The Effects of Manifest Residual Variances, Indicator Communalities, and Sample Size on the χ^2 -Test Statistic of the Metric Invariance Model

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Abstract

In this contribution, we investigate the effects of manifest residual variance, indicator communality and sample size on the χ^2 -test statistic of the metric measurement invariance model, i.e. the model with equality constraints on all loadings. We demonstrate by means of Monte Carlo studies that the χ^2 -test statistic relates inversely to manifest residual variance, whereas sample size and χ^2 -test statistic show the well-known proportional relation. Moreover, we consider indicator communality as a key factor for the size of the χ^2 -test statistic. In this context, we introduce the concept of signal-to-noise ratio as a tool for studying the effects of manifest residual variance and indicator communality and demonstrate its use with some examples. Finally, we discuss the limitations of this contribution and its practical implication for the analysis of metric measurement invariance models.

Keywords: metric measurement invariance, indicator communality, manifest residual error, signal-to-noise ratio, Monte Carlo simulation

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Introduction

Measurement invariance analysis has become a widely used tool in many areas of applied research, e.g., psychology, sociology, and organizational research (Davidov, Muthen, & Schmidt, 2018; Greiff & Scherer, 2018; van de Schoot, Schmidt, & de Beuckelaer, 2015) and received a lot of attention in methodological research (e.g., Millsap, 2011). The goal of measurement invariance (MI) analysis is to investigate whether measurement properties of a latent variable's indicators are the same across two or more groups.¹ Usually, there are several levels of MI that are successively investigated (e.g., Brown, 2015, pp. 242-243). The lowest level is configural MI. At this level, the factor structure is the same in all groups. At the next level, metric MI, not only the factor structure is the same in all groups but the loadings of all indicators are invariant across groups. Thus, metric invariance entails configural invariance. A third level is scalar MI, which means that the factor structure, the loadings, and the intercepts are invariant across groups.

MI analysis draws on testing nested models by means of χ^2 -difference tests. For instance, once it is established that configural MI holds, i.e., a multi-group model with equal structure in each group fits the data well, equality constraints are placed on all loadings so that they are invariant across all groups. This model with equality constraints on the loadings is called the *metric MI model*. In case this metric MI model fits the data well, too, a χ^2 -difference test is conducted. In case this χ^2 -difference test is significant, the hypothesis of equal loadings is discarded because the metric invariance constraints deteriorate model fit significantly. When the χ^2 -difference test is not significant, the metric MI model is retained. A detailed description of this process can be found in, e.g., Jiang, Mai, and Yuan (2017).

In this contribution, we only consider the metric MI, so we do not elaborate on further MI levels. Moreover, we do not consider the χ^2 -difference test to compare the configural and metric MI model. *The focus of this contribution is on the metric MI model and its χ^2 -test statistic.* We demonstrate that the metric MI model's χ^2 -test statistic depends on the manifest residual variance, indicator communality and sample size. Whereas the relation between sample size and χ^2 -test statistic is well-known, the effects of manifest residual variance and indicator communality have not been considered yet. We introduce the concept of signal-to-noise ratio as a joint measure of manifest residual variance and indicator communality. Moreover, we contrast the concept of signal-to-noise ratio with the usually used concept of scale reliability. The paper is structured as follows: In the next section, we use an example from Raykov, Marcoulides, Harrison, and Zhang (2019) to characterize the dependence of

¹In this contribution, we will only consider a two group case and do not explicitly refer to longitudinal MI analysis, but the argumentation applies straightforwardly to settings with more than two groups and longitudinal models.

the metric MI model's χ^2 -test statistic on the manifest residual variance and sample size. In the following section, we introduce the signal-to-noise ratio and contrast signal-to-noise ratio, which refers to each indicator separately, with scale reliability. In the following section, we use further examples and elaborate on how the signal-to-noise ratio and χ^2 -test statistic relate. The last section discusses the results from the previous sections and concludes.

A first example

In Raykov et al. (2019), the authors provide an example that consists of a population model with two groups and one latent variable. In detail, they used the following data generating process (dgp)

$$\mathbf{y}_g = \boldsymbol{\alpha}_g + \boldsymbol{\Lambda}_g \cdot \boldsymbol{\eta}_g + \boldsymbol{\delta}_g \quad (1)$$

with $g \in \{1, 2\}$ indexing the group with $g = 1$ indicates the reference group and $g = 2$ indicates the focal group (see Holland & Thayer, 1988). Let p be the number of manifest variable and n the number of latent variables, then $\mathbf{y}_g$ is a $p \times n$ vector of manifest variables, $\boldsymbol{\alpha}_g$ is a $p \times n$ vector of latent intercepts, $\boldsymbol{\Lambda}_g$ is a $p \times n$ loading matrix, $\boldsymbol{\delta}_g$ is a vector with manifest residual variances and zero mean. Under standard assumptions (e.g., Bollen, 1989), this yields the well-known representation

$$\boldsymbol{\Sigma}_g = \boldsymbol{\Lambda}_g \cdot \boldsymbol{\Phi}_g \cdot \boldsymbol{\Lambda}_g^\top + \boldsymbol{\theta}_g \quad (2)$$

$$\text{E}(\mathbf{y}_g) = \boldsymbol{\alpha}_g + \boldsymbol{\Lambda}_g \cdot \boldsymbol{\tau}_g \quad (3)$$

of the covariance matrix of the manifest variables and the respective manifest means in group G . In this representation, $\boldsymbol{\Sigma}_g$ is a $p \times p$ covariance matrix of the manifest variables, $\boldsymbol{\Phi}_g$ is a $p \times p$ covariance matrix of the latent variables, $\boldsymbol{\theta}_g$ is a $p \times p$ diagonal matrix with the manifest residual variances, and $\boldsymbol{\tau}_g$ is a $n \times 1$ vector of latent means.

In their example, Raykov et al. (2019) used $p = 12$ manifest variables and one latent variable η_G , $n = 1$. The actual model parameters were:

$$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = (1, 1.25, 1.25, 1.25, 1.5, 1.5, 1.5, 1.5, 1.75, 1.75, 1.75, 2)^\top,$$

$$\boldsymbol{\Lambda}_1 = (1, 1.25, 1.25, 1.25, 1.5, 1.5, 1.5, 1.5, 1.75, 1.75, 1.75, 2)^\top,$$

$$\boldsymbol{\Lambda}_2 = (1, 1.25, 1.25, 1.25, 1.5, 1.5, 1.5, 1.5, 1.75, 1.75, 1.75, 2.5)^\top,$$

$$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \mathbf{0},$$

$$\phi_{1,1,1} = \phi_{1,1,2} = 1,$$

$$\theta_{i,i,g} = 2, 1 \leq i \leq p.$$

Thus, the last indicator is non-invariant with $\lambda_{12,1,1} = 2$ in the reference group and $\lambda_{12,1,2} = 2.5$ in the focal group. As in this example, the model includes the indicators intercepts' and the factor mean, thus, there are 130 degrees of freedom for scaled model. For the usual $\alpha = .05$ significance level, the critical value of the χ^2 -test statistic to reject the metric MI model is $\chi^2_{crit} = 157.610$. As the model violates metric MI, the metric MI models' χ^2 -test statistics would be expected to be significant.

We extend this example and vary the manifest residual variance with $\theta_{i,i,g} \in \{0.5, 1, 2, 4\}$ for all indicators and the sample size with $N \in \{200, 400, 600, 800, 1000\}$ yielding a total of 20 conditions. We conduct a Monte Carlo study with 10,000 replications using Mplus (Muthén & Muthén, 1998-2019) in version 8.3. Estimator was ML and to scale the latent variable, we used the Fixed Marker Method with the first indicator being the referent indicator (Kline, 2016, pp. 404-405)². In order to explore the effects of manifest residual variance and sample size, we consider the *average of the metric MI models' χ^2 -test statistic over all replications*.

The results are presented in Figure 1 and Table 1. They show that the average of the χ^2 -test statistics depends on the manifest residual variance and the sample size. For $\theta_{i,i,g} = 4$, which is the condition with the largest manifest residual variance, Figure 1 shows that for all sample sizes the average of the χ^2 -test statistics is below the critical χ^2_{crit} -value. For $\theta_{i,i,g} = 2$, starting with a sample size of $N = 600$, the average of the χ^2 -test statistic is above the critical χ^2_{crit} -value. For $\theta_{i,i,g} = 1$, Figure 1 reveals that only for the smallest sample size of $N = 200$ the average of the χ^2 -test statistic is below the critical χ^2_{crit} -value, and for $\theta_{i,i,g} = 0.5$, which is the condition with the smallest manifest residual variance, the average of the χ^2 -test statistic is above the critical χ^2_{crit} -value for all samples sizes.

Taken together, the results demonstrate the well-known relation of sample size and χ^2 -test statistic: the greater the sample size the greater the average χ^2 -test statistic. In addition, the results demonstrate that there is an inverse relation between manifest residual variance and χ^2 -test statistic: the greater the manifest residual variance the smaller the average χ^2 -test statistic. Stated otherwise, the metric MI models' ability to detect the violation of equal loadings in both groups depends not only on the sample size, which is quite obvious and relates to the question of power, but also on the size of the manifest residual variance, a quantity which is commonly not investigated in the context of measurement invariance analysis. In the next section, we introduce the concept of signal-to-noise ratio of an indicator as the ratio of indicator communality over manifest residual error variance and explore how the quantities

²Originally, Raykov et al. (2019) were concerned about using the Fixed Marker Method with the referent indicator being the non-invariant indicator. The supplementary material contains simulations that demonstrate that regardless the use of the choice of a scaling method, the resulting fit for the metric MI model is always the same.

involved in calculating the signal-to-noise ratio and sample size relate to the average χ^2 -test statistic.

Signal-to-noise ratio and sample size

In psychology and other social sciences, an unobserved latent variable reflects an underlying cause that yields the covariation between the observed manifest variables (e.g., Borsboom, Mellenbergh, & Heerden, 2003). Mathematically, this covariation is represented by the covariance matrix Σ_g . According to Equation (2), the variance of the indicator y_i , $1 \leq i \leq p$, loading on the latent variable η_j , $1 \leq j \leq n$, in the group g is given by

$$\text{var}(y_{i,j,g}) = \underbrace{(\lambda_{i,j,g})^2 \cdot \psi_{j,j,g}}_{=: h_{i,j,g}^2} + \theta_{i,i,g}. \quad (4)$$

The first term on the right-hand-side represents the indicator communality $h_{i,j,g}^2$. Indicator communality $h_{i,j,g}^2$ refers to the portion of an indicator's variance that is due to the latent variable (Mulaik, 2010, p. 134). The second term denotes the portion of variance that is due to the manifest measurement error. In general, this error variance may contain systematic error as well as random error. In the following, we assume that we have only random error.

From another point of view, latent variables form a signal that causes the observed manifest data, or respectively, its covariation. In addition to the signal, there is some independent background noise that additively contributes to the manifest variables (Owen & Wang, 2016). The signal is generated by the latent variables whereas the background noise is generated by random errors. Thus, the variation of the signal is indicated by the indicator communality. By defining the ratio of indicator communality over manifest residual variance, we receive a measure that is similar to the concept of signal-to-noise-ratio (SNR). The SNR indicates how noisy a measurement is and is given by the ratio of the power of a signal over the power of background noise. The power of signal and the power of the background noise may be indicated by the variances of these measurements. In the context of latent variable models, the variance that is due to the latent variable can be considered as power of a signal, i.e., meaningful information, and the variance that is due to measurement error can be considered as noise, i.e., unwanted background noise. Thus, we can define the SNR for indicator y_i , $1 \leq i \leq p$, and the latent variable η_j , $1 \leq j \leq n$, in group g as

$$\text{SNR}_{i,j,g} := \frac{h_{i,j,g}^2}{\theta_{i,i,g}} = \frac{(\lambda_{i,j,g})^2 \cdot \psi_{j,j,g}}{\theta_{i,i,g}}. \quad (5)$$

Obviously, according to the preceding formula, when the manifest residual variance is greater than the indicator communality, then $\text{SNR}_{i,j,g} < 1$, and the measurement is considered a rather noisy. Vice versa, when indicator communality is greater than manifest residual variance, $\text{SNR}_{i,j,g} > 1$, then the power of the signal is greater than the power of the background noise. A $\text{SNR}_{i,j,g} > 1$ indicates that the power of the signal equals the power of the noise. Hereafter, we consider a threshold $\text{SNR}_{i,j,g} \geq 1.5$ as reflecting an amount of variance due to the latent variable that is *considerably greater* than 1. This threshold represents the belief that the signal should be at least 50% larger than the noise.

We now use the concept of SNR and formula (5) to study the simulation results from the previous section, see Table 2. This table shows the SNRs for the various settings of $\theta_{i,i,g}$. For the conditions with $\theta_{i,i,g} = 4$, only the last, non-invariant, indicator in the focal group has a SNR that is just above 1.5. For $\theta_{i,i,g} = 2$, a third of indicators have SNRs that are considerably larger than 1 in both groups. In the conditions with $\theta_{i,i,g} = 1$, all but the first indicator have SNRs considerably greater than 1 in both groups. Finally, in the condition with $\theta_{i,i,g} = 0.5$, all indicators in both groups have SNRs considerably larger than 1. This consideration mirrors the results from the Monte Carlo study. As the SNR of the non-invariant indicator and the SNRs of the other indicators increase, the average of the metric MI models' χ^2 -test statistic increases as well.

Before we consider sample size as a determinant of χ^2 -test statistic, we want to relate the concept SNR, which is commonly not used in confirmatory factor analysis settings, with the concept of reliability. The SNR indicates the noisiness of *a single indicator*, in other words, it indicates how reliable the meaningful signal could be separated from the noise. In psychometrics, it is more common to evaluate how reliable *the entire set of indicators*, i.e. the scale, is. Usually, reliability indices, which express the ratio of true variance (or variance explained by the latent variable) over the total variance, are calculated for this purpose. For instance, the ω -reliability index (Raykov & Marcoulides, 2011, p. 161) for the latent variable η_j , $1 \leq j \leq n$, in group g is defined as

$$\omega_{j,g} := \frac{(\sum_{i=1}^p \lambda_{i,j,g})^2 \cdot \psi_{j,j,g}}{(\sum_{i=1}^p \lambda_{i,j,g})^2 \cdot \psi_{j,j,g} + \sum_{i=1}^p \theta_{i,i,g}} \quad (6)$$

with p being the number of indicators. The reliability coefficient gives the part of the variance due to the latent variable in comparison to the total variance for *the whole scale*, whereas the SNR indicates the part of the variance due to the latent variable in comparison to the manifest residual variance *for one indicator*. As can be seen in the example, it is possible that a lot of indicators with a rather low SNR add up to a scale with a good reliability. Consequently, the information provided by the SNR could not be given by the reliability coefficient and a scale

with low reliability does not necessarily preclude to reliably detect non-invariance. In case of the previous example, it is $p = 12$ and the ω -reliabilities are printed in Table 2. Thus, from the usual psychometric perspective, the scales are considered to have a good reliabilities even for large values of $\theta_{i,i,g}$. This happens to be the case because in this condition, there are a lot of indicators that have low SNRs but add up to a scale with an overall good reliability.

As it is well known, sample size directly relates with the χ^2 -test statistics. If F_{ML} is the minimum of the discrepancy function and N the sample size, then according to Kline (2016, p. 207)³ the χ^2 -test statistic is given by

$$\chi^2 = 2 \cdot N \cdot F_{ML}. \quad (7)$$

This relation can be seen in Figure 1, which shows an (almost) linear relation of the average χ^2 -test statistics with sample size. In this figure, the slope is given by the average $2 \cdot F_{ML}$. As can be seen, for larger manifest residual variances, the slope is rather flat and its steepness increases with decreasing manifest residual variance.

Besides this relation with sample size, in the context of measurement invariance analysis, Meade and Bauer (2007) summarized research (Hogarty, Hibes, Kromrey, & Mumford, 2005; Gerbing & Anderson, 1987; Marsh, Hau, Balla, & Grayson, 1998; MacCallum, Widaman, Zhang, & Hong, 1999) demonstrating that the precision of estimated factor loadings increases as sample size increases, which is moreover a quite obvious result. Increasing precision of the estimated loadings in turn is also a reduction of noise, which is this time caused by the decrease of sample fluctuation and should therefore also yield the metric MI model to better detect indicators violating MI. Thus, the sample size “counteracts” the effects of large manifest residual variances.

Further Examples

In this section, we want to further elaborate on the SNR, sample size and manifest residual variances using two more examples. The goal is to demonstrate the effects of latent variances and manifest residual variances on the SNR. The examples are similar to the one by Raykov et al. (2019), i.e., there is a reference and a focal group, one latent variable, and the last indicator is non-invariant and therefore, the model violates the metric MI model. But these new examples only contain $p = 6$ manifest variables and we set all intercepts to 0. The dgp is according to Equation (1) with $\alpha_g = \mathbf{0}$ and $\tau_g = \mathbf{0}$ in both groups. The estimated model has

³As we used Mplus, the relation between the test statistic and the discrepancy function calculated in this software is according to Equation (7), Muthén (2004, p. 21).

28 *df*. Each example consists of three subexamples that are variations of a certain dgp. The parameters of the dgps are shown in Table 3. The manifest residual variances are the same for each indicator and from the set $\theta_{i,i,g} \in \{0.200, 0.100, 0.075, 0.050\}$. Sample sizes are from the set $N \in \{200, 400, 600, 800, 1000\}$ for each group. The resulting SNR and ω -reliabilities are shown in Tables 4 and 5. For all upcoming examples, we used a Monte Carlo setting as in the first example. The critical value of the χ^2 -test statistic to reject the metric MI model is $\chi^2_{crit} = 41.337$ for a .05 significance level.

In Example 2, the loading of the first indicator is twice as large as the other loadings in the reference group and twice as large as the loadings of the following four indicators in the focal group. In the Subexamples 2a and 2b the non-invariance of the last indicator is also numerically quite small and the examples differ in the variances of the latent variables. While in Subexample 2a the variances are invariant between the groups, they are non-invariant in Subexample 2b. In particular, the variance of the latent variable in the focal group is twice the variance of the latent variance in the reference group. This relation of the variances is kept in Subexample 2c, but the amount of non-invariance of the last indicator is increased. The results of the MC study for Example 2 are depicted in Table 6 and Figure 2. Regarding the SNRs in Example 2, as can be seen in the Table 4, only the first indicator has an SNR that is considerably greater than 1 for all $\theta_{i,i,g}$ in both groups and in all examples. Table 4 also shows that the proportion of indicator having SNRs considerably larger than 1 increases as the manifest residual variances decrease. Moreover, in all subexamples, the ω -reliability ranges from at least good to excellent.

In Subexample 2a, we would expect that the proportion of χ^2 -test statistic averages above the critical value is quite low over all conditions. This is exactly what can be seen in Figure 2 (top panel). This example shows that although the reliability of the indicators is good, the overall low level of SNR yields a low average rejection rate of the metric MI model. The only exception to this is a setting with overall low manifest residual variances and large sample sizes. In Subexample 2b, from the pattern of SNR we would expect that in general, there should be χ^2 -test statistic averages above the critical value than in Example 2a. This can be seen in Figure 2 (middle panel). In Subexample 2c, the amount of non-invariance of the sixth indicator is increased. Drawing on the theoretical reasoning put forward in this contribution, we would expect this increased amount of non-invariance to be beneficial for the average rejection of the metric MI model. Figure 2 (bottom panel) shows that the average χ^2 -value is above its critical value for all conditions of manifest residual variance and sample sizes.

Example 2 demonstrates that a ω -reliability does not ensure a high rejection rate of metric MI model in sense of the average χ^2 -test statistic. Example 2 also demonstrates the role of the latent variance and the role of the amount of non-invariance. Increasing latent variances

lower the sample size necessary to detect violations of metric MI. The same applies when the amount of non-invariance increases. Example 2c especially shows that the numerical difference of the non-invariance can compensate for a low SNR of the non-invariant indicator. Additionally, the plots show that with increasing manifest residual variances, the average of the χ^2 -test statistic decreases.

The setting of Example 3 is different from the setting of Example 3. Overall, the factor loadings are quite small and the non-invariance of the last indicator is also quite small in Subexamples 3a and 3b, whereas the difference between the loadings is larger in Subexample 3c. Additionally, the latent variances are non-invariant and rather large in Subexamples 3a. However, in the Subexamples 3b and 3c, the latent variances are smaller and invariant. With respect to the SNRs, Table 5 shows that for the reference group the SNRs are smaller than 1 in each condition. The same holds true for the first five indicators in the focal group. For Subexample 3a, only the non-invariant indicator has a SNR considerably greater than 1 in the condition with $\theta_{i,i,g} \leq 0.100$. In Subexample 3b, also the SNRs of the sixth indicator are smaller than 1 in each condition. This is because the variances are lower and are in addition invariant. Using the conventional standards, ω -reliabilities, apart from two exceptions, are not satisfactory.

The results of the MC study for Example 3 are depicted in Table 7 and Figure 3. As could be expected from the description above, for Subexample 3a only in the condition with very large manifest residual variance all χ^2 -test statistic averages are below the critical value for all sample sizes, see Figure 3 (top panel). In Subexample 3b, which has in contrast to Subexample 3a equal and rather low latent variances in both groups, Figure 3 (middle panel) shows that the rate of χ^2 -test statistic averages that are above the critical value drops. This example demonstrates the effect of lower latent variances. According to Equation (5), the latent variance contributes to the numerator of the SNR and therefore lower latent variances yield lower SNRs that in turn have negative effects on detecting the violation of metric MI on average. Only the SNR of the non-invariant indicator approaches 1, and given a sample size of at least $N = 600$, the average of the χ^2 -test statistics is above the critical value. Increasing the amount of non-invariance in Subexample 3c should in turn increase the rate of χ^2 -test statistic averages that are above the critical value. In fact, this can be seen in the bottom panel of Figure 3, at least for a small $\theta_{i,i,g} = 0.050$ the metric MI model has an average of the χ^2 -test statistics that is above the critical value. Again, as the left-hand graphs in Figure 3 show, an increasing level of manifest residual variance yield a lower χ^2 -test statistic of the the metric MI model.

Example 3 once again highlights the role of the latent variances. Moreover, it demonstrates that low scale reliabilities do not necessarily mean low rejection rates of metric MI test. Rather, the SNR matters and especially, the SNRs of the non-invariant indicators matter.

Discussion

In this contribution we used an example introduced by Raykov et al. (2019) and two own examples to demonstrate the importance of the SNR in the analysis of metric MI. As the SNR is defined as the ratio of indicator communality over the manifest residual variance of the indicators, it focuses on quantities that are only seldom reported in MI analysis. As demonstrated in the examples, the quantities involved all show relations with the average of the χ^2 -test statistic of the metric MI model in Monte Carlo studies.

Although the three examples in this contribution demonstrated the effects of indicator communality, latent variance and the manifest residual variance of the indicator on the average of the χ^2 -test statistic of the metric MI model, the models investigated do not represent the models typically investigated in practical applications. The dgps were constructed to demonstrate certain issues and therefore, the results are in some way limited. In all examples, there was only one non-invariant indicator and only one latent variable in a setting with only two groups. Further studies should target on more complex and more realistic settings.

Notwithstanding the limitations of these examples, it is obvious that finding referring to two groups generalize to settings with more groups. Regarding more complex factor structures, the usual setting in a multiple-group confirmatory factor analysis setting is an *independent cluster model* in which each indicator only loads on one latent variable and therefore constitutes a simple structure. In such structures the SNR is always calculated according to Equation (5) because of the absence of cross-loadings. In the same way as for the number of groups, the confirmatory factor analysis models can be broken down in one-factor models. In summary, although we only considered some simple and selected dgps, the results are valid to highlight the importance of indicator communality and manifest residual variance in the analysis of metric MI.

Two further limitations apply to the Monte Carlo studies in this contribution. Firstly, in all MC studies we only considered the ML estimator. It is obvious that different estimators yield different effects of indicator communality and manifest residual variances on the average of the χ^2 -test statistic for the metric MI model. Further studies should therefore consider different estimators. Secondly, the main focus in this contribution was on the χ^2 -test statistic for the metric MI model. As mentioned in the introduction, the analysis of MI consists in a series of nested model comparisons. Thus, the question for the average of the χ^2 -difference-

test is obvious. Further studies should also include the typically considered set of fit indices, i.e., the CFI, SRMR and RMSEA.

As a last limitation, we want to mention that we only studied the average behavior of the χ^2 -test statistic. This was implemented by comparing the average of the χ^2 -test statistic to its critical value. Further studies should consider the power of the metric MI models' χ^2 -test directly. Moreover, further studies should also take the χ^2 -difference test between the configural and metric MI model and its power into account.

Regarding practical applications, an important feature in this paper was the use of Monte Carlo studies. In this setting, the *dgp* in the population is known and we calculated the SNRs drawing on the population quantities of the parameters. This is not possible in applied settings. In applied settings, the estimated parameters have to be used. To get unique parameters, the model has to be scaled. As the SNRs are calculated in the configural model, where there are no across-group equality constraints, each group is scaled separately (usually by means of the same scaling method). In such a setting, the SNR is invariant with respect to the choice of a particular scaling method. The product of estimated loading times estimated latent variance is invariant and the manifest residual variances are always estimated as their population counterpart (Klößner & Klopp, 2019; Klopp & Klößner, 2019).

Moreover, this contribution also questions the usual reporting practice. In almost all cases of MI analyses, the authors report the results for the nested model comparison and common reliability indices as an measure for the scale's quality. Drawing on the simulation studies in this contribution, it seems necessary to report the SNR, or the quantities involved in calculating the SNR, i.e., the loadings, latent variance and manifest residual variance in the configural model so that a judgment regarding the power can be made. A more elaborate way would be not only to report the SNR, but to use the results from a given metric MI model as *dgp* and evaluate the power of the model using a simulation.

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Tables

Table 1: Average χ^2 -test statistics for Example 1, $df = 130$ in all conditions.

$\theta_{i,i,g}$	N	Mean	SD
0.500	200	172.163	20.677
	400	208.478	23.279
	600	246.368	26.614
	800	284.059	29.115
	1000	321.870	31.882
1.000	200	153.135	18.905
	400	170.194	20.104
	600	188.878	22.216
	800	207.508	23.731
	1000	226.344	25.513
2.000	200	143.301	17.874
	400	150.394	18.205
	600	159.114	19.431
	800	167.957	20.235
	1000	176.977	21.329
4.000	200	138.330	17.311
	400	140.411	17.174
	600	144.078	17.805
	800	148.052	18.167
	1000	152.101	18.797

Table 2: Signal-to-noise ratios and ω -reliabilities for various $\theta_{i,i,g}$ in the first example.

$\theta_{i,i,g}$	Referent group				Focal group			
	4.000	2.000	1.000	0.500	4.000	2.000	1.000	0.500
SNR _{1,1,g}	0.250	0.500	1.000	2.000	0.250	0.500	1.000	2.000
SNR _{2,1,g}	0.391	0.781	1.562	3.125	0.391	0.781	1.562	3.125
SNR _{3,1,g}	0.391	0.781	1.562	3.125	0.391	0.781	1.562	3.125
SNR _{4,1,g}	0.391	0.781	1.562	3.125	0.391	0.781	1.562	3.125
SNR _{5,1,g}	0.562	1.125	2.250	4.500	0.562	1.125	2.250	4.500
SNR _{6,1,g}	0.562	1.125	2.250	4.500	0.562	1.125	2.250	4.500
SNR _{7,1,g}	0.562	1.125	2.250	4.500	0.562	1.125	2.250	4.500
SNR _{8,1,g}	0.562	1.125	2.250	4.500	0.562	1.125	2.250	4.500
SNR _{9,1,g}	1.000	2.000	4.000	8.000	1.000	2.000	4.000	8.000
SNR _{10,1,g}	1.000	2.000	4.000	8.000	1.000	2.000	4.000	8.000
SNR _{11,1,g}	1.000	2.000	4.000	8.000	1.000	2.000	4.000	8.000
SNR _{12,1,g}	1.000	2.000	4.000	8.000	1.562	3.125	6.250	12.500
$\omega_{1,1,g}$	0.879	0.936	0.967	0.983	0.885	0.939	0.969	0.984

Table 3: Population models for Example 2 and 3, a-c. All intercepts and latent means are set to 0. RG: referent group, FG: focal group.

Parameter	RG	FG	RG	FG	RG	FG
	Exp. 2a		Exp. 2b		Exp. 2c	
$\lambda_{1,1,g}$	0.600	0.600	0.600	0.600	0.600	0.600
$\lambda_{2,1,g}$	0.300	0.300	0.300	0.300	0.300	0.300
$\lambda_{3,1,g}$	0.300	0.300	0.300	0.300	0.300	0.300
$\lambda_{4,1,g}$	0.300	0.300	0.300	0.300	0.300	0.300
$\lambda_{5,1,g}$	0.300	0.300	0.300	0.300	0.300	0.300
$\lambda_{6,1,g}$	0.300	0.250	0.300	0.250	0.300	0.100
$\phi_{1,g}$	1.000	1.000	1.000	2.000	1.000	2.000
	Exp. 3a		Exp. 3b		Exp. 3c	
$\lambda_{1,1,g}$	0.100	0.100	0.100	0.100	0.100	0.100
$\lambda_{2,1,g}$	0.100	0.100	0.100	0.100	0.100	0.100
$\lambda_{3,1,g}$	0.100	0.100	0.100	0.100	0.100	0.100
$\lambda_{4,1,g}$	0.100	0.100	0.100	0.100	0.100	0.100
$\lambda_{5,1,g}$	0.100	0.100	0.100	0.100	0.100	0.100
$\lambda_{6,1,g}$	0.100	0.300	0.100	0.300	0.100	0.600
$\phi_{1,g}$	1.000	2.000	0.500	0.500	0.500	0.500

Table 4: Signal-to-noise ratio for each indicator and ω -reliability for the latent variable in Example 2, a-c.

	Example 2a				Example 2b				Example 2c			
Parameter θ	0.200	0.100	0.075	0.050	0.200	0.100	0.075	0.050	0.200	0.100	0.075	0.050
Referent group												
SNR _{1,1,1}	1.800	3.600	4.800	7.200	1.800	3.600	4.800	7.200	1.800	3.600	4.800	7.200
SNR _{2,1,1}	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800
SNR _{3,1,1}	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800
SNR _{4,1,1}	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800
SNR _{5,1,1}	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800
SNR _{6,1,1}	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800	0.450	0.900	1.200	1.800
$\omega_{1,1}$	0.786	0.880	0.907	0.936	0.786	0.880	0.907	0.936	0.786	0.880	0.907	0.936
Focal group												
SNR _{1,1,2}	1.800	3.600	4.800	7.200	3.600	7.200	9.600	14.400	3.600	7.200	9.600	14.400
SNR _{2,1,2}	0.450	0.900	1.200	1.800	0.900	1.800	2.400	3.600	0.900	1.800	2.400	3.600
SNR _{3,1,2}	0.450	0.900	1.200	1.800	0.900	1.800	2.400	3.600	0.900	1.800	2.400	3.600
SNR _{4,1,2}	0.450	0.900	1.200	1.800	0.900	1.800	2.400	3.600	0.900	1.800	2.400	3.600
SNR _{5,1,2}	0.450	0.900	1.200	1.800	0.900	1.800	2.400	3.600	0.900	1.800	2.400	3.600
SNR _{6,1,2}	0.312	0.625	0.833	1.250	0.625	1.250	1.667	2.500	0.100	0.200	0.267	0.400
$\omega_{1,2}$	0.778	0.875	0.903	0.933	0.875	0.933	0.949	0.966	0.857	0.923	0.941	0.960

Table 5: Signal-to-noise ratio for each indicator and ω -reliability for the latent variable in Example 3, a-c.

	Example 3a				Example 3b				Example 3c			
Parameter θ	0.200	0.100	0.075	0.050	0.200	0.100	0.075	0.050	0.200	0.100	0.075	0.050
Referent group												
$\text{SNR}_{y,1,1}$	0.050	0.100	0.133	0.200	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,2,1}$	0.050	0.100	0.133	0.200	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,3,1}$	0.050	0.100	0.133	0.200	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,4,1}$	0.050	0.100	0.133	0.200	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,5,1}$	0.050	0.100	0.133	0.200	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,6,1}$	0.050	0.100	0.133	0.200	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\omega_{1,1}$	0.231	0.375	0.444	0.545	0.130	0.231	0.286	0.375	0.130	0.231	0.286	0.375
Focal group												
$\text{SNR}_{y,1,2}$	0.100	0.200	0.267	0.400	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,2,2}$	0.100	0.200	0.267	0.400	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,3,2}$	0.100	0.200	0.267	0.400	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,4,2}$	0.100	0.200	0.267	0.400	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,5,2}$	0.100	0.200	0.267	0.400	0.025	0.050	0.067	0.100	0.025	0.050	0.067	0.100
$\text{SNR}_{y,6,2}$	0.900	1.800	2.400	3.600	0.225	0.450	0.600	0.900	0.900	1.800	2.400	3.600
$\omega_{1,2}$	0.516	0.681	0.740	0.810	0.211	0.348	0.416	0.516	0.335	0.502	0.573	0.669

Table 6: Average χ^2 -test statistics for Example 2, $df = 28$ in all conditions.

$\theta_{i,i,g}$	N	Example 2a		Example 2b		Example 2c	
		Mean	SD	Mean	SD	Mean	SD
0.050	200	32.685	8.649	34.132	8.972	110.299	18.640
	400	36.630	9.535	39.457	10.094	191.864	25.157
	600	40.433	10.316	44.648	11.040	272.909	30.043
	800	44.698	11.046	50.329	11.955	354.931	34.357
	1000	48.856	11.881	55.920	12.979	437.043	38.737
0.075	200	31.191	8.304	32.160	8.531	83.810	16.182
	400	33.667	8.886	35.535	9.296	138.901	21.514
	600	36.019	9.427	38.802	9.955	193.502	25.531
	800	38.786	9.922	42.506	10.603	249.029	29.124
	1000	41.452	10.537	46.123	11.368	304.587	32.761
0.100	200	30.443	8.127	31.172	8.302	69.754	14.597
	400	32.194	8.543	33.579	8.870	110.806	19.139
	600	33.827	8.950	35.887	9.365	151.399	22.582
	800	35.847	9.311	38.603	9.855	192.862	25.681
	1000	37.768	9.798	41.232	10.468	234.330	28.818
0.200	200	29.328	7.854	29.698	7.949	47.754	11.510
	400	30.024	8.008	30.683	8.194	66.860	14.374
	600	30.607	8.192	31.579	8.410	85.593	16.614
	800	31.523	8.328	32.824	8.624	105.032	18.627
	1000	32.342	8.587	33.983	8.960	124.446	20.723

Table 7: Average χ^2 -test statistics for Example 3, $df = 28$ in all conditions.

$\theta_{i,i,g}$	N	Example 2a		Example 2b		Example 2c	
		Mean	SD	Mean	SD	Mean	SD
0.050	200	45.915	11.802	32.639	8.684	37.849	10.083
	400	63.114	14.844	37.150	9.741	47.133	11.928
	600	80.735	17.417	41.757	10.583	56.681	13.702
	800	98.239	19.664	46.468	11.587	66.316	15.146
	1000	115.656	21.856	50.987	12.426	75.783	16.628
0.075	200	37.261	9.861	29.920	7.969	32.785	8.830
	400	45.839	11.679	32.259	8.581	37.243	9.822
	600	54.705	13.266	34.479	9.019	41.849	10.833
	800	63.590	14.723	36.816	9.633	46.573	11.711
	1000	72.332	16.117	38.957	10.064	51.114	12.597
0.100	200	33.740	8.999	28.595	7.626	30.698	8.260
	400	38.848	10.196	30.224	8.053	33.366	8.904
	600	44.167	11.261	31.613	8.326	36.099	9.547
	800	49.568	12.297	33.074	8.769	38.940	10.140
	1000	54.811	13.257	34.322	9.007	41.589	10.714
0.200	200	29.707	7.972	25.637	6.877	27.503	7.387
	400	31.119	8.341	27.133	7.184	28.949	7.772
	600	32.560	8.667	27.867	7.367	29.812	8.012
	800	34.155	9.072	28.627	7.617	30.741	8.252
	1000	35.584	9.358	29.063	7.635	31.433	8.345

Figures

Figure 1: Results from the Monte Carlo study for the example in Raykov et al. (2019). Top panel: The mean χ^2 -test statistic is plotted against the sample size for the various $\theta_{i,i,g}$. Lower panel: The mean χ^2 -test statistic is plotted against $\theta_{i,i,g}$ for the various sample sizes. In both panels, the dashed line indicates the critical χ^2 -test statistic for 130 df .

Figure 2: Results from the Monte Carlo study for Example 2, a-c (from top to bottom). On the right, the mean χ^2 -test statistic is plotted against the sample size for the various $\theta_{i,i,g}$. On the left, the mean χ^2 -test statistic is plotted against $\theta_{i,i,g}$ for the various sample sizes. In all plots, the dashed line indicates the critical χ^2 -test statistic for 28 df .

Figure 3: Results from the Monte Carlo study for Example 3, a-c (from top to bottom). On the right, the mean χ^2 -test statistic is plotted against the sample size for the various $\theta_{i,i,g}$. On the left, the mean χ^2 -test statistic is plotted against $\theta_{i,i,g}$ for the various sample sizes. In all plots, the dashed line indicates the critical χ^2 -test statistic for 28 df .