

# Multilayer networks as embodied consciousness interactions. A formal model approach.

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Received: date; Accepted: date; Published: date

**Abstract:** An algebraic interpretation of multigraph networks is introduced in relation to conscious experience, brain and body. These multigraphs have the ability to merge by an associative binary operator  $\odot$ , accounting for biological composition. We also study a mathematical formulation of splitting layers, resulting in a formal analysis of the transition from conscious to non-conscious activity. From this construction, we recover core structures for conscious experience, dynamical content and causal constraints that conscious interactions may impose. An important result is the prediction of structural topological changes after conscious interactions. These results may inspire further use of formal mathematics to describe and predict new features of conscious experience while aligning well with formal tries to mathematize phenomenology, phenomenological tradition and applications to artificial consciousness.

Accepted for publication at *Phenomenology and the Cognitive Sciences*

Link: <https://link.springer.com/article/10.1007/s11097-024-09967-w>

Doi: [10.1007/s11097-024-09967-w](https://doi.org/10.1007/s11097-024-09967-w)

**Keywords:** Artificial Intelligence; Conscious Experience; Category Theory; Multilayer Network; Phenomenology; Radical Embodiment.

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## 1. Introduction

Mathematical consciousness science is becoming a promising research field [1]. This novel approach aims to mathematize the phenomenology of conscious experience pointing to potential applications to Artificial Intelligence (AI). This mathematical approach focuses either on observable and phenomenal aspects of experience [2–6], or on the mathematical relationships between the physical substrate and consciousness [7–9]. Although the former promises to enlighten our understanding of the formal structures of conscious experience [10,11], the latter strategy still presents reminiscences of reductionism: consciousness is explained only by virtue of physical computations, with conscious experience playing no role in such explanation.

In this article, we react by offering a general framework to study irreducible phenomenal aspects of an experience together with irreducible brain-body couplings under an enactive and embodied approach. This aligns well with discussions about naturalised and mathematised phenomenology [2,10–12]. We first wonder what are specific, but yet simple scalable common principles from cells to assemblies and from assemblies to consciousness and cognition, without reducing higher levels to the lower ones. Our tentative answer invokes compositionality and biological couplings [13], in accord with radical embodiment [14,15]. Secondly, to make these ideas precise and model these couplings formally, we translate such principles into axiomatic mathematics and network theory.

Taking a phenomenological perspective, the explanation of subjective conscious experience is given by studying its structure [16], instead of a mechanistic explanation that would claim only causal neural/cellular processes as explanatory [17,18]. By *studying its structure* we mean accounting for relational invariants across experiences [19–21], that is, several experiences sharing some structural similarities and/or having

their particular structure (e.g. temporality, colour, etc). The phenomenology of conscious experience, then, refers to the structure of relations that can be extracted from a descriptive analysis of the experience in question. Following phenomenological tradition, we assume that experience also constitutes the world around us [22,23], as a meshwork of interactions. Our approach, as well as our understanding of radical embodiment, acknowledges the phenomenological point of departure, i.e. the existence of conscious subjective experience and the need to study its structure to give a better account of science [24–26] and the world in interaction with us. Therefore, in the following, we offer descriptions and analyses of some aspects of its hypothetical structure, avoiding reduction to its physical/material constituents only. This approach also avoids potential tautological circularities: We start in experience and we end in experience, as someone would analyse social interactions through social concepts and social mechanisms, but not reducing societal behaviour to the personality of their individuals. Therefore, our explanatory target is the structure of embodied experience and our method uses mathematical concepts and descriptions, as well as acknowledges experience itself as a point of departure [22].

Our mathematical structure assumes a non-reductive set of *experiential* brain-body layers and their composition. As a mathematical consequence of the *decoupling* operation in our mathematical structure, this type of composition also imposes causal constraints in the structure, one of the most distinguished features of radical embodiment [15]. Therefore, starting from a basic phenomenology of brain and body organization in the context of conscious experience, our mathematical structure enforces mathematical conditions, that we interpreted here as the co-dependence between conscious experience and the brain-body organization.

To target these mathematical relationships, we present an algebraic interpretation of multilayer networks [27–29]. The main concepts are introduced across the text with formal definitions and simple examples<sup>1</sup>. Our point of departure is the theory of multilayer networks [27] as an extension of graph theory, being the latter the mathematical substrate, explicitly or implicitly, of the most influential neuroscientific models of consciousness today [30,31]. Therefore, multilayers might be the simplest and most natural choice to generalize mathematical relationships across brain, body and conscious experience. Moreover, multilayer networks present several advantages: they allow us to consider signals of different types, simplify seemingly complicated problems within single-network, and describe better emergent phenomena (see examples and applications in [27] and [32]).

Together, our mathematical structures present theoretical and empirical benefits to reason about experience, its biological substrate and eventual applications to AI.

## 2. Embodiment and composition

In this section, we introduce the core philosophical concepts and biological principles for our mathematical formalization.

### 2.1. Radical embodiment

Most scientific models of consciousness fall into one type of monism (e.g. physicalism, dual monism, idealism) [33] while a foundational problem of consciousness science assumes two opposed substance-ontologies, i.e. two different substances having constant properties and existing each one by itself [34,35]. On the one side is conscious experience (qualia), on the other side, is the physical body (with its structure, functions, and mechanisms).

A different approach is the metaphysically neutral standing point of continental phenomenology [26,36]. Within this stream of thought, neurophenomenology and an enactive approach convey a concrete scientific and compelling philosophical model for the biology of conscious experience [15,23,37–40]. Following Buddhist philosophy [14], this theoretical framework supposes that conscious experience corresponds to one mode

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<sup>1</sup> Readers unfamiliar with abstract mathematics will be able to grasp these benefits by understanding the examples we provide, without any lack of rigour or generality.

of bodily processes, the lived body, which in turn requires a biological living body. According to enactive readings and the connection with Husserl's phenomenology "lived" and "living" corresponds to two modes of existence of the same body. One is experienced by the same body as a way of cognition (i.e. from a first-person subjective perspective, involving several cognitive capacities) and the second is being experienced by another body (i.e. as an object of cognition from a third-person objective perspective) [13,41].

Lived and living body mutually constrains each other, having the former also "causal" influence on the latter<sup>2</sup>. Such lived body (experience) can not be reduced to only neural and physical interactions, because experience is assumed to be always embodied into a mesh of more complex biological, social and environmental exchanges [43,44]. Moreover, since the lived body and living body are two modes of the same body, it is also wrong to assume that the lived body emerges from the living body (including the brain). The expression *two modes of the same body* can be understood with a domestic artefact metaphor. A television, for example, can be either on or off, i.e. the same television has two main modes. In our case, the claim of two modes is slightly more complicated, at least regarding its ontological status and modes of existence [13,41], but in simple terms, it might be interpreted similarly. Under this framework, ontological existence is only given by interdependent transformations. In our specific case, the lived (mental) and the living (biological) body become related to each other: they are two different modes of the same existence [4,42]. Therefore, this irreducible relationship between the lived and living body is better understood as a co-dependence, i.e. in order to define one, we require the other, and vice versa [13,42].

To formalize these ideas, we require a mathematical model that takes into account: i) living properties, such as biological autonomy and embodiment, together with the possibility of describing different types of biological interactions, ii) the irreducible nature of brain-body interactions, iii) the irreducible nature of conscious experience, and iv) time-evolving systems (dynamics). One would also expect to recover and/or predict important features from a mathematical formalization, such as the constraints that conscious experience exercises in its substrate.

## 2.2. Autonomy and embodiment

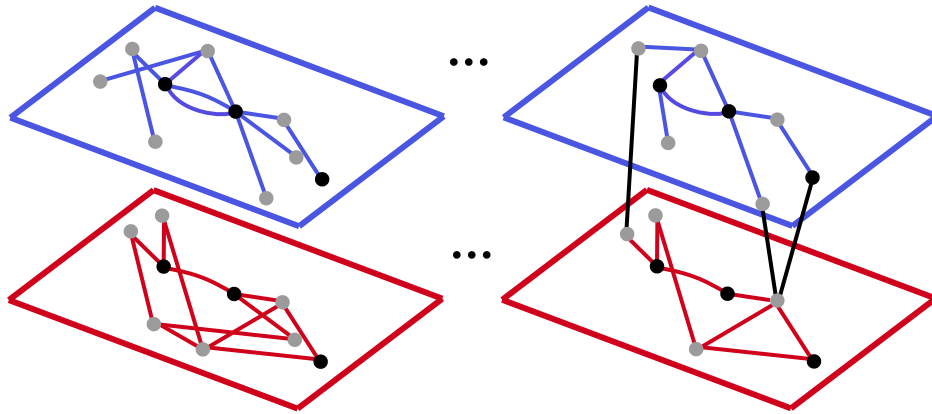
The radical embodiment approach urges us to account for the various biological processes that relate to conscious experience, without reducing it to neural and/or only physical systems [13]. The main challenge is doing so in a specific, simple and scalable way. In the rest of this article, we approximate some of these requirements by extending the recent work presented in [13,45].

In order to describe biological autonomy and the embodiment of conscious experience, we first introduce the mathematical concept of multilayer networks. We understand a multilayer as a set of multigraphs where each multigraph (or network) is contained in an independent *layer*. In Figure 1), each graph is represented by a colour, red and blue. The set of edges is partitioned into *intra-layer edges* and *inter-layer edges*. Intra-layer edges correspond to the interactions inside layers (e.g. red and blue edges, Figure 1). The inter-layer edges are those which correspond to interactions between layers (e.g. black edges, Figure 1). Nodes that appear only in one layer are depicted in grey colour, and overlapping nodes, i.e. nodes that appear in more than one layer, as depicted in black (Figure 1). This phenomenon is also called multiplexity [27]. The main advantage of multilayer networks is to decompose what seems to be a whole network into sub-networks that can be described by particular interactions. Biologically, it allows, for instance, the distinction between electrical signal propagation and neurotransmitter gradients (e.g. a map of neurotransmitter receptors) and/or physiological signals coming from different biological systems (e.g. vestibular system, visual system, immune system, etc). Within a formal and general setup, multilayers simplify the study of the coupling and evolution across complex systems of interaction.

In the following, we will assume that each of these layers represents independent or semi-independent biological autonomous systems [13]. Key conditions for biological autonomy are *self-production* and

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<sup>2</sup> Here we will understand causal influence in a broader sense, allowing structural and dynamical influences beyond direct interventions. See [42] for details.



**Figure 1.** Multilayer Structure. A multilayer corresponds to a set of graphs organized by layers. In the figure, intra-blue-edges correspond to a group of layers  $L_{blue} = \{L_{blue}^1, \dots, L_{blue}^j\}$ . Intra-red-edges correspond to a set of layers  $L_{red} = \{L_{red}^1, \dots, L_{red}^k\}$ . Some layers can interact through inter-edges, e.g.  $L_{blue}^j$  and  $L_{red}^k$ .

*self-distinction* [39,46], such that a biological autonomous system is always *operationally closed* [47–49]. In the ideal case, layers act as continuous biological membranes, which are abstracted in section 3.3 as the elements of **Biobrane**. Some of these layers might correspond to specific neural networks, while others might account for bodily systems, such as the metabolic system, the cholinergic system, immune system, among others. The elements inside layers (nodes) represent biological processes, cells, neurons, and/or their assemblies, e.g. brain regions and organs such as the gut, liver, heart, lungs, etc. In [13], these systems are introduced as functional, anatomical and metabolic membranes of brain-body organization, acting and reacting as an independent living system (e.g. two amoebas interacting).

For example, let's consider a set of elementary layers, each attached to a colour (or *aspect*) into a set of colours (e.g. types of interactions) and a time index  $t$  (not considered as an aspect and formally introduced in section 3), we then have a multilayer according to the formal definition of a multilayer network of [27]. Every layer in this network contains a multigraph in it.

A multigraph  $G$  on a set  $V(G)$  of vertices is a multisubset  $E(G)$  of pairs of elements of  $V(G)$ , called edges, with a function  $m : E(G) \rightarrow \mathbb{N}$  that calculates the multiplicity<sup>3</sup> of every edge. In the sequel, every graph will be a multigraph in which loops are not allowed and we colour edges rather than vertices (which gave different models for a multilayer network).

**Definition 1.** An *edge-colored multigraph* is a multigraph  $G$  together with a (non-necessarily surjective) function  $col : E(G) \rightarrow \mathcal{P}(\mathcal{C})$ , where  $\mathcal{P}(\mathcal{C})$  denotes the set of subsets of the colour multiset  $\mathcal{C}$ . A graph is said to be *monochromatic* if all its edges are of the same colour.

Notice that the function  $m$  gives us the number of times that two specific vertices are connected, while the function  $col$  tells us how to assign the colours in those connections. For instance, if we have two layers  $G$  and  $H$  with just two nodes  $u$  and  $v$  each and having  $k$  edges between them in  $G$  and  $k'$  in  $H$ , we obtain  $m_G(uv) = k$  and  $m_H(uv) = k'$ . Similarly, these edges could have different colours, e.g.  $col_G(uv) = \{c_1, \dots, c_s\}$  and  $col_H(uv) = \{c'_1, \dots, c'_q\}$  where  $s \leq k$  and  $q \leq k'$ .

**Definition 2.** Let  $MG(n)$  be the set of multigraphs whose vertices are indexed by the set  $\mathbf{n} = \{1, \dots, n\}$ . Now we define the set of graphs  $(MG(n), +)$  where  $+$  is multiset sum, that is, the addition of multiplicities of edges sharing the same vertices.

<sup>3</sup> A multiset is a set in which some its elements can appear more than once. Then, *multiplicity* of a multiset is the number of times a certain element appears in the multiset.

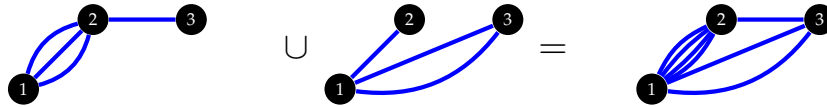
In the following  $(MG(n), +)$  is denoted simply by  $MG(n)$ . The graphs within are monochromatic and we call them indistinctly layers and graphs in this paper. We identify a layer with a 1-coloured graph, and all of the edges within every layer are assigned a single colour  $c_i$  from a multiset of colours  $\mathcal{C}$ . We now endow  $MG(n)$  with a monoidal algebraic structure (see 7.1 for details).

By endowing  $MG(n)$  with the so-called *overlay operation*:

$$MG(n) \times MG(n) \xrightarrow{\cup} MG(n)$$

we obtain a commutative monoid  $(MG(n), \cup, \emptyset)$  for which  $\emptyset$  is the undirected multigraph with  $n$  nodes and no edges and behaves as the *identity element* (i.e.  $G \cup \emptyset = G = \emptyset \cup G$ ).

**Example 3.** If  $n = 3$  we calculate  $G \cup H$  as



where the multiplicity and colour between nodes 1 and 2 are given by  $m_G(12) = 3$  and  $col_G(12) = \{blue\}$ . For the other pair of nodes:  $m_G(23) = 1$ ,  $m_H(12) = 1$ ,  $m_H(23) = 2$ ,  $m_{G \cup H}(12) = 4$ ,  $m_{G \cup H}(23) = 3$  etc. as well as  $col_H(12) = \{blue\}$ ,  $col_{G \cup H}(12) = \{blue\}$  etc.

In the following, we need to consider a set of layers. With this in mind, we construct a colour-indexed tensor product of  $MG(n)$ , each of them labelled initially with a single aspect (colour).<sup>4</sup> Elements of these tensors are called *multilayers*. That is, we *put together* a number of layers as defined above, where every colour could appear in more than one layer if they share the same aspect. The product of  $|\mathcal{C}|$  copies of the monoid  $MG(n)$  is denoted by  $MG^{\otimes \mathcal{C}}(n)$ , being  $|\mathcal{C}|$  the cardinal of the multiset of colours  $\mathcal{C}$  (the cardinality of a multiset is constructed by summing up the multiplicities of all its elements.). The elements of  $MG^{\otimes \mathcal{C}}(n)$  are  $\mathcal{C}$ -tuples of 1-coloured multigraphs with  $n$  nodes each. Every graph into  $MG^{\otimes \mathcal{C}}(n)$  is called a  $|\mathcal{C}|$ -coloured *multilayer*.

The tensor product of sets of multigraphs is constructed here by means of the usual cartesian product of monoids as

$$[MG^{c_1} \otimes MG^{c_2}](n) = MG^{c_1}(n) \times MG^{c_2}(n)$$

for the case of  $\mathcal{C} = \{c_1, c_2\}$ . In the sequel, we denote  $[MG^{c_1} \otimes MG^{c_2}](n)$  simply as  $MG^{c_1 \otimes c_2}(n)$ .<sup>5</sup>

At the initial time, every layer into a multilayer is identified both with a colour and a position in the tensor product, such as in Figure 1. They contain edges of a single colour each.

**Remark 1.** Recall that the function  $col$  from Definition 1 is not surjective. It means that some layers may have a multiplicity zero in every node. That is, we admit graphs  $G$  with empty sets  $E(G)$  for certain colours or, in other words, we interpret that monochrome layers are multicoloured layers with some missing colours.

### 2.3. Layers of experience

We now interpret layers as bringing forth an *identity* through their *agency* and convey several isolated or minimal sets of possible experiences. In this work, we understand agency as the experience of taking control of our actions, and from them, the interaction with the environment [50]. In this sense, Embodiment refers to the idea that being conscious is always understood in the context of a brain “nested within a body, which, in turn, is nested within a particular environmental context” [51]. Therefore, a conscious entity is a *being in the world* at a concrete moment and place, while its experience varies across different contexts

<sup>4</sup> This is based on the categorical networks models of [28]

<sup>5</sup> In [28], the authors explicitly describe the symmetric monoidal structure on the category of coloured network models that allow us to consider coloured multilayers.

[51]. For schematic purposes, we associate each colour with one sense, such as layers account for internal (organs) and external (sense) minimal *unimodal* experiences. Each layer by itself is a closed unity, and the experience becomes embodied by intrinsic internal/external layer definition.

Therefore, each colour represents an experiential aspect that may have an associated set of layers representing specific instances of such aspects of experience. For example, let's say that the blue edge colour represents auditory experiences. Then, particular configurations of blue aspect layers would account for specific distinguishable sound experiences like pleasant sound, loud, and noise (e.g. blue layers in Figure 1). Formally, we have a set  $L_{blue}$  of layers  $L_{blue}^1, L_{blue}^2, L_{blue}^3, \dots, L_{blue}^j$ , with  $j$  the total number of configurations for the aspect represented by the colour blue. For instance, in example 3, we have two layer configurations of the same auditory aspect generating a new more complex configuration (the result of the overlay operation). We might have several of these layer configurations depending on how much detail we add to the experiential description.

In this model, the brain and the body admit a meaningful and non-trivial decomposition into living experiential systems. Notably, to give a first mathematical account, we do not need to specify the nature of these nodes and layers. Here, we recognize that a simple and scalable principle is only the interaction between layers.

#### 2.4. Composition of layers

Layers are not, by themselves, conscious experiences such as what we, humans, encounter conscious experience. Indeed, our specific kind of conscious experience will result from the composition of these layers, as we introduce in this section. According to the embodiment interpretation, we treat layers as a minimal bodily organization (living body), as well as a minimal set of possible experiences (lived body). The fact that both are co-dependent is given by the use of the same mathematical structure. In our framework, the conscious brain-body acts as an entangled system of experiential layers.

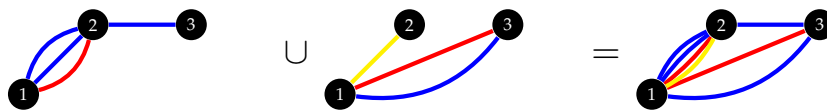
We now model operations among experiential layers and describe key features of our layers. For that, we fix a multiset of colours  $\mathcal{C} = \{c_1, \dots, c_d\}$  (from now on every set of colours will be a multiset). We proceed endowing  $MG^{\otimes \mathcal{C}}(n)$  with three different operations.

First, consider the overlay operation introduced before example 3. This operation is extended to the general case with several colours, and the algebraic structure appears as inherited from  $MG(n)$  with the overlay operation:

$$MG^{\otimes \mathcal{C}}(n) \times MG^{\otimes \mathcal{C}}(n) \xrightarrow{\cup} MG^{\otimes \mathcal{C}}(n)$$

again with addition for multiplicity.

**Example 4.** For  $n = 3$ ,  $d = 3$  with  $c_1 = \text{blue}$ ,  $c_2 = \text{red}$ ,  $c_3 = \text{yellow}$ :



In this case, a 2-coloured layer with two experiential aspects merges with a 3-coloured layer. Both multilayers have the same number of nodes. It accounts for experiences that are experienced as one single experience, without a clear distinction of all the different elements and aspects of such experiences. It brings more and new experiential relationships into a new unified experience. If each colour aspect corresponds to a particular sensory experience, we might have, for instance, blue for auditory experience, red for gustatory, and yellow for visual experience. Then, a concrete example may be the construction of experiential *things/objects* like a red sweet apple that is entirely silent. Although we can distinguish all the qualities of that "apple" (colour edge configurations), we are unable to disentangle them from the apple itself. Therefore, the overlay operation accounts for experiences that are indistinguishable or entirely mixed. The graphical example shows that the overlay operation works for layers with both the same and different experiential aspects (colour edges).

A different restriction conveys the definition of a *disjoint union operation*:

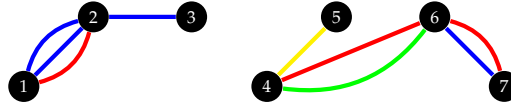
$$MG^{\otimes \mathcal{C}}(n) \times MG^{\otimes \mathcal{C}}(m) \xrightarrow{\sqcup} MG^{\otimes \mathcal{C}}(n+m)$$

with addition for multiplicity and where  $\sqcup$  works as renaming nodes.

**Example 5.** For  $d = 4$ , one 2-coloured layer and one 4-coloured layer:



with  $n = 3$  and  $m = 4$  respectively we obtain:



This operation also works for more general cases where layers have different experiential aspects (colour edges). In this specific example, the disjoint operator combines 2-coloured and 4-coloured layers in one new single experience with separate elements inside such an experience. Although the new experience is one unified experience (a new multigraph), we can distinguish the elements from the original layers, i.e. they are experienced in parallel. Taking the comments about particular configurations from section 2.2, the first multilayer might correspond to an auditory and gustatory experience (e.g. lower sound and sweet), while the second multilayer to auditory, visual, gustatory, and olfactory experience, where now green represent smell. Then, different configurations account for particular instances of such aspects of experience. For instance, the second multilayer might be a high sound, red, bitter and fruity experience. Both multilayers combine to give a new more complex experience. In an unimodal set-up, the disjoint operation also accounts for cases where we experience the same aspect simultaneously (e.g. bitter and sweet when tasting orange marmalade), such that we are able to distinguish them.

**Remark 2.** Observe that  $MG^{\otimes \mathcal{C}}(n)$  and  $MG^{\otimes \mathcal{C}}(m)$  contain, as maximum, all the edges of the same colours since  $\text{col}$  is not necessarily surjective (that is, not every colour is necessarily present in every layer). This is because one or more of the factors into a product  $MG^{\otimes \mathcal{C}}(-)$  could be the empty set on edges.

Finally, we define an operation that subsumes our previous  $\sqcup$  accounting for irreducible entangled experiences. In this case, “irreducible entangled experiences” speak of the impossibility of disentangling or fully separating all the aspects of a given experience. For example, an experience of a red apple, involves the colour of the apple and the apple, shapes, etc, but also a certain degree of temporality, memory, and others. Our framework acknowledges that in the mixing of layers, there are always partial descriptions of the structure of an experience that needs to be instantiated in advance by the scientist. Whether such mixing of layers is expected to decompose later becomes an important constraint in the mathematical structure (see section 3.3 for details).

**Definition 6.** Let  $\mathcal{C}$  be a fixed set of colours and  $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}$ . For every  $s$ -coloured layer  $G$  in  $MG^{\otimes \mathcal{C}_1}(n)$  and  $q$ -coloured layer  $H$  in  $MG^{\otimes \mathcal{C}_2}(m)$  the operation:

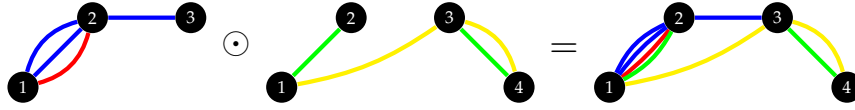
$$MG^{\otimes \mathcal{C}_1}(n) \times MG^{\otimes \mathcal{C}_2}(m) \xrightarrow{\odot} MG^{\otimes \mathcal{C}_1 \cup \mathcal{C}_2}(n+m-p)$$

produces a new  $(s+q)$ -colored layer  $G \odot H$  in  $MG^{\otimes \mathcal{C}_1 \cup \mathcal{C}_2}(n+m-p)$  where  $p = |V(G) \cap V(H)|$ .

Taking our example of  $G$  and  $H$  as layers, we now have  $m_{G \odot H}(uv) = k + k'$  and  $\text{col}_{G \odot H}(uv) = \{c_1, \dots, c_s, c'_1, \dots, c'_q\}$ , where some of the colours may repeat.



**Example 7.** For  $n = 3, m = 4, s = q = 2$  and  $p = 3$ :



Now we combine all the notions introduced so far while keeping colours associated with different sensory modalities<sup>6</sup>. Here,  $\odot$  combines multilayers to account for an experience that combines all modalities, as experiences mixing them in entangled forms. This new composition is an edge-coloured multilayer with multiple aspects. In example 7, the result of  $\odot$  gives a new edge-coloured multilayer with four aspects (e.g. audition, taste, smell and vision). This particular  $\odot$ -direct product of multigraphs is chosen as a way of preserving all edges and nodes from original layers, whenever forming new multigraphs.

That is, several experiential layers compose to model simultaneous experiences or multi-experiential layers where some elements may be distinguishable and others not (disjoint and overlay operation respectively). This new experience is unified but given by both, parallel and mixed experiences accounting for various experiential elements at the same time. Blue layers are bitter or sweet according to their configurations, different red layers correspond to particular smells, and we merge layers to account for complex combinations of further experiences. This operation becomes a more general case, from which we emphasise later the possibility of obtaining a meaningful decomposition (see section 3).

Note that, in the sense of [27] and section 2.2, all three operations above do not generate any inter-layer edge among layers but rather the intra-layer edges of a  $(s + q)$ -aspect layer from the previous  $s$ -aspect and  $q$ -aspect, in the case of  $\odot$ . In other words, our layers are composed in such a way that the intra-interaction of one layer becomes part of the intra-interactions of the other layer, and vice versa.

This is a relevant difference from other types of multilayer models since it accounts for the embodiment of different systems forming a new systemic whole. Its importance relies not only on the description it gives of the difficulty to isolate brain-body interactions under states of conscious experience but also because it tells us how monochromatic unisensory layers combine to construct experiences involving several aspects and modalities: One can identify a single  $|\mathcal{C}'|$ -coloured layer into a  $|\mathcal{C}|$ -coloured multilayer for  $\mathcal{C}' \subseteq \mathcal{C}$  from the interaction through  $\odot$  among all the different layers. Then,  $\odot$  shows the way by which a multilayer can combine minimal experiences by the process of *merging layers*.

**Remark 3.** Observe that, whenever  $n = m$  and  $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ ,  $\odot$  reduces to the internal operation  $\cup$ . In other words,  $\odot$  is a generalization of  $\cup$ . In fact, we may have considered  $\odot$  as the internal operation for that the different  $MG^{\otimes \mathcal{C}}(n)$  become monoids. One should note that not only  $n$  is fixed but also  $\mathcal{C}$ , where the latter appears as a set of colours where some of the elements could be missing.

### 3. Dynamical interactions

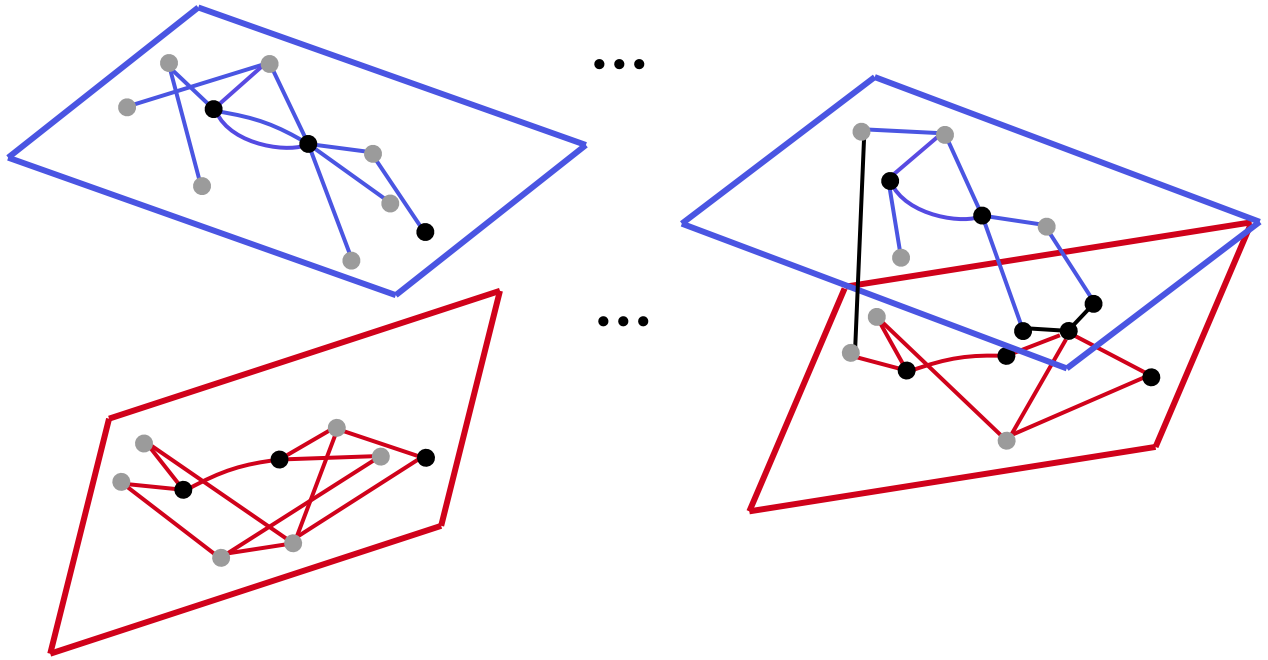
The final requirement for our model is to incorporate dynamics. The addition of this feature gives a richer account of different states and contents of conscious experience. Moreover, it allows us to predict causal constraints imposed by conscious interactions.

#### 3.1. Rotation layers

Rotation layers are mathematical abstractions to account for functional and dynamical overlapping regions and networks [52,53] (Figure 2). This dynamical configuration is described by multigraphs with a *rotation angle*. This angle is time-indexed, making every layer dependent on a time parameter, it allows us to formalize agency as a will to interact with other layers. Therefore, bigger angles represent a greater

<sup>6</sup> Bear in mind that modalities can be generalized to more aspects than only the five senses.





**Figure 2.** Rotation Multilayers. Layers are allowed to interact changing their angles and forming a multilayer interacting structure.

willingness to interact. We formally introduce the notion of a multigraph endowed with a rotation angle as follows:

**Definition 8.** A *rotation graph* is a pair  $[G, \alpha]$  where  $G$  is an edge-coloured multigraph and  $\alpha \in [-\pi/2, \pi/2]$ .

Every rotation graph in our model is now a layer with the ability to rotate. In time  $t = 0$  every layer into a multilayer contains a unique colour and the layers are in parallel non-interacting position, such as exhibited in Figure 1. Given an interval  $T \subseteq \mathbb{R}^+$ , we consider angles in the form of continuous functions  $\alpha : T \rightarrow [-\pi/2, \pi/2]$ . Then:

**Definition 9.** Two rotation graphs  $[G, \alpha]$  and  $[H, \beta]$  *interact in a time  $t$*  if  $\alpha(t)$  and  $\beta(t)$  have a different sign, considering zero as a sign in itself, and  $|\alpha(t)| + |\beta(t)| \geq \pi/2$ .

Since we are interested in layers for which the interaction in definition 9 takes place, one would like all the operations for multigraphs in section 2.4 to be subject to that condition. That is, every operation is a *partial operation* (see formal definition in 7.3). Partiality means that  $\oplus$  is an internal operation for a set when *some condition* has been specified. In our case, let be the sets

$$RMG^t(n) = \{\text{rotation graphs } [G, \alpha] \text{ with } |V(G)| = n \text{ and rotation angle } \alpha(t)\}$$

by setting  $RMG^0(n)$  to contain only 1-coloured multigraphs when there is no interaction.

The above allows us to consider multilayers with rotation multigraphs on their constituent layers. Every element of  $RMG^t(n)$  is then called a *rotation layer* or simply a layer. A *rotation multilayer network* for multigraphs with coloured edges is

$$[RMG^t]^{\otimes \mathcal{C}}(n)$$

i.e, a product of  $|\mathcal{C}|$  copies of  $RMG^t(n)$ .

Every element of  $[RMG^t]^{\otimes \mathcal{C}}(n)$  is called a  $|\mathcal{C}|$ -coloured rotation multilayer or simply a  $|\mathcal{C}|$ -coloured multilayer. A more general form of coloured rotation multilayer is

$$[RMG^t]^{\otimes \mathcal{C}_1}(n_1) \times \dots \times [RMG^t]^{\otimes \mathcal{C}_k}(n_k)$$

We consider an internal operation on every  $[RMG^t]^{\otimes \mathcal{C}}(n)$  that makes use of the previous operator  $\cup$ :

**Definition 10.** For every interacting pair of layers  $[G, \alpha]$  and  $[H, \beta]$  in  $[RMG^t]^{\otimes \mathcal{C}}(n)$  we define

$$[RMG^t]^{\otimes \mathcal{C}}(n) \times [RMG^t]^{\otimes \mathcal{C}}(n) \xrightarrow{\cup} [RMG^t]^{\otimes \mathcal{C}}(n)$$

where

$$[G, \alpha] \cup [H, \beta] = [G \cup H, \min(\alpha(t), \beta(t))]$$

This operation endows  $[RMG^t]^{\otimes \mathcal{C}}(n)$  with the structure of a *partial monoid*, whose identity is given by  $[\emptyset, \pi/2]$ , where  $\pi/2$  means the undirected multigraph with  $n$  vertices and no edges and  $\pi/2$  is the constant function.<sup>7</sup>

Finally, the  $\odot$  operation in the non-rotation case is extended to the rotation (rotation operators are denoted with bold and slightly bigger symbols) and it is partial (it works only in the case that layers interact). In a time  $t > 0$  rotation layers compose as follows:

**Definition 11.** For every interacting  $s$ -coloured layer  $[G, \alpha]$  in  $[RMG^t]^{\otimes \mathcal{C}_1}(n)$  and  $q$ -coloured layer  $[H, \beta]$  in  $[RMG^t]^{\otimes \mathcal{C}_2}(m)$

$$[RMG^t]^{\otimes \mathcal{C}_1}(n) \times [RMG^t]^{\otimes \mathcal{C}_2}(m) \xrightarrow{\odot} [RMG^t]^{\otimes \mathcal{C}_1 \cup \mathcal{C}_2}(n + m - p)$$

produces a new  $(s + q)$ -coloured layer containing a graph

$$[G, \alpha] \odot [H, \beta] = [G \odot H, \min(\alpha(t), \beta(t))]$$

in  $[RMG^t]^{\otimes \mathcal{C}_1 \cup \mathcal{C}_2}(n + m - p)$  where  $p = |V(G) \cap V(H)|$ .

Differently than in the initial time, layers are not tied to any particular position in the tensor product. The final number of vertices after a long product such as

$$[G_1, \alpha_1] \odot \dots \odot [G_k, \alpha_k]$$

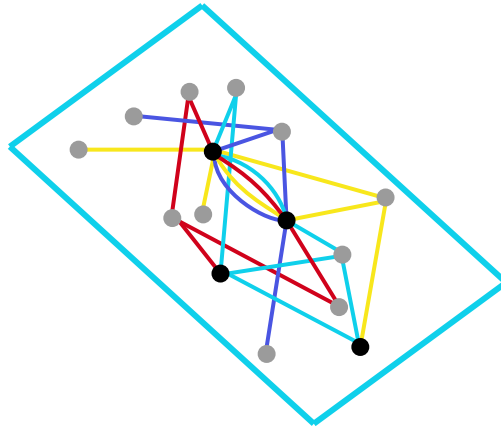
comes from an application of the *Inclusion-Exclusion Counting Principle* (see section 7.7 for details).

### 3.2. Content structure for dynamical experiences

In the dynamical case, every rotation layer mixes colours as in the previous section 2.4. However, this new mixing involves the notion of *coupling*: the  $\odot$  commutative operator preserves the individuality of layers' colours at the cost of losing their angle independence. Coupling of layers, in this sense, signifies the entangled interaction between layers, such as they are indistinguishable from each other, up to a certain degree (given by the angle of interaction) [13]. We represent the result of dynamical coupling and possible transitions by the picture in Figure 3. Note that in the rotation, new edges (and possibly new colours) appear in a layer only after the interaction of two layers where the union  $\mathcal{C}_1 \cup \mathcal{C}_2$  is the multiset colouring  $G \odot H$ .

In this picture, edges whose constituent nodes appear in more than one layer (coloured more than once) are allowed. They are called *coupling edges* and admit an underlying *coupling graph* containing a recurrent configuration of the system. These nodes should be seen as part of a *core structure* of the brain-body. That is, in the extreme case that a coupling edge is coloured with all colours into  $\mathcal{C}$ , we have a pair of nodes that participate in every conscious operation that such a system might have. In fact, as a purely mathematical consequence of our formalization, the existence of a core structure for conscious experience appears as given by nodes that are coloured by all coloured edges.

<sup>7</sup> The operator  $\cup$  is also useful to check that our previous 1-coloured sets  $RMG^t(n)$  are *partial commutative monoids*.



**Figure 3.** Consciousness interaction and phenomenology. After all transitions and compositions take place, we end up with one multi-colour layer. Different dynamical layer configurations may represent different phenomenological instantiations of conscious experience.

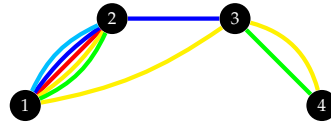
**Postulate 1. Core structure.** A pair of nodes that are coloured by coupling edges with all colours in  $\mathcal{C}$  define a *core structure* for conscious experience. The core structure for a multilayer is the subset of all nodes such that whenever they are connected, the connection consists of at least one colour edge in  $\mathcal{C}$ .

Formally:

**Definition 12.** Let  $\mathcal{C} = \{c_1, \dots, c_d\}$  a set of colors,  $\mathbf{L} = \{L_k\}_{k=1}^d$  a set of layers and  $[G, \alpha] \in [RMG^t]^{\otimes \mathcal{C}}(-)$  a rotation graph obtained by the interaction of graphs in  $\mathbf{L}$  and for which every color in  $\mathcal{C}$  is present at least once. Its *core structure* is the subset  $V(Core) \subseteq V(G)$  where *Core* is a subgraph of  $G$  such that for every  $(uv) \in E(Core)$  one has  $col_{Core}(uv) = \{c_1, \dots, c_d\}$ .

Observe that not every multigraph obtained through interactions contains a core structure in it. The following does, indeed.

**Example 13.** For  $d = 5$  with  $c_1 = \text{blue}$ ,  $c_2 = \text{red}$ ,  $c_3 = \text{yellow}$ ,  $c_4 = \text{green}$ ,  $c_5 = \text{lightblue}$  and the following multigraph  $G$ :



its core structure would be the set  $\{1, 2\} \subseteq V(G)$ .

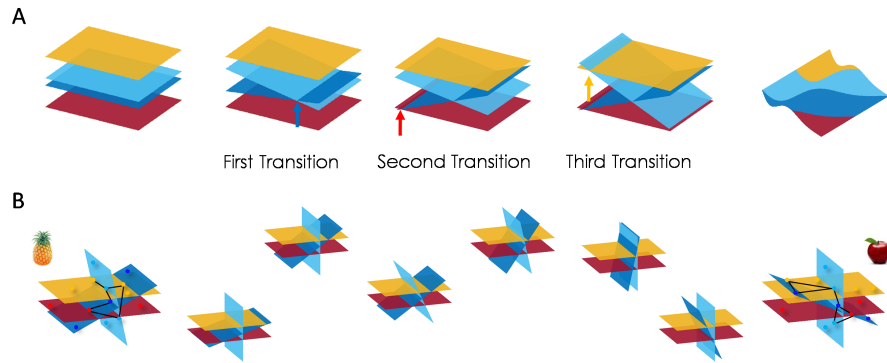
These nodes might correspond to observed brain-body regions whose contribution to the conscious experience is diverse (section 4). In our model, the core structure is a consequence of conscious processes interacting, as new coupling edge activities appear. This is different from other models claiming core structures for consciousness (e.g. global workspace, dynamical core, etc). In such models, the core structure is the *effective cause*, or the one from which conscious experience would *emerge*. That is not our case here since, as discussed also in section 4, the core structure is not the starting construct but one of the more recurrent configurations produced by the interaction of former processes which are already conscious.

Furthermore, the content of conscious experience is given by dynamical configurations of rotation layers interacting and composing whole systems. One toy example is shown in the example 13. Nodes in the coloured layers are part of the different processes according to their network affiliation at a given time. As explained, in certain cases, they define a dynamical core structure that we interpret as co-arising with the phenomenal content of conscious experience.

**Postulate 2. Phenomenal content.** The aspects, configurations and degree to which layers interact with each other co-arise with the dynamical structure and content of experience. Formally:

- if  $uv$  is an edge in a monochrome layer, the value of  $col_G(uv)$  accounts for a single colour relating to a unisensory experience (e.g. gustatory layers in which each configuration corresponds to salty, bitter, sweet, etc.)
- if  $uv$  is an edge in a multichromatic layer, the value of  $col_G(uv)$  accounts for a subset of the more general set of diverse experiences (e.g. salty, green, loud, etc, altogether).

Informally, an interaction for which  $|\alpha(t)| \geq \pi/4$  or  $|\beta(t)| \geq \pi/4$  is viewed as a *strong interaction*, while a situation such as  $|\alpha(t)| = \pi/2$  and  $|\beta(t)| = 0$  is interpreted as a *total interaction* for  $\alpha$  while  $\beta$  stays in a non-interacting position or, geometrically, as  $\alpha$  being *perpendicular* while  $\beta$  stays *parallel*. The in-between configurations represent the dynamical evolution of these layers (Figure 4). For example, in Figure 4, the blue layers change their position while the others remain fixed. These changes may inform about different content states. The experience dynamically evolves and the whole system may feel in one or another way. The structure and content of experiences depend on the degree (strong, total, etc), aspects and configurations of layers interacting but this structure and content of experience also affect and defines the degree, aspects and configurations of the interactions on which they depend, i.e. these layers, as modes of lived and living, co-determine each other definition.



**Figure 4.** Phenomenal content. **(A)** Different layers may dynamically interact generating different transitions and local-global new multilayers. First transitions may correspond to wakefulness, followed by a transition representing phenomenal experience, reflective and pre-reflective awareness, and other transitions related to more complex phenomenal content structure. **(B)** Different dynamical layer configurations may co-arise with the phenomenology of conscious experience, understood as the structure of relations that can be extracted from a descriptive analysis of the experience in question. Here is an example of layer configuration from the content of pineapple to apple, and their dynamical changes. The colour of every layer represents the colour of the edges contained in it.

### 3.3. Splitting layer of experience

After showing how layers combine we address the issue of splitting layers from a certain configuration. For this purpose, we take a particular decomposition of graphs for which we recover all the nodes involved in the interactions. As we will see, we compose using  $\odot$  but decompose using  $\cup$ .

Once we have at our disposal infinite sets of rotation graphs in the form  $[RMG^t]^{\otimes \mathcal{C}}(n)$ , we aim to isolate a particular subset in the initial time, giving a formal account of the experiences and thus performing our model for brain-body. This will consist of a set of sets of layers, monochrome in time zero and ready to merge through  $\odot$ , producing richer multicoloured experiences. In order to split those experiences, the mathematical structure requires a set of indecomposable experiences which turn out to be unique up to the rotation angle (see 2.4).

**Definition 14.** Consider a fixed  $\mathcal{C} = \{c_1, \dots, c_d\}$ . Let be **Biobrane** =  $\{B_k\}_{k=1}^d$  where:

- every  $B_k = \{B_k^i\}_{i=1}^{m_k}$  is a set of layers with cardinality  $m_k$  and
- every multigraph  $[G, \alpha]$  in a layer  $B_k^i$  belongs to  $[RMG^0]^{c_k}(n_i)$  where  $n_i$  is the number of vertices.

That is, **Biobrane** is the initial multilayer of experiences (minimal indecomposable and irreducible set of experiences) where no interaction has taken place yet. In a certain time  $t > 0$  interacting layers merge into multicoloured multigraphs  $[RMG^t]^{c_1 \otimes \dots \otimes c_d}(p)$  (Section 3.1). Then, let  $\{[G_1, \alpha_1], \dots, [G_m, \alpha_m]\}$  be the set of irreducible layers or *atoms* such that

$$[G, \alpha] = \odot_{j=1}^m [G_j, \alpha_j]$$

where every  $[G_j, \alpha_j]$  belongs to a certain  $B$  into **Biobrane**.

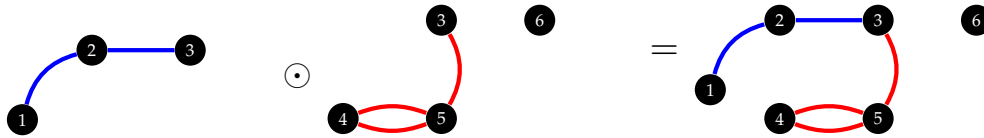
Now, given these  $[G_j, \alpha_j]$  in **Biobrane**, there exists a decomposition set  $\{[H_1, \beta_1], \dots, [H_m, \beta_m]\}$  such that

$$[G, \alpha] = \cup_{j=1}^m [H_j, \beta_j]$$

where every  $[H_j, \beta_j]$  belongs to a certain  $[RMG^t]^{c_k}(n)$  such that  $t > 0$  (See section 7.8 for details). Some essential features of this decomposition are:

1. every  $[H_j, \beta_j]$  is an atom for a monoid  $[RMG^t]^{c_k}(n)$  since every graph was constructed out of atoms in **Biobrane**,
2. edges in every initial layer appear together to the end in every further combination and
3. given a set of atoms in **Biobrane**, the decomposition described is unique up to rotation angle.

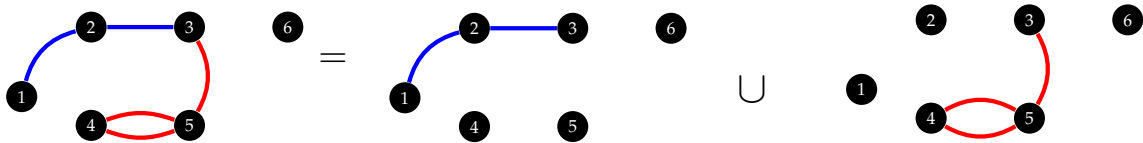
**Example 15.** We take, for the sake of clarity, *usual sets* (that is, no multisets)  $B$  in **Biobrane**. Merge two graphs by using  $\odot$  as:



where the left-hand term is  $[G_1, \alpha_1] \odot [G_2, \alpha_2]$  for interacting:

- $[G_1, \alpha_1]$  in  $[RMG^0]^{c_1}(3)$
- $[G_2, \alpha_2]$  in  $[RMG^0]^{c_2}(4)$

and  $c_1 = \text{blue}$  and  $c_2 = \text{red}$ . Now, starting from the merged multigraph, we make use of  $\cup$  to split it as:



where the right-hand term is  $[H_1, \beta_1] \cup [H_2, \beta_2]$  for non-interacting:

- $[H_1, \beta_1]$  in  $[RMG^t]^{blue}(6)$
- $[H_2, \beta_2]$  in  $[RMG^t]^{red}(6)$

with  $t > 0$  and  $|\beta_1| + |\beta_2| < \pi/2$ . Observe that this satisfies

$$[G_1, \alpha_1] \odot [G_2, \alpha_2] = [H_1, \beta_1] \cup [H_2, \beta_2]$$

#### 4. Implications and discussion

Across previous sections, we introduced a simple algebraic model of layers and multilayers representing experiential and biological brain-body systems. In the beginning, each system is constituted by a set of

one-colour graphs, each contained in a layer. These multilayers can interact in various ways, via composition, disjoint union, and also via a rotation mechanism. This framework allows us to reason about the brain-body organization and describe key configurations related to human conscious (lived body) and unconscious (living body) experiences.

#### 4.1. Irreductive experiences

In our formalism, monochrome layers represent both minimal experiences and living bodily systems, such as richer forms of the former are given by the coupling of the latter. Indeed, conscious and unconscious experiences are described as two modes or configurations of the same system, here represented by a group of layers in a multilayer structure. Therefore, conscious experience would co-arise with the interaction of these biological layers, such that intra-edges of one layer become new intra-edges in a richer layer via the  $\odot$  operation.

It is important to notice that the lived experience (mind) and living experience (body) do not convey an identity theory. Identity theories claim that conscious experience either emerges from the biological configurations of the body or identically corresponds to those configurations, following one-to-one relationships. Therefore, explaining a particular conscious experience is equivalent to explaining the particular configuration, either in the brain or the full body. This approach is ultimately intractable since it is impossible to track simultaneously all the elements in such systems and consequently determine a unique state of it (see [54,55]).

Contrary, enaction and radical embodiment would claim that the lived body is neither reduced to, nor emerges from the living body, but both are dynamic modes of the same existence. This approach does not imply that describing all "free parameters" of a living system will end in a unique description of conscious experience. Quite the contrary, the embodiment implies that this is never the case. The degeneracy of physiological systems (the fact that multiple micro-configurations end up in similar global or macro-configurations) and the irreductive aspect of experiential decompositions do not allow such methodological reduction. In radical embodiment, the description of the system and experiences are given by a mutual or contextual constraint, i.e. the decompositions are chosen together with the experiences we want to describe, and therefore, the decomposition is co-dependent with such experiences. Then, in our mathematical model, these configurations become irreducible to each other. Therefore, conscious lived experience does not emerge from these interactions. It is just described as a more complex composition of layers, such that we can incorporate the transcendental intuition: "lived experience is always already presupposed by any statement, model, or theory, and the lived body is an a priori invariant of lived experience" [39].

#### 4.2. An enactive approach to conscious experience

Layers model biological autonomous systems and irreductive experiential units that bring an agency through the sense of the environments, significance and value of their interactions (self-making) [38,50,51]. These systems are "minimally decomposable", or nondecomposable at all [15]. It means that layers represent global processes that subsume their components such that they are nonseparable. For example, local neural activity might emerge from the whole dynamic patterns of large-scale activity as much as the whole might emerge from their local components [39]. The distinction between parts and whole vanishes. Moreover, our mathematical results point out that these layers are contextual decompositions (section 3.3), but not ontological ones. In other words, we emphasise the irreducibility of relations where layers become patterns that exist in relationship with each other: all are equal interacting processes and none has ontological primacy over the others.

This ontological neutral approach, however, presents several advantages. From very simple, yet scalable principles, we obtain several important results. For example, the postulate 1 is a consequence of the formal treatment and deeper assumptions. Several models of consciousness, to more or less extent, also claim core structures. Some examples are the global workspace (GW) [56], the global neuronal workspace (GNW) [57], the dynamical core [58], and the memory evolutive neural systems (MENS) [59], among many others. All these models treat conscious experience as emerging and reduced to the core structure, while this

structure is postulated as a key initial assumption. One exception is MENS, where the core structure is also a consequence of the mathematical formalization. Enaction and radical embodiment, however, would claim that the core structure arises as a consequence of embodiment and it is not an initial assumption, contrary to GNW and others.

The postulate 2 is akin to the relationships between biological substrates and conscious experience presented in the dynamical core and integrated information theory (IIT) [60]. For example, through the definition of the "maximum irreducibility of cause-effect repertoires" and their "complexes". Concretely, the variety of layers and their couplings would trigger the complex configurations and global brain-body dynamics observed in awake conditions (e.g. functional connectivity patterns). Nevertheless, the radical embodiment does not claim an identity between physical substrates and experience (section 4.1) nor is a computational theory reducing body interactions to only one type of neural interaction. In radical embodiment, layers are contextual upon observers, making the framework more general, but also more tractable than IIT's current methodology [60,61].

Our formal model of enaction and radical embodiment is a general framework for the study of conscious experience that also subsumes certain aspects of other models of conscious experience, acknowledging biological and environmental constraints. Therefore, another advantage of our approach is that it seems to be a model which is more complete than proposals in the field that do not naturally subsume other models' key aspects.

#### 4.3. Causal experiential constraints

As a mathematical consequence of the irreducible nature of conscious experiences, such experiences impose causal constraints in the layer system. These effects are given by the splitting layer constraint, predicting a type of interacting memory after layer composition. Specifically, the new nodes after the splitting process may be interpreted as a memory of the layer-interaction that we further understand as a causal constraint of conscious activity into the biological substrate [62].

To detail the consequences of section 3.3, we note that, although splitting layers in  $[RMG^t]^{\otimes C}(n)$  is also possible, we are not reaching the same original picture we had at time  $t = 0$ . Indeed, we are recovering all the edges in separate layers at the cost of carrying a copy of every node that is found in the *maximum-coloured* multilayer from which they arise. That is, after the splitting process every monochrome layer appears with the same edges as it had, but also carries a copy of the vertices in the layers with which interaction has taken place. With the notation of section 3.1, such a new number of vertices is the result of an application of the Inclusion-Exclusion Counting Principle. In short, we make use of the operation  $\odot$  for merging layers but in the process of splitting them back, they need to appear as a  $\cup$ -product.

This mathematical constraint fits well with constraint types of explanation [42] and perhaps with the idea of causal experiential efficacy, sometimes called downward causation [39,63]. The multilayer structure requires constraints as causal second-order topological and contextual restrictions in the individual layers that constitute the whole system. Importantly, in our mathematical treatment, this is not a "force" that acts on the biological system, but the mathematical consequence of interconnectedness or relatedness among layer processes after composing and splitting.

In summary, the model predicts that layers taking part in conscious experience compound in such a way that the intra-interactions of one become intra-interactions of the others, and quite surprisingly, it also predicts that the interaction associated with conscious experience constrains the biological substrate, imposing a sort of structural memory after conscious interactions.

#### 4.4. Implications for neuroscience of consciousness

Once the reciprocal actions are in place through composition and coupling layers, each layer receives the influence of the other via the new coloured edges. Because we impose that these interactions need to have experiential meaning, each layer might care about its and others' processing, as part of its biological/experiential requirements and the other layer's requirements, which can directly affect them. It should be seen in direct analogy to how any living being acts and reacts to different types of contexts,



ensuring its survival. In other words, layers ultimately act as biological entities through their agency, and not only as "informational units".

As a consequence, any biological entity, after interactions with its pairs, also needs periods of non-interaction (e.g. by means of sleep or rest). The whole system and its parts try to keep their balance, coupling and decoupling from time to time, as part of their own biological/experiential demands. This decouple correlates with a reduction of awareness even under global awake conditions. Interestingly, reduction of brain activity seems to be present between two moments of consciousness [64,65], as some dynamical brain analyses associated with unconsciousness predict during conscious resting-states [66]. Our theoretical model explains this reduction through the natural and biological need to go back to the intrinsic dynamics of splitting layers. In other words, this reduction is due to the decoupling of layers, or the recovery of intrinsic layer dynamics against the detriment of interactions that would interfere with those intrinsic layer dynamics. In our model, all these imply that if one process layer disappears, the awareness associated with that process also disappears.

This consequence of the model has implications and become a potential explanation for conditions such as sleep, disorders of consciousness and anaesthesia. In all these conditions, where conscious experience is globally modified, we would expect different layer configurations and coupling interactions. The configuration of layers during deep sleep might be similar to anaesthesia and vegetative state, while during REM sleep, these layers' configuration could be similar to minimally conscious states (MCS) and vice versa. In all these cases, however, the potential impairments of layers might also be different case by case, such as one anaesthetics act in different pharmacological pathways (i.e. impairments during anaesthesia may affect Layer A, but under MCS conditions, the impairment might be partial or total in another Layer B). The great advantage of our approach is subsuming all potential, specific, and arguably "different" mechanisms in one overall comprehensive metaphor: coupling and decoupling layers interacting.

Finally, in other impaired neurological conditions such as frontotemporal dementia, in which the content and way of experiencing are modified (e.g. personality and behavioural changes), the coupling and decoupling of bodily layers might, through allostatic overload for example, impact the coupling and decoupling of layers' usual configurations [67]. These multilevel layers and their corresponding measurements might offer a global profile for different experiential conditions.

#### *4.5. Implications for AI and artificial consciousness*

Phenomenology calls for a suspension of our assumptions and careful reading of our system of beliefs, to account for the "how" things appear to us [21,23]. In this line, the possibility of synthetic conscious experience is open to philosophical, mathematical and experimental investigations. It can not be discarded in the first place. The issue is also linked with animal consciousness. The default position nowadays is slowly moving towards the consideration that all animals are conscious, in their respective particular ways [68,69]. Since our framework is based on axiomatic mathematics, it resonates with the suspension of the natural attitude [5,6,70], questioning hidden assumptions and making them explicit through formal definitions and instantiations of concrete mathematical structures. In this line, axiomatic mathematics and our framework may help to formalize, generalize and make explicit key assumptions to study the boundaries of our subjective conscious experience, applications and implications to AI.

For instance, the whole project of a synthetic conscious system is far from evident. According to [71]: "It is still unclear whether building a machine with the behavioural advantages of a conscious machine will entail building a machine with real feelings and real phenomenal experience. Most of the people currently working in the field of Artificial Consciousness would be agnostic on this topic." Accordingly, our model, indeed, can accommodate the main possibilities: i) there is nothing unique regarding conscious experience within biological systems (i.e. conscious experience can be instantiated in artificial systems), ii) there is something unique regarding conscious experience within biological systems (i.e. there is no conscious experience beyond biology), iii) there is something unique regarding conscious experience within biological systems that can be extracted and instantiated as synthetic conscious experience [72,73]. In all these cases, layers can be implemented to test and study, mathematically, the main consequence of these general hypotheses.

In the first case, attempts to design a synthetic conscious system point mainly to the existence of a conscious experience that is different from the conscious experience of humans. This leads to the distinction between *phenomenal* and *access consciousness* [74], being the second amenable to implementations based on information availability. This can be modelled and studied through the metaphor of a “global workspace” [57]: the global availability of information through the workspace “refers to a self-referential relationship in which the cognitive system can monitor its processing and obtain information about itself” [75]. In our model, a global workspace corresponds to nodes that appear recurrently across layers of interactions. Other attempts for access consciousness incorporate a series of “incremental layers connecting perception to action” which is formalized to design functional units in the context of what was called a “subsumption architecture” [76]. It entails a hierarchical structure for which higher levels can subsume lower levels by a sort of *integration* or *combination*. This operation resembles the process of mixing layers in our model (although, we do not need reference to any type of hypothetical hierarchy). Finally, in [77] a synthetic model of conscious experience is built in the context of an “attention schema theory”, including a layered set of cognitive models, the schema layer the one where phenomenological features are implemented. The attention schema refers to a conscious artefact having “a parallel set of information processes that can objectively monitor and report on how the whole system is put together” [78]. Our model can incorporate all these architectures avoiding, as well, their reductive stand.

As pointed out in this article, conscious subjective experience cannot be treated as a merely emergent property, since there is no way to track all the possible cellular, neural, and environmental contexts from which a determined experience may arise. In our paper, and following the enactive and radical embodiment approach, we consider conscious and unconscious to be modalities in transition of the same existence. As argued, this points to the suppression of the boundaries between the parts (body) and the whole (experience), where none of the processes have priority over the others. Additionally, the observer, i.e. the subjective experience of the scientist, plays a crucial role in instantiating a layer model (differently than other models). It is, therefore, in the interaction across layers itself, but also up to the definitions of the layers by a scientist, where conscious activity lies. Under these assumptions, the project of implementing a synthetic artificial consciousness has to consider a form of that interaction.

Following, in the second and third cases, one would argue that biological properties such as plasticity, adaptation, and autonomy, among others, play key roles in the unfolding of conscious experiences. In this case, the assumption is a life-mind continuity thesis [79], in which experience is assumed at the level of cells and gets complexified on the aggregation of their units. This is, for example, the stand of some proponents of radical embodiment. On the one hand, we can assume these properties can not be disentangled from biological substrates. On the other hand, we could hypothesise that this is possible (and experiment with different artificial synthetic structures that replicate such biological properties). In both cases, our framework offers mathematical tools that complement and formalize the study of living systems, such as hypothesising what elements are missing or left over in our current simple structure. The motivation for our work was this in the first place (see section 2).

In any of the cases above, researchers on *synthetic phenomenology* would argue that “addressing questions about the phenomenology of perceptual experience has great advantages for the computational description, the algorithmic design and the implementation of artificial agents” [80]. The present paper shares in its way the same target: several current and practical projects aligned with the ideas above are related to, and can benefit from our formalism, making our formalism a meta-framework for reasoning about hypothetical structures of conscious experience and eventual applications in AI.

## 5. Conclusions

Across this article, a simplified version of enactive and radical embodiment was introduced. Their main concepts were generalized by using symmetric monoidal categories out of sets of multigraphs. We formally defined biological experiential layers, and their interactions, and studied their mathematical consequences. According to the chosen set-up, we introduced the  $\odot$  operation, modelling the composition of experiential layers, and studying the implications of the uniqueness (up to rotation) of the layer decompositions.

Importantly, we have recovered extra ingredients, such as core structures for conscious experience and dynamical content, as well as predicted causal topological constraints that conscious interactions impose on its substrate.

Our approach allows us to study the coupling and decoupling of the whole brain-body system and its mathematical consequences without reducing them to any particular ontology. It aligns well with embodiment [23,39,40,81], new theoretical and experimental framework to study subjective experience [13,45], and with the recent push for formal and mathematical accounts to study conscious experience [2–5,8,9,82].

Our results are also an example of how the combination of the phenomenological tradition (through the embodied and enactive perspective), mathematical methods and network theory can provide new ways of understanding subjective experience with eventual implications for AI. For example, one could instantiate the multilayer model to approach toy models of artificial consciousness. These toy artificial models might instantiate some structural and invariant aspects of subjective experience, and potentially replicate such aspects to improve human-machine interactions. In that sense, a reciprocal bridge and two-way interaction between phenomenological concepts, methods and applications, and AI developments may bring new ways of understanding both: what are structurally relevant aspects of subjective experience and how to implement, improve and develop new generations of AI systems. Nevertheless, the question of whether such systems might become truly conscious is an open issue [72,83]. Taking a phenomenological perspective, we might suspend any assumption regarding artificial consciousness, animal consciousness and others, leaving it open to further and rigorous research [17,84].

Finally, future developments are expected as long as the characterization of multiple features of these layers is possible. Extensions may be based on more concrete phenomenological and biological accounts to describe specific layers and their configurations. One would like to include the use of sheave theory [85]: the multigraph structure may admit a pre-sheave definition and composition of layers being characterized in terms of local and global topological paths. Alternatively, another development could incorporate a mathematical measure of the features of every multigraph obtained by interaction across time (the Betti numbers). In this way we could characterize the degree of similarity among conscious experiences using persistent homology (see [86]). In short, the mathematical structure introduced in this article admits further generalizations and connections with phenomenological works on conscious experience [4,5] and the abstract structure of monoidal categories can also help us with the generalization of other influential models of consciousness.

## 6. Acknowledgments

Through this work C.M.S. was supported by the grant: Categorical Theories of Consciousness: Bridging Neuroscience and Fundamental Physics (FQXi-RFP-CPW-2018) and currently by FNRS postdoctoral grant Embodied-Time—40011405. The authors also thank the Association for Mathematics of Consciousness Science (AMCS) for creating spaces to discuss and share ideas regarding this research, in particular the Cabin workshop, where the two authors of this research meet for the first time.

## 7. Appendix

In this supplementary section, we briefly discuss our multilayer implementation in light of its algebraic structure through Category Theory and provide some algebraic definitions used in the main text. We first introduce some content on algebraic structures used in the paper, some formal definitions on Multilayer Networks and finally categorical concepts for a formal account of multilayer networks (see [5,27,87] for other introductions).

### 7.1. Monoids

We start by defining a monoid as a particular algebraic structure:

**Definition 16.** A *monoid* is a tuple  $(M, \oplus, \epsilon)$  where  $M$  is a set and for every  $m, n, p \in M$ :

- $\oplus$  is a total associative binary operation on  $M$ , that is

$$(m \oplus n) \oplus p = m \oplus (n \oplus p)$$

- $\epsilon$  is the identity for  $\oplus$ , that is

$$\epsilon \oplus m = m = m \oplus \epsilon$$

A monoid is said to be *commutative* if  $\oplus$  is commutative, that is, if

$$m \oplus n = n \oplus m$$

### 7.2. Tensor product of Monoids

We define the tensor product of (commutative) monoids starting from the well-known concept of *cartesian product of sets*: given two sets  $F$  and  $S$  their *cartesian product* is defined as

$$F \times S = \{(f, s) / f \in F, s \in S\}$$

We can think of the sets of first and second dishes when ordering in a restaurant:  $F$  would be all available first dishes and  $S$  the second ones, then  $F \times S$  represent all possible combinations one can order.

Now the cartesian product of two given monoids  $(M_1, \oplus, \epsilon_1)$  and  $(M_2, \otimes, \epsilon_2)$  is defined through the componentwise operation  $\star$  given by

$$(m, n) \star (m', n') = (m \oplus m', n \otimes n')$$

for  $m, m' \in M_1$  and  $n, n' \in M_2$ . Now  $M_1 \times M_2$  with  $\star$  is a two-fold monoid.

Finally, the tensor product of monoids, written  $M_1 \otimes M_2$ , is the quotient of their cartesian product by the relations

$$\begin{aligned} (m, n) \star (m', n) &\sim (m \oplus m', n) & (m, n) \star (m, n') &\sim (m, n \otimes n') \\ (\epsilon_1, n) &\sim (m, \epsilon_2) \end{aligned}$$

### 7.3. Partial Monoids

**Definition 17.** A *partial monoid* is a monoid  $(M, \oplus, \epsilon)$  where  $\oplus$  is a partial operation. A partial monoid is said to be *commutative* if  $\oplus$  has this property whenever defined.

### 7.4. Concepts on Multilayer Networks

The following is a sketch of some of the concepts involved in the theory of Multilayer Networks belonging to [27], they are mentioned and used in the paper and given here in a more formal presentation. If a node  $u$  has different aspects  $a_1, \dots, a_d$  in layers into the sets  $(L_1, \dots, L_d)$  respectively, a widely used notation is:

$$(u, \mathbf{a}) \equiv (u, a_1, \dots, a_d)$$

We note here that in this paper we avoided this notation in favour of a more appropriate way to express the features we want to describe.

The set of edges is partitioned into *intra-layer edges* and *inter-layer edges*. Intra-layer edges correspond to the interactions inside layers (e.g. yellow and light blue edges, Figure 1A) and belong to sets

$$E_A = \{((u, \mathbf{a}), (v, \mathbf{a}')) \in E_M \mid \mathbf{a} = \mathbf{a}'\}$$

The inter-layer edges are those in  $E_C = E_M - E_A$ , which correspond to interactions between layers.

### 7.5. Categorizing Multilayers

Category theory is a branch of highly abstract mathematics. One of the main features is its power to generalize relationships across different mathematical structures. This generalization is due to its general definition of objects as mathematical types, morphisms as maps or transformations from objects/types to other objects/types, and the composition operation, which may respect associative and identity law. It can be consulted [88] as a source of the main categorical concepts.

In order to give a categorical interpretation of our multilayer networks of section 2.2, we make use of the notions introduced in [28]. Following that reference, we consider a functor that produces many-coloured networks out of non-directed multigraphs.

Denote by  $\mathcal{S}$  the *permutation groupoid*, that is, a skeleton of the groupoid of finite sets and bijections.  $\mathcal{S}$  is a (strict) symmetric monoidal category.

**Definition 18.** A one-coloured network model is a lax symmetric monoidal functor  $F : \mathcal{S} \rightarrow \text{Mon}$  where  $\text{Mon}$  is the category of monoids.

**Example 19.** Let  $MG(n)$  be the set of multigraphs on  $\mathbf{n} = \{1, \dots, n\}$ . Then, a network model is a functor  $MG : \mathcal{S} \rightarrow \text{Mon}$  with values  $(MG(n), +)$  where  $+$  is a multiset sum, that is, the addition of multiplicities of edges sharing the same vertices.

The category of network models over a fixed colour set  $\mathcal{C}$  is denoted by  $\text{NetMod}_{\mathcal{C}}$  (see [28]).  $\text{NetMod}_{\mathcal{C}}$  is a symmetric monoidal category and therefore we can have tensor functors such as those introduced above.

**Definition 20.** A network model for multigraphs with coloured edges is a functor

$$MG^{\otimes \mathcal{C}} : \mathcal{S} \rightarrow \text{Mon}$$

where  $MG^{\otimes \mathcal{C}}(n)$  is a product of  $|\mathcal{C}|$  copies of the monoid  $MG(n)$ , being  $|\mathcal{C}|$  the cardinal of the set  $\mathcal{C}$ .

It is crucial that, in these tensor models, a network is not only a  $\mathcal{C}$ -tuple of multigraphs: it may also be viewed as a multigraph with as many edges of each colour between any pair of distinct vertices as they appear in every layer. The fact that the functor  $MG^{\otimes \mathcal{C}}$  is indeed a network model comes from the disjoint union operation:

$$MG^{\otimes \mathcal{C}}(n) \times MG^{\otimes \mathcal{C}}(m) \xrightarrow{\sqcup} MG^{\otimes \mathcal{C}}(n + m)$$

The particular kind of *network model* considered for the rotation multigraphs of section 5 depends precisely on the rotation angle and, subsequently, a variable on time. However, it does not satisfy one of the requirements introduced in [28], namely, being lax monoidal (see [28]). This is due to the fact that the operation analogous to  $\sqcup$  in non-rotation models, turns out to be partial in the rotation ones (i.e. subject to the condition  $|\alpha(t)| + |\beta(t)| \geq \pi/2$ ).

**Definition 21.** Given  $t \in T$  and  $\text{PMon}$  the category of partial monoids, a *network rotation model* is a functor

$$\text{RMG}^t : \mathcal{S} \rightarrow \text{PMon}$$

**Example 22.** A rotation network model for multigraphs with coloured edges is a functor

$$[RMG^t]^{\otimes \mathcal{C}} : \mathcal{S} \rightarrow PMon$$

where  $[RMG^t]^{\otimes \mathcal{C}}(n)$  is a product of  $|\mathcal{C}|$  partial monochrome monoids  $RMG^t(n)$  (recall that  $\mathcal{C}$  can be a multiset and then  $|\mathcal{C}|$  refers to the multiset cardinal).

### 7.6. Distributivity

The  $\odot$  operation is distributive over  $\times$ . That is, we obtain:

$$\left(\prod_{i=1}^{d_1} G_i\right) \odot \left(\prod_{j=1}^{d_2} H_j\right) = \prod_{i,j} (G_i \odot H_j) = G_1 \odot H_1 \times \dots \times G_{d_1} \odot H_{d_2} \quad (1)$$

whenever we calculate over expressions of multilayers containing monochrome layers such as

$$G_1 \times \dots \times G_{d_1} \in MG^{c_1}(n_1) \times \dots \times MG^{c_{d_1}}(n_{d_1})$$

$$H_1 \times \dots \times H_{d_2} \in MG^{c'_1}(m_1) \times \dots \times MG^{c'_{d_2}}(m_{d_2})$$

The expression 1 contains all combinations in the form  $G_i \odot H_j$  for  $i = 1, \dots, d_1$  and  $j = 1, \dots, d_2$  and lives in

$$MG^{c_1 \otimes c'_1}(n_1 + m_1 - p_{11}) \times \dots \times MG^{c_{d_1} \otimes c'_{d_2}}(n_{d_1} + m_{d_2} - p_{d_1 d_2})$$

where  $p_{ij} = |V(G_i) \cap V(H_j)|$ .

### 7.7. Inclusion-Exclusion Principle

Counting the number of vertices into a graph

$$[G_1, \alpha_1] \odot \dots \odot [G_k, \alpha_k]$$

means considering the known as *Inclusion-Exclusion Principle* in the form:

$$\begin{aligned} & \sum_{1 \leq i_1 \leq k} |V(G_{i_1})| - \sum_{1 \leq i_1 < i_2 \leq k} |V(G_{i_1}) \cap V(G_{i_2})| + \sum_{1 \leq i_1 < i_2 < i_3 \leq k} |V(G_{i_1}) \cap V(G_{i_2}) \cap V(G_{i_3})| - \dots \\ & \dots + (-1)^{k+1} \left| \bigcap_{i=1}^k V(G_i) \right| \end{aligned}$$

where  $k$  corresponds to the number of graphs.

### 7.8. Decomposition in **Biobrane**

The decomposition of graphs in **Biobrane** is determined by the equation:

$$\odot_{j=1}^m [G_j, \alpha_j] = \cup_{j=1}^m [H_j, \beta_j]$$

where we impose that  $E(G_j) = E(H_j)$  for  $j = 1, \dots, m$  but  $|V(H_j)| = n$  and, moreover,  $|\beta_j| + |\beta_l| < \pi/2$  for  $j, l = 1, \dots, m$ .

$H_j$  are then defined exactly as  $G_j$  but adding an isolated vertex for every missing one in  $G_j$  up to  $n$ .  $[G_j, \alpha_j]$  are the monochrome graphs in **Biobrane** from which  $[G, \alpha]$  was constructed and  $[H_j, \beta_j]$  are defined from them satisfying the following conditions:

- every  $[H_j, \beta_j]$  is monochromatic
- every  $[H_j, \beta_j]$  has  $n$  vertices

- $m$  is the number of layers used to build and decompose  $[G, \alpha]$  by applications of  $\odot$  and  $\cup$  respectively
- $|\beta_j| + |\beta_l| < \pi/2$  for  $j, l = 1, \dots, m$
- $m_1 + \dots + m_d = m$ .

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