A primer on measuring social preferences with linearly-additive decomposed 2×2 games

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Abstract

2×2 games (such as the Prisoner's Dilemma) are economic games for studying cooperation and social decision-making. Linearly-additive decomposed games are variants of 2×2 games that can change the framing of the game and thereby provide researchers with additional flexibility for measuring preferences and social cognition that would not be possible with standard (matrix-form) 2×2 games. In this paper, we provide a systematic overview of linearly-additive decomposed 2×2 games. We show which 2×2 games can be decomposed in a linearly-additive way and how to calculate possible decompositions for a given game. We close by suggesting for which experiments decomposed games might be more conducive than matrix games.

Keywords: 2×2 games, allocation, decomposed games, economic games, framing, social dilemma

1 Introduction

In this article, we explain what linearly-additive decomposed 2×2 games are, how they work, and for which types of experiments they might be most useful. In our opinion, they are a versatile but underused tool for studying social preferences and social interactions. For readability, we will omit the phrase 'linearly additive' (such that we refer to linearly-additive decomposed 2×2 games as 'decomposed games') and we put all relevant mathematics in the Appendix.

1.1 Social interactions and 2×2 games

Cooperation and social decisions have been studied for decades across a wide range of disciplines [1, 2, 3, 4, 5, 6, 7, 8]. To tease apart various aspects of cooperation, different tasks have been developed, such as the Prisoner's Dilemma, which models conflict between individual and collective. The Prisoner's Dilemma might be the most famous 2×2 game, but it is only one of many. In a 2×2 game, 2 players each make a binary decision, leading to 4 possible outcomes. The different games can be defined by the order of their payoffs. Of all ordinal 2×2 games with strict preferences, 12 are symmetric (both players have the same payoffs; names from [9]):

Game	1 >	2 >	3 >	4
Chicken	T	R	S	P
Battle	T	S	R	P
Hero	T	S	P	R
Compromise	T	P	S	R
Deadlock	T	P	R	S
Prisoner's Dilemma	T	R	P	S
Stag Hunt	R	T	P	S
Assurance	R	P	T	S
Coordination	R	P	S	T
Peace	R	S	P	T
Harmony	R	S	T	P
Concord	R	T	S	P

Table 1: The 12 symmetric ordinal 2×2 games with strict preferences. The '1' column indicates the highest payoff, which is larger than the payoffs for the '2' column, and so forth. The letters stand for the outcomes as named in the context of the Prisoner's Dilemma [10]: T = Temptation to Defect when other Cooperates (DC); R = Reward for mutual Cooperation (CC); P = Punishment for mutual Defection (DD); S = Sucker's payoff for Cooperating when other defects (CD). While the abbreviations don't make much sense for other games (e.g., in Harmony there's no temptation to Defect on a Cooperator), we use them here because using the alternative of CC/CD/DC/DD leads to an unreadable mess of Cs and Ds

The only difference between different ordinal 2×2 games is the rank of their payoffs and the resulting strategic decisions for the players (e.g., Nash equilibria). Consider the 4 games presented in matrix-form in Table 2: the Prisoner's Dilemma models conflict between what's best for the individual (defect) and what's best for both players combined (cooperate); for Chicken (also known as Hawk-Dove or Snowdrift), getting exploited is not the worst option, mutual defection is, making defection the riskier option and cooperation safer; the reverse is true for Stag-Hunt, where cooperation can lead to the highest or the lowest payoff; Concord contains no real conflict: mutual cooperation is the best payoff and mutual defection the worst. Thus, 2×2 games collectively map various interdependent decisions and can be used for studying social interactions.

	C	D		C	D		C	D		C	D
C	5/5	1/7	C	5/5	3/7	C	7/7	1/5	C	7/7	3/5
D	7/1	3/3	D	7/3	1/1	D	5/1	3/3	D	5/3	1/1
(a) Pris. Dilemma			(b) Chicken			(c) Stag-Hunt			(d) Concord		

Table 2: Exemplary payoffs of 4 different 2×2 games in matrix form

1.2 Matrix games and decomposed games

In most empirical studies, 2×2 games are displayed as the outcomes of the interdependent decision: if you choose C and the other person chooses D, you get 1 dollar and the other gets 7 dollars (see Table 2a). This is usually displayed in a matrix and called the 'matrix form'. But the outcomes can be decomposed into actions with consequences for self and other that are independent of what the other person chooses: if you choose A, you get 0 dollar and the other person gets 5 dollars, but if you choose B, you get 2 dollars and the other person gets 1 dollar. If both players decide between such decomposed options, the matrix form and the decomposed form describe the same 2×2 game, only changing the way the game is presented. For example, Figure 1 shows how if both players decide between the decomposed options just mentioned, the outcomes are identical to the Prisoner's Dilemma displayed in Table 2:

Figure 1: How the decomposed options add up to the final outcomes. In the (Prisoner's Dilemma) example on the left, Player 1 chose D and gave himself 2 points and the other player 1 point. Player 2 chose C and gave herself 0 points and the other player 5 points. Added together, Player 1 gets 7 points and Player 2 gets 1 point. The right side shows the generic variables, and an example where both players Defect

The central difference between the matrix form and the decomposed form thus lies in their different emphasis on action and outcome. While the matrix form shows only the consequence of the players' choices, the decomposed form shows only the value of the actions themselves, without explicitly stating the final outcomes. In matrix form, a decision has no intrinsic value and can only be evaluated in relation to the other's decision; in decomposed form, each action carries an intrinsic value for oneself and the other. Decomposed games may thus be psychologically similar to allocation tasks: formally, they are 2×2 games, but they appear like a forced-choice allocation task, where each player chooses 1 of 2 possible allocations options for oneself and another person. The tasks of the social-value orientation literature are often described as 'decomposed games' [11], but in these tasks usually only one person makes a decision, such that they are not games in the strict sense. In this article, we use the term 'decomposed games' to refer to a 2×2 game in which both players decide between 2 options. Decomposed games thus allow changing the way a game is framed, manipulating the story behind the game.

2 Decomposing different 2×2 games

2.1 Symmetric game, symmetric decomposition

This section answers the questions: 1) 'which games can be decomposed?' and 2) 'if a game can be decomposed, what are possible decompositions?'.

From previous studies, we know that the Prisoner's Dilemma can be decomposed. For example, Pruitt [12] empirically tested the effect of different decompositions of the same Prisoner's Dilemma on human cooperation rates. Pruitt decomposed the payoff matrix $T = 18$, $R = 12$, $P = 6$, $S = 0$ into several decomposed games, including the 2 displayed in Table 3. These decompositions are psychologically quite different (in decomposition 1, you can either split \$12 in half, or take \$6 away from the other to get \$12 yourself, but in decomposition 2 the decision lies between either giving \$12 to the other player or taking \$6 oneself), but if both players choose between the same decomposed options, the resulting game is the same. Pruitt found that participants had a 55% cooperation rate for the matrix-form and Decomposition 1 from Table 3, and a 70% cooperation rate for Decomposition 2.

	C	D		Self	Other		Self	Other
C	12/12	0/18	C	6	6	C	0	12
D	18/0	6/6	D	12	-6	D	6	0
(a) Matrix-form			(b) Decomposition 1			(c) Decomposition 2		

Table 3: Pruitt (1967) decomposed the same payoff matrix (a) into several different decomposed games, 2 of which are displayed in (b) and (c). Both players have to use the same decomposition to add up to (a)

2.1.1 Which symmetric 2×2 games can be decomposed symmetrically?

Pruitt [12] found that for any Prisoner's Dilemma to be decomposable, it has to fulfil the necessary condition $R - T = S - P$. As we demonstrate in the Appendix, this necessary condition is independent of the order of payoffs: any 2×2 game can be decomposed symmetrically if and only if $R - T = S - P$. Thus, we can use this rule to find out which games are decomposable in principle by taking $R - T = S - P$ (equivalent to $R + P = S + T$ and $P - T = S - R$) and searching for contradictions. As Table 4 shows, 4 games do not inherently contradict $R - T = S - P$: Deadlock, Prisoner's Dilemma, Harmony, and Concord. These are the only strict ordinal symmetric 2×2 games that can be decomposed symmetrically.

This logical explanation of why only 4 symmetric 2×2 games can be decomposed symmetrically can be complemented in a visually more intuitive way (Figure 2): Fix one of the two decomposed options at an arbitrary point (the black dot in the centre) and let the other decomposed option vary freely. The type of game resulting from the decomposed game depends only on the relative position of both options: for example, if the second point lands in the lowest right triangle (as depicted in blue) or highest left triangle, the resulting game is a Prisoner's Dilemma; if it lands in the second lowest right triangle or the second highest left triangle, the game is Deadlock, and so on. If the freely-varying point were to land on the diagonal, horizontal, or vertical lines, the resulting game would not be a ordinal 2×2 games with strict preferences: e.g., if the freely-varying point were to land on the diagonal between Prisoner's Dilemma and Deadlock, the resulting payoff matrix would be $T > R = P > S$. In other words, the game would be between the Prisoner's Dilemma and Deadlock.

Game	1 >	2 >	3 >	4	Potentially decomposable?
Chicken	T	R	S	P	No: $T > R \Rightarrow R - T < 0$; $S > P \Rightarrow S - P > 0$
Battle	T	S	R	P	No: $T \& S > R \& P \Rightarrow T + S \neq R + P$
Hero	T	S	P	R	No: same as for Battle
Compromise	T	P	S	R	No: $T > P \Rightarrow P - T < 0$; $S > R \Rightarrow S - R > 0$
Deadlock	T	P	R	S	Yes
Prisoner's Dilemma	T	R	P	S	Yes
Stag Hunt	R	T	P	S	No: $R > T \Rightarrow R - T > 0$; $P > S \Rightarrow S - P < 0$
Assurance	R	P	T	S	No: $R \& P > T \& S \Rightarrow R + P \neq S + T$
Coordination	R	P	S	T	No: same as for Assurance
Peace	R	S	P	T	No: $P > T \Rightarrow P - T > 0$; $R > S \Rightarrow S - R < 0$
Harmony	R	S	T	P	Yes
Concord	R	T	S	P	Yes

Table 4: Which strictly ordinal 2x2 games are decomposable? Deadlock, Prisoner's Dilemma, Harmony, and Concord and the only games that don't inherently contradict $R - T = S - P$ (or its equivalents)

Figure 2: The 4 strictly ordinal 2x2 games that can be decomposed symmetrically. The decomposed options for C and D are displayed by the two dots, connected by a line. The black dot is fixed (at an arbitrary point), the blue dot varies freely. The resulting game depends on the relative position of the decomposed options

2.1.2 What decompositions are possible, and why are there infinitely many?

Pruitt [12] also mentioned that if a Prisoner's Dilemma payoff matrix is decomposable, then there are infinitely many possible decompositions. But how do the infinite decompositions relate to each other; can we choose freely which decomposition to use, or are these decompositions related in some systematic way? As above, the full explanation is in the Appendix, and the approach isn't defined by the order of payoffs in the Prisoner's Dilemma, so the other 3 decomposable games also have infinitely many decompositions once the necessary condition is fulfilled. From the algebraic formulation in the Appendix, we can create a table with the generic formula for symmetrically decomposing any

symmetric 2×2 game (if it fulfills the necessary condition):

	Self	Other
C	$S - \gamma$	$R - S + \gamma$
D	$T - R + S - \gamma$	γ

Table 5: The generic formula for symmetrically decomposing a symmetric 2×2 game. γ can be chosen freely to create different decompositions with (potentially) different psychological effects

The decomposed options are defined by the payoffs of the game and γ , which can be chosen freely, and is subtracted from payoffs for the self and added to payoffs for the other. Mathematically, the infinity of decomposed options per 2×2 game is trivial ($x - x = 0$), but from a practical perspective, this provides flexibility when designing experiments to alter the framing of the decomposed option.

As an example, the following Table displays 3 different decompositions of the same payoff matrix. We use the Prisoner's Dilemma with the payoff matrix $T = 7$, $R = 5$, $P = 3$, $S = 1$, and then use γ values of $(-1, 0, 1)$. The resulting decomposition lead to the same payoff matrix specified above:

	Self	Other
C	2	3
D	4	-1
(a) $\gamma = -1$		

	Self	Other
C	1	4
D	3	0
(b) $\gamma = 0$		

	Self	Other
C	0	5
D	2	1
(c) $\gamma = 1$		

Table 6: 3 exemplary decompositions of the payoff matrix $T = 7$, $R = 5$, $P = 3$, $S = 1$. (c) is also in Figures 1 and 2

2.2 Asymmetric games and decompositions

So far, we have only dealt with symmetric games and symmetric decompositions: both players have the same options and the same potential outcomes. But real life doesn't consist exclusively of symmetric situations. Outcomes can differ between people, the actions available to them differ, or both. To account for decomposed games with asymmetries, either in payoff matrix or the decomposed options, we expand our previous section.

Conceptually, we can distinguish between 2 different kinds of asymmetries in the payoff structures of 2×2 games: first, both players' outcomes have the same payoff structure but with different values; second, the ordering of the payoffs differs between both players. Table 7 displays both situations: first, in Table 7a, the payoffs of both players are that of the Prisoner's Dilemma, but the row player's payoffs are multiplied by 10. Thus, for each player, the standard game-theoretic strategic considerations are the same as in a symmetric Prisoner's Dilemma, but psychologically it might feel quite different. For example, if we consider different aspects of social-value orientation [11, 13], joint gain is now highest if the row player defects and the column player cooperates – but if a player cares most about reducing inequality, then the opposite would yield the best result (if the row player cooperates and the column player defects, the absolute difference between both players is only 3). Second, in Table 7b, both players have the same values for their payoffs, but the payoffs of the column player are no longer in the order of the Prisoner's Dilemma, but instead in the order of Concord. This combination of strict ordinal games constitutes a new game [9].

Thus, there are two different kinds of asymmetries of the payoff structure that we could incorporate into decomposed games. Additionally, we would like to incorporate situations in which the decompositions differs between the players (asymmetric decompositions). We can use the same

	C	D		C	D
C	50/5	10/7	C	5/7	1/5
D	70/1	30/3	D	7/3	3/1

(a) Same game, different values

	C	D		C	D
C	5/7	1/5	C	5/7	1/5
D	7/3	3/1	D	7/3	3/1

(b) Diff. game, same values

Table 7: Two different kinds of asymmetries (a): Both players's payoffs have the same structure but with different values; (b): Both players' payoffs have a different structure but with the same values. Combining both types of asymmetry (different structure and different values) is also possible

approach as before, but need to specify independent variables for each player, both for the payoffs and for the decomposition. We thus expand Figure 1 to Figure 3:

Figure 3: How decomposed values add up for asymmetric games and decompositions. The principle is the same as for symmetric games, but now R for Player 1 does not necessarily equal R for Player 2 (same for T, S, and P), and that the decomposed values can now be separate for the two players (so instead of x_1 to x_4 for both players we now have x_1 to x_4 for Player 1 and x_5 to x_8 for Player 2). The blue and red arrows indicate an example of how R1 and R2 come about through both players choosing C

Table 8 provides the summary of how to decompose any 2×2 game. As before, the full explanation is in the Appendix. The game can be symmetric or either type of asymmetric, and the decompositions can be symmetric or asymmetric. 2 parameters (α and β) can be chosen freely: as before, the self and other columns have to add up to a constant for the ultimate payoffs to be constant. The necessary condition $R - T = S - P$ still holds, but separately for the payoff matrices of each player.

	Self	Other		Self	Other
C	$S_1 - \beta$	$T_2 - \alpha$	C	$R_2 - T_2 + \alpha$	$R_1 - S_1 + \beta$
D	$T_1 - R_1 + S_1 - \beta$	$S_2 - R_2 + T_2 - \alpha$	D	α	β

(a) Player 1

(b) Player 2

Table 8: The generic formula for decomposing a 2×2 game in a linearly additive way. This can be used for both asymmetric and symmetric 2×2 games and decompositions

As examples, we show 3 different situations: Table 9 shows a symmetric Prisoner's Dilemma decomposed into asymmetric decompositions, where player 1 contributes more than player 2; Table 10 shows decompositions for when the payoff structure differs between both players: player 1's payoffs are of a Prisoner's Dilemma, and player 2's payoffs are of Concord; Table 11 shows a combination of the two, where both players have different payoff structures and where Player 1 will get higher payoffs than Player 2 (independent of how both players decide):

	Self	Other		Self	Other
C	3	7	C	-2	2
D	5	3	D	0	-2

(a) Player 1 (b) Player 2

Table 9: Both players play a Prisoner's Dilemma with the payoff matrix $T = 7, R = 5, P = 3, S = 1$. We set $\alpha = 0$ and $\beta = -2$ to create a situation where player 1 always contributes more than player 2, but both get the same outcomes

	Self	Other		Self	Other
C	1	4	C	3	4
D	3	0	D	1	0

(a) Player 1 (b) Player 2

Table 10: Player 1's payoffs are of a Prisoner's Dilemma with the payoff matrix $T = 7, R = 5, P = 3, S = 1$, and player 2's payoffs are of Concord with $T = 5, R = 7, P = 1, S = 3$. We set $\alpha = 1$ and $\beta = 0$ so that both players contribute the same amount of possible points

	Self	Other		Self	Other
C	5	0	C	7	45
D	25	-4	D	5	5

(a) Player 1 (b) Player 2

Table 11: Player 1's payoffs are of a Prisoner's Dilemma with the payoff matrix $T = 70, R = 50, P = 30, S = 10$, and player 2's payoffs are of Concord with $T = 5, R = 7, P = 1, S = 3$. We set $\alpha = 5$ and $\beta = 5$

2.3 Games with ties

So far, we only considered ordinal games with strict preferences, such that per player 2 payoffs cannot be the same. For example, for a game to be considered a Prisoner's Dilemma, T has to be larger than R , the player cannot be indifferent about the order of the two outcomes. But just as we generalized symmetric games to asymmetric games, in real life different outcomes can be equally appealing. Thus, our final expansion includes games with ties between payoffs (e.g., $T > R = P > S$).

As before, for a game to be decomposable, the necessary condition still holds for each players' payoffs: $T + S = R + P$, even if outcomes are equal. Take a game with the payoff matrix ($T = 3, R = 2, P = 2, S = 1$). This game is between the Prisoner's Dilemma ($T > R > P > S$) and Deadlock ($T > P > R > S$). Because $T + S = R + P$, the game is decomposable. Table 12 shows two possible decomposed versions of this game, one symmetric and the other asymmetric:

	Self P1 & P2	Other P1 & P2		Self P1	Other P1	Self P2	Other P2
C	1	1	C	0	1	1	2
D	2	0	D	1	0	2	1

(a) Symmetric

(b) Asymmetric

Table 12: Exemplary symmetric and asymmetric decompositions for the same game with ties ($T = 3$, $R = 2$, $P = 2$, $S = 1$). In a) both players ($P1 = \text{player 1}$, $P2 = \text{player 2}$) play with the same decomposed options, whereas in b) the players play with different decomposed options; the resulting payoff matrix is identical for a) and b)

Thus, decomposed games can not only incorporate symmetric and asymmetric games for ordinal games with strict preferences, but also for games with ties. This allows for even more flexible and nuanced experimental designs and thus expands the range of questions one can answer with decomposed games.

3 Conceptual differences between matrix games and decomposed games

When could one use the matrix-form and when could one use the decomposed form? Any specific response depends on the specific experimental question, but we can provide general guidelines that might aid deciding between these two ways of presenting 2×2 games in an experiment. To our knowledge, not much research has systematically compared games in matrix form and decomposed form. Although some early studies compared the two [12, 14, 15], these study focused on individual aspects: Evans and Crumbaugh compared participants' cooperative choices in the matrix-form and one decomposed form and found that for that particular decomposition and payoffs, participants cooperated more in the decomposed than in the matrix form; Pruitt generalised symmetric decomposed games in the Prisoner's Dilemma and found differences in cooperation rate between the matrix-form and decomposed form, and differences in cooperation rate between different decomposed forms of the same payoff matrix; Messick and McClintock found that decomposed games could be used to assess different motivational aspects of their participants. Those studies deal only with the symmetric Prisoner's Dilemma (no other 2×2 games and no asymmetries between players) and mainly show that decomposed games can affect people's decisions, without attempting to systematically study how or why decomposed games can affect people's behaviour. Similarly, although the term 'decomposed games' has been used extensively in social-value orientation, these studies are not games but allocation decisions because only one person decides [11] - and thus these studies also do not tell us anything about the differences between matrix form and decomposed form. Many of the differences we point out are thus 'potential differences', rather than 'established differences', and could be tested empirically in future studies.

First, the main conceptual difference between matrix form and decomposed form lies in their different foci: the matrix form emphasises outcomes, the decomposed form emphasises the actions. In matrix form, players decide between different outcomes, but in decomposed games players can base their decision on either the intrinsic value of an action itself (e.g., I prefer giving equally to both and will choose the action with the smallest difference between both players) or on the outcome (as in the matrix form). This distinction is related to the distinction between deontological ethics and consequentialism, where the former values the action itself higher and the latter values the outcome higher, independent of the action that went into the outcome [16]. This difference between action

and outcome is also reminiscent of the distinction between procedural fairness and outcome fairness [17]. Decomposed games might provide a new angle to study open questions in those fields.

Second, games might be more ecologically valid in decomposed form than in matrix form. In many situations in life, actions have a direct effect on oneself and someone else, independent of what the other person does. In the matrix form, however, actions only exist in the interdependent context; thus, the decomposed form might be more ecologically valid because they let people decide between actions with inherent value, which add up to a specific 2×2 game [12].

Third, from a practical perspective, games in decomposed form might be easier to understand for participants (and animals). Thus, for any experimental design that might benefit from a simpler task (e.g., if the participants/animals might struggle with the instructions; if there is little time in the experiment to explain the task; when testing children; if the rest of the experiment is already very complex), it might be useful to use the decomposed form. The simpler task structure might facilitate research that would otherwise not be possible. Studying cooperation and social decision-making in animals often requires a relatively simple experimental design (e.g., the car-driving task with rhesus macaques [18] or the rope-pulling task with elephants [19]); using decomposed games may allow novel variants of these tasks. Caution is advised: any decomposition will affect the framing of the situation, which might have unintended side-effects.

Linearly-additive decomposed 2×2 games have existed for more than 50 years in the context of the Prisoner's Dilemma but have been studied relatively rarely and, to our knowledge, have never been explored systematically. This is surprising, given the flexibility that they provide in assessing various aspects of social preferences, particularly in relation to inequality between two players in terms of actions and outcomes. In this article, we highlighted linearly-additive decomposed games by explaining their logic and providing a way to calculate possible decompositions for a given payoff matrix, for symmetric and asymmetric games and decompositions. Decomposed games allow for more flexible experimental designs, potentially enabling for ecologically more variable and realistic experimental set-ups.

4 Appendix

To keep the main article short, we present the relevant mathematics here. Any introductory course on Linear Algebra, such as the one by Gilbert Strang (<https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/index.htm>), should cover the following.

Why is $R - T = S - P$ a necessary condition for a Prisoner's Dilemma payoff matrix to be decomposable? In a decomposed game, the standard options of C and D indicate a specific amount of points each player gets (see Figure 1, right example). We can calculate the standard payoffs $\{T, R, P, S\}$ the following way:

$$\begin{aligned} x_1 + x_2 &= R \\ x_3 + x_2 &= T \\ x_1 + x_4 &= S \\ x_3 + x_4 &= P \end{aligned} \tag{1}$$

This can be written as:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} R \\ T \\ S \\ P \end{bmatrix} \tag{2}$$

The first column stands for x_1 , the second column for x_2 , and so forth. To simplify the notation, we write A and b as a single *augmented matrix*. We can now test whether this set of equations has a solution, and if so how many and under which conditions. We use Gaussian elimination to bring this matrix into the upper triangular form:

$$\begin{aligned} & \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 1 & 0 & 0 & 1 & S \\ 0 & 0 & 1 & 1 & P \end{array} \right] \xrightarrow{R3-R1 \rightarrow R3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 0 & -1 & 0 & 1 & S-R \\ 0 & 0 & 1 & 1 & P \end{array} \right] \xrightarrow{R2+R3 \rightarrow R3} \\ & \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 0 & 0 & 1 & 1 & S+T-R \\ 0 & 0 & 1 & 1 & P \end{array} \right] \xrightarrow{R4-R3 \rightarrow R4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 0 & 0 & 1 & 1 & S+T-R \\ 0 & 0 & 0 & 0 & P+R-S-T \end{array} \right] \end{aligned}$$

The last row now states that $0x_1 + 0x_2 + 0x_3 + 0x_4 = P + R - S - T$. Therefore, this system of equations only has a solution (i.e., the matrix is decomposable) if:

$$0 = P + R - S - T.$$

Which is equivalent to:

$$R - T = S - P.$$

Therefore, this set of equations does not have a solution if $R - T \neq S - P$. If this condition were not to be fulfilled, then the set of equations would run into internal contradictions. This is why Pruitt said that $R - T = S - P$ is a necessary condition for a Prisoner's Dilemma payoff matrix to be decomposable.

To find out what all possible decompositions for a given payoff matrix is, we need to solve the set of equations. We take the upper triangular matrix and bring it to the row-reduced echelon form. Given that we are only interested in games that are actually decomposable, we know that the necessary condition of $R - T = S - P$ is fulfilled. To simplify the set of equations, we can substitute $P + R - S - T$ with 0:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 0 & 0 & 1 & 1 & S + T - R \\ 0 & 0 & 0 & 0 & P + R - S - T \end{array} \right] \xrightarrow{P+R-S-T=0} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 0 & 0 & 1 & 1 & S + T - R \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

We now bring this form into the reduced row echelon form by using Gauss-Jordan elimination:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 1 & 0 & T \\ 0 & 0 & 1 & 1 & S + T - R \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R2-R3 \rightarrow R2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & R \\ 0 & 1 & 0 & -1 & R - S \\ 0 & 0 & 1 & 1 & S + T - R \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R1-R2 \rightarrow R1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & S \\ 0 & 1 & 0 & -1 & R - S \\ 0 & 0 & 1 & 1 & S + T - R \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

By finding the particular and the special solution, we can now find the generic solution to this set of equations:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} S \\ R - S \\ S + T - R \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad (3)$$

To make it easier to use, we reformatted this equation in Table 5. This solution also explains why if there is one decomposition, there are infinitely many decompositions: we can take a particular decomposition, and then shift the decomposition such that x_1 and x_3 are decreased by γ to the same degree that x_2 and x_4 are increased. In other words, if for both options the self-payoff is decreased as much as the other-payoff is increased, the resulting payoff matrix remains constant.

At no point did we specify the order of the payoffs. This means that the results from this section hold true independent of which game is being used. As we show in the main text, only some games can abide by the necessary condition, but for those games, the solutions offered here apply equally.

So far we have assumed that both players play the same game with the same decomposed options. Using the same approach as for symmetric games, we can now solve the system without those assumptions and thus incorporate asymmetric games and asymmetric decompositions. We skipped the steps here, but the approach is the same as above, just with more variables:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & R_1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & R_2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & S_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & T_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & T_1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & S_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & P_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & P_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & S_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & T_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & T_1 + S_1 - R_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & T_2 + S_2 - R_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & R_2 - T_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & R_1 - S_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 + S_2 - R_2 - P_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_1 + S_1 - R_1 - P_1 \end{bmatrix}$$

As before, we now get the necessary condition that $T + S - R - P = 0$, but for both players separately. This means that both players can have different payoff matrices, but they still have to each abide by the necessary condition for a 2×2 game to be decomposable.

Again, to get a generic formula for calculating the decompositions, we solve this set of equations:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ T_2 \\ T_1 + S_1 - R_1 \\ T_2 + S_2 - R_2 \\ R_2 - T_2 \\ R_1 - S_1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

As before, we reformatted this into a more usable table format in Table 8.

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