

a **b**