Differences and Ratios in a Nonsymbolic ‘Artificial Algebra’: Effects of Extended Training

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Abstract

Grace et al. (2018) showed that humans could estimate ratios and differences of stimulus magnitudes by feedback and without explicit instruction in a nonsymbolic ‘artificial algebra’ task, but that responding depended on both operations even though only one was trained. Here we asked whether control by the trained operation would increase over several sessions, that is, if perceptual learning would occur. Observers ($n = 16$) completed four sessions in which feedback was based on either ratios or differences for stimulus pairs that varied in brightness (Experiment 1) or line length (Experiment 2). Results showed that control by the trained and untrained operations increased and decreased, respectively, over the sessions, indicating perceptual learning. For about two thirds of individual sessions, regressions indicated significant control by both differences and ratios, suggesting that the perceptual system automatically computes two operations. The similarity of results across experiments with both intensive (brightness) and extensive (line length) stimulus dimensions suggests that differences and ratios are computed centrally, perhaps as part of a general system for processing magnitudes (cf. Walsh, 2003).

Keywords: magnitude comparison, brightness, line length, differences, ratios, perceptual learning, artificial algebra, Torgerson’s conjecture, psychophysics, humans.

How we compare stimulus magnitudes - for example, which of two lines is longer, or which of two lights is brighter - is a question of fundamental interest. Most theorists have assumed that comparisons are based on either differences or ratios of magnitudes, but which operation is correct has been controversial. According to a conjecture by Torgerson (1961), observers use a single operation to compare magnitudes, either a difference or a ratio, but which one cannot be determined empirically. Because differences and ratios are related by simple transformations (i.e., ratios can be converted to differences by a logarithm function; differences to ratios by exponentiation), it is possible to have multiple mathematical models to describe the same data. Torgerson noted that a similar ambiguity had led to the controversy about the psychophysical law (i.e., the Fechner-Stevens debate; Krueger, 1989). Questions about whether magnitudes should be scaled linearly or logarithmically may reflect the same fundamental indeterminacy (e.g., Dehaene, 2003; Dehaene, Izard, Spelke, & Pica, 2008; Gibbon & Church, 1981; Tan & Grace, 2012). Models for nonhuman choice behavior have also disagreed about whether differences or ratios of reinforcement-related variables control responding (Mazur, 2002; Savastano & Fantino, 1996).

Torgerson's conjecture was the focus of considerable research in the 1970s and 1980s. Typical studies used magnitude estimation procedures, in which observers were asked to produce numeric estimates of the subjective difference or ratio of stimulus magnitudes for a range of dimensions, including (among others) heaviness (Birnbaum & Veit, 1974; Rule, Curtis, & Mullin, 1981), loudness (Birnbaum & Elmasian, 1977; Schneider, Parker, Farrell, & Kanow, 1976), line length (Parker, Schneider, & Kanow, 1975), and remembered distance between cities (Birnbaum, Anderson, & Hyman, 1989). Overall, results were mixed. Although the majority of studies indicated that observers used a difference operation, Birnbaum (1978) noted that when the dimension had a well-defined zero point (e.g.,

Birnbaum et al., 1989), they were also able to estimate ratios. More recent research by Masin and colleagues (Masin, 2014; Masin, Brancaccio, & Tomassetti, 2019) also suggests that observers struggle to estimate ratios, particularly for intensive dimensions such as brightness and heaviness (cf. Rule, Curtis & Mullin, 1981).

Grace, Morton, Ward, Wilson, and Kemp (2018) developed a novel, nonsymbolic procedure for testing Torgerson's conjecture that did not require explicit instruction about ratios or differences. They argued that previous research might have found mixed results because observers were required to give numerical responses and thus to access their mathematical knowledge. But because the perceptual system evolved long before the invention of mathematics, it should be possible to test for difference or ratio operations with a procedure that did not require numeric responses. In Grace et al.'s task, observers viewed a pair of stimuli (e.g., circular disks that varied in brightness, numbers of dots, or area) and were asked to compare their magnitudes by clicking on a horizontal bar beneath the stimuli. Feedback was provided by a colored oval with a location based on the difference or ratio of the nominal stimulus values, mapped linearly along the response bar (left end = minimum difference or ratio, right end = maximum difference or ratio). If the response location was within the oval, the color was green, otherwise it was red. Results showed that observers quickly learned to respond accurately (correlations with correct values averaged $r = .95$) regardless of whether ratios or differences were trained. Notably, regression analyses showed that the untrained operation consistently accounted for significant variance in responding after controlling for the trained operation. In other words, responding seemed to reflect a compromise between differences and ratios, even though only one operation was trained.

Based on this result, Grace et al. (2018) proposed that the perceptual system automatically computed both differences and ratios when comparing stimulus magnitudes, so that Torgerson's conjecture was false. They suggested that two operations might indicate that

the perceptual system represents an algebraic field, which is an abstract mathematical concept that generalizes the symmetries underlying arithmetic. Because their experimental task can be used to infer structure in responding from training algebraic combinations of stimulus magnitudes, it might be described as an ‘artificial algebra’, similar to how ‘artificial grammars’ in psycholinguistic research are used to infer grammatical structure underlying observers’ responding (e.g., Pothos, 2007; Reber, 1989).

Grace et al. (2020) developed some implications of these results. They suggested that a field representation might be necessary to support ‘computationally-complex behavior’, that is, behavior which appears to require the equivalent of algebraic computation. Spatial navigation by path integration is a good example: Many species (from insects to mammals) can apparently keep track of their current location while foraging relative to a home base, such that they are able to return by most direct route, as if able to calculate the homeward vector algebraically (e.g., Etienne & Jeffery, 2004; Srinivasan, 2015). Also, because a field includes two operations (corresponding to additive and multiplicative groups), it would provide flexibility not only for both differences and ratios to be computed, but also linear and logarithmic scaling of magnitudes. Thus, controversies like that regarding the psychophysical law, which hinge on which group is primary (i.e., Fechner: additive; Stevens : multiplicative), may overlook the possibility that the perceptual system is flexible enough to include both.

Although the ‘artificial algebra’ task used by Grace et al. (2018) is novel, it has some connections to other experimental and theoretical work that are worth exploring. As a procedure, it resembles a function learning task, in which participants learn by feedback to make an analogue response with a specified mathematical relationship to a stimulus magnitude (Lucas, Griffiths, Williams, & Kalish, 2015; McDaniel, Cahill, Robbins, & Wiener, 2014). As an example of a function learning task, Koh and Meyer (1991) trained

participants (over six experimental sessions) to make a keypress duration that was a power, logarithmic or linear function of the distance between two vertical lines presented on a screen, and found that power functions were most easily learned. Grace et al.'s task is similar except that participants learned to make an analogue response that was linearly related to an algebraic combination of stimulus magnitudes.

Perceptual learning refers to an increase in discrimination performance for a stimulus dimension with feedback and training (Doshier & Lu, 2017). Performance in the artificial algebra task could be regarded as demonstrating perceptual learning to the extent that accuracy of responding improved with continued training. However, participants in Grace et al.'s (2018) study only completed single sessions (four blocks of 84 trials each) in the difference or ratio condition. They reported that both operations were learned rapidly; performance improved from the first to second block, but not thereafter. Thus, it would be interesting to test whether performance in the artificial algebra task will improve over multiple sessions of training.

Another question is how performance in Grace et al.'s (2018) task relates to research on numerical cognition. As proposed originally by Dehaene (2011), humans have a nonverbal 'number sense' for dealing with numerosity that provides a foundation for later learning about numbers and arithmetic (Feigenson, Dehaene, & Spelke, 2004). Two core systems that support number processing have been proposed, an 'approximate number system' (ANS) that provides estimates of larger quantities, and an 'object tracking system' (OTS) that allows for automatic identification of small quantities (e.g., 1-4 items). Both of these systems are shared with nonhumans, and can be distinguished by their variability signatures: The ANS shows scalar variability (i.e., standard deviation increases proportionally with the mean estimate), whereas the OTS has constant variability. The OTS and ANS may be part of a larger system for processing both discrete and continuous

magnitudes (Leibovich, Katzin, Harel, & Henik, 2017; Walsh, 2003). As applied to our task, we were interested in whether the possibility of automatic computation of ratios and differences by the perceptual system might lead to a variability footprint similar to the OTS.

Thus in the present study we planned to test the effects of extended training in Grace et al.'s (2018) task, and also to examine if variability signatures were more consistent with the ANS or OTS. We report two experiments, in which observers were trained to respond based on differences or ratios of circles that varied in brightness (Experiment 1) or pairs of lines that varied in length (Experiment 2). Both experiments used a randomization procedure to generate stimulus values, rather than a limited set of stimulus pairs as in Grace et al. (2018). We predicted that control over responding by the trained operation would increase over sessions – that is, perceptual learning would occur – and results would show that both differences and ratios influenced responding, contrary to Torgerson's conjecture. If differences and ratios are computed automatically by the perceptual system then we would expect results to be similar to the OTS, that is, constant variability.

Experiment 1

In Experiment 1, stimuli were circles that varied in brightness. Observers completed four sessions, on separate days, and were randomly divided into difference ($n = 4$) and ratio ($n = 4$) groups, depending on the operation used to generate feedback. Rather than using a limited set of exemplars as in Grace et al. (2018), stimuli were generated pseudo-randomly so that a different pair of brightness values was presented on each trial. We planned to test whether measures of accuracy and relative control of responding by differences and ratios changed over sessions, and if patterns of response variability were more consistent with the ANS or OTS.

Method

Participants. A total of 8 adults served as participants (6m, 2f). Observers Db, Ja, Jo, and Ma were randomly allocated to the difference condition, and Ar, K, L, and Wa to the ratio condition. All were naïve to the purpose of the research, and received a NZ\$15 voucher for each session completed. All reported normal or corrected-to-normal vision.

Materials. A custom Javascript software run on a Lenovo Thinkpad T520 laptop computer was used to display stimuli and record responses. The laptop had a 15.6" (39.62 cm diagonal) LCD display screen, with a resolution of 1600 x 900 pixels, a maximum brightness of 231 cd/m², and a contrast ratio of 670:1. The screen settings were unaltered throughout all sessions. Participants were seated at a desk at an approximate distance of 40cm from the laptop screen.

Pairs of circles in varying grayscale, each 6 cm in diameter, were used as stimuli in Experiment 1. They were displayed side-by-side, 8 cm from the top of the screen and separated by a 3.3 cm gap. The circles were identical except in brightness. An 8-bit monochrome palette ranged from 0 - 255 (0 = minimum luminance, 0.33 cd/m²; 255 = maximum luminance, 231 cd/m²) simulated the greyscale values for the circles. The brightness values were randomly sampled for each trial between a minimum of 50 and maximum of 205, with the lower value always displayed on the left side of the screen. A horizontal grey response bar which measured 16.5 cm in length and .2 cm in width was located 6.5 cm below the stimuli and contained no markings. All items were displayed in front of a black background.

Differences and ratios were calculated for each stimulus pair based on the nominal brightness values. Nominal values were linearly scaled along the response bar from left to

right, so that 0% (left) corresponded to the minimum difference (or ratio) and 100% (right) to the maximum difference (or ratio). Feedback was provided for each response based on the scaled difference or ratio value. Correct responses were defined as those within $\pm 7\%$ of the trained value. Red and green ovals centred on the trained value with a width of $\pm 7\%$ of the response bar were used as feedback for incorrect and correct responses, respectively. The 7% value was the same as Grace et al. (2018), and was based on pilot testing to result in a reasonable proportion of correct responses (~50%).

Procedure. Participants were tested individually in a quiet room. At the beginning of each session, observers were seated with the laptop and presented with the following instructions on the computer screen: "In this experiment, you will see a series of pairs of gray circles on the screen. For each pair, you need to compare the brightness of the circles by clicking on the horizontal bar. The purpose is to learn how to compare the brightnesses as accurately as you can. You will receive feedback for each correct response, but if your response is incorrect, you will have a chance to repeat with the same pair of circles. You should respond at whatever pace feels comfortable and natural for you, and will have several chances to take a break during the session." Each participant completed four sessions, which were conducted on separate days (usually consecutive). Each session contained four blocks of 84 trials, giving 336 trials per session. A rest break was provided between each block.

For each trial, observers clicked along the response bar using the mouse. A black dot with a white circular outline 10 mm in diameter appeared immediately on the response bar marking the location of the response. Following a 100 ms delay, feedback showing the trained value was presented. If the response was correct, a green oval centred on the trained value on the response bar persisted for 500 ms. After feedback, all items on the screen were removed and the next trial was presented following a fixed 2-s inter-trial interval. Incorrect responses produced a red oval centred on the trained value as feedback, followed by a single

correction trial in which the same stimuli were repeated. Correction trials are only given once; they are not repeated if the correction trial was failed. Responses on correction trials were omitted for analyses. The task was self-paced; there was no time limit. Sessions typically required about 30-35 minutes.

Results

Accuracy of performance was assessed with four measures: Average percent correct (i.e., the percentage of responses that produced green feedback ovals), average absolute deviation of responses from correct values, root-mean-square (RMS; i.e., the square root of the average squared deviation of responses from correct values), and correlation of responses with correct values (r). These measures were computed for each session.

Results are shown in Figure 1 for both difference and ratio groups. Overall, responding became more accurate across sessions, and there were no significant differences between groups. These observations were confirmed by a series of repeated-measures ANOVAs with session as a within-subjects factor and group as a between-subjects factor.

For average percent correct, the effect of session was significant, $F(3, 18) = 4.535$, $p = .015$, $\eta_p^2 = .430$, but the effects of group and the interaction were not ($ps > .244$). Post-hoc tests (Newman-Keuls) confirmed that the average percent correct trials for Sessions 1 and 2 were significantly lower than for Sessions 3 and 4. Similar results were obtained with average absolute deviation and RMS. For average absolute deviation, the effect of session was significant, $F(3, 18) = 4.277$, $p = .019$, $\eta_p^2 = .416$, while the effect of group and the interaction were not ($ps > .658$). For RMS, the decrease across sessions was significant, $F(3, 18) = 4.685$, $p = .014$, $\eta_p^2 = .438$; whereas the effects of group and interaction were not

($ps > .199$). For both average absolute deviation and RMS, post-hoc tests (Newman-Keuls) showed that average values for Session 1 were significantly greater than for Sessions 3 and 4.

Correlations of responses with correct values significantly increased across sessions, $F(3, 18) = 4.017, p = .024, \eta_p^2 = .401$, but again, the effects of group and the interaction were not ($ps > .210$). Post-hoc tests (Newman-Keuls) confirmed that the average correlation was significantly greater for Session 4 compared to Session 1. Overall, these results suggest that responding became gradually more accurate across sessions, and there were no significant differences in accuracy depending on whether observers were trained with difference or ratio relations. Table 1 further supports these findings; it shows the averaged means and standard deviations of sessions 1 and 4 of the four measures. The ‘% improved’ column shows the percentage of improvement of each these four measures from the first to the fourth session. All four measures in both groups showed improvement in performance after training, as indicated by an increase in percentage correct and r , and a decrease in absolute deviation and RMS.

Figures 2 and 3 shows representative scatterplots of the responses plotted as a function of the scaled trained values for the difference and ratio group, respectively. Sessions 1 through 4 are shown in the rows (top to bottom), while the left and centre columns show the best and worst performing observer, respectively, in terms of average correlation between response and trained value, and the right column shows the group mean data. Individual session scatterplots were generated by grouping data together in bins based on sorting trials with trained correct values from .00 to .99 in bins with a width of .01. The average response and trained value were then calculated for each bin. The scatterplots’ y- and x-axis correspond to these binned averages. The dotted lines show the lines of best fit and the solid line represents perfect correlation. Figures 2 and 3 show that responses were generally highly correlated with trained values. All observers except one (Wa; Figure 3, centre column)

showed a strong positive correlation between responses and correct values. Results for Observer Wa are unusual and will be further discussed later. For the remaining observers, correlations were strongly positive ($> .7$) in each session. Given no explicit instructions, observers were able to quickly learn and make relatively accurate estimates via simple feedback. Furthermore, averaged correlations improved to well over .9 in the fourth session for both groups.

An important question posed was the degree of control by the two operations, and whether responding was consistent with control by one or two operations (i.e., Torgerson's conjecture). To quantify this, multiple regression was used to determine the degree of control that differences and ratios of brightness values had over responding. Unstandardized regression coefficients (B) measured control of participants' responses by the trained and untrained relations. Figure 4 shows the averaged regression coefficients for the two groups. A prominent feature shown is that the ratio group showed greater variability, as evidenced by wider error bars. The bottom right panel shows the ratio group data presented with the exclusion of observer Wa, and shows that the wide error bars were due to this observer. Figure 4 is generally representative of individual data. A repeated measures ANOVA with session and relation (trained vs untrained) as within subjects factors, and group (difference vs ratio) as between subjects factor was conducted. The effect of session was statistically significant, $F(3, 18) = 3.171, p = .050$. Also, the interaction between session and relation was found to be statistically significant, $F(3, 18) = 4.950, p = .011$. There was no effect of relation. The interaction effect suggested that control by the trained relation increased, whereby control by the untrained relation decreased over sessions. Figure 5 depicts the individual multiple regression analysis of the 8 observers. Over sessions, the regression coefficients for the trained relation increased for all observers. Moreover, the opposite trend is found in the untrained relation whereby their coefficients decreased over sessions.

A more detailed report of the multiple regression results is shown in Table 2. For each observer, unstandardized regression weights of both relations and overall R^2 s are shown for each session. Presence of significant control by the difference or ratio coefficients are noted with asterisks (e.g., * for $ps < 0.05$). Among the 32 sessions combined between observers, 21 (65.62%) of them showed significant control ($p < 0.05$) by both relations. Additionally, all observers had shown significant control by both relations in at least one session. These results suggest that responding cannot be accounted by solely one operation, contrary to Torgerson's conjecture. In line with previous analyses on measures of accuracy, R^2 increased in the fourth session from the first session for all observers. Over sessions, observers' responding generally became more closely aligned with the trained relation.

Finally, we examined whether patterns of variability in responding were more consistent with predictions of the ANS or OTS. For these analyses, the trained correct value for each stimulus pair was sorted into one of 20 bins by truncating to intervals of .05, and we planned to calculate standard deviations for responding with correct values in each bin. However, variability for responses close to 0 or 1 might have been attenuated because the response bar was bounded. To avoid this problem, for trained values that were $\leq .50$, we estimated the standard deviation as twice the standard deviation of responses that were greater than or equal to the median response for that bin, and for trained values $> .50$, as twice the standard deviation of responses that were less than the median response for that bin. This approach assumes that the response distribution for each bin is symmetrical, but should produce estimates of variability that are not compromised by the response bar boundaries.

Estimated standard deviations for each session, averaged over bins and observers, are shown in Figure 6. A repeated-measures ANOVA with session and group as factors found a significant effect of session, $F(3, 18) = 9.703$, $p < .001$, $\eta_p^2 = .62$, but the effect of group and the interaction were not significant ($ps > .50$). Pairwise comparisons found that standard

deviations for session 1 were greater than for sessions 2-4, but the latter did not systematically differ. These results suggest that responding became more consistent after session 1, but did not change substantially thereafter.

To test whether patterns of variability were more consistent with responding based on the ANS or OTS, estimated standard deviations were plotted against trained values for each bin. Because the ANOVA in Figure 6 found that variability did not decrease after session 1, results from sessions 2-4 were used for this analysis. As Figure 7 shows, standard deviations did not change systematically with trained values. For both ratio (left panel) and difference (right panel) groups, regression slopes were not significantly different from zero, $R^2 = .05$ and $.03$, $p = .40$ and $.48$, respectively. These results are consistent with responding based on the OTS, which predicts that variability should be constant, rather than the ANS which predicts that variability should increase with as the magnitude of the trained ratio or difference increases.

Experiment 2

In Experiment 2, observers were trained to respond based on ratio or differences of line length pairs. Line length has not previously been used as a stimulus dimension in Grace et al.'s (2018) procedure, so Experiment 2 allows us to assess Torgerson's conjecture with this dimension. Previous research (Parker et al., 1975; Schneider & Bissett, 1988) suggests that observers can accurately judge ratios of line lengths, and that when asked to judge differences may respond based on ratios. A recent report by Masin, Brancaccio, and Tomassetti (2019) also found that observers could judge line length ratios accurately. We were interested to compare results for observers trained to respond based on differences or ratios of line length in Grace et al.'s task.

Method

Participants. Eight observers were recruited for participation (3m, 5f). For Experiment 2, B, Mm, Mi, and R were in the difference condition, and Al, An, De, Wi were in the ratio condition. All were naïve to the purpose of the research, and received either a NZ\$60 voucher or course credits for completing all four sessions. All had reported normal or corrected-to-normal vision.

Materials. Pairs of straight lines with differing lengths were used as stimuli. The lines measured 1 mm in width. The lengths varied between 50 and 350 and were always integers, equating to 22 mm to 154 mm in length. The lengths were sampled randomly to make the stimulus sets for each session. The midpoints of the lines were 18 cm apart and 10 cm from the top of the screen. Lines were yellow (R255, G255, B0) to avoid confusion with the response bar. Stimuli were displayed side-by-side and the shorter line was always on the observer's left side. Additionally, the lines were tilted at randomised angles with a difference of at least 30°. This was made to prevent strategies such as directly comparing the length of the lines (such as subtracting the length of the two lines), which would bias responses based on stimuli differences. As both luminance and line lengths can be defined numerically, correct responses are defined as those within the range of $\pm 7\%$ of the trained value as in the same way as Experiment 1. The response bar, background, and feedback oval in Experiment 2 were all identical to Experiment 1.

Procedure. The procedure for Experiment 2 was the same as Experiment 1. The instructions were similar but referred to line lengths.

Results

Measures of accuracy for Experiment 2 are shown in Figure 8. Overall, accuracy levels increased over sessions for both groups.

For average percent correct, the effects of session, $F(3, 18) = 16.006, p < .00, \eta_p^2 = .727$, and group, $F(1, 6) = 21.086, p < .00, \eta_p^2 = .778$, were both significant, but the interaction was not ($p = .236$). Post-hoc analyses (Newman-Keuls) confirmed that the average percent correct trials for Session 1 were significantly lower than for Sessions 2, 3, and 4. From the first to final session, percent correct improved for both groups, but was overall greater for the ratio group.

For absolute deviations, the effects of session, $F(3, 18) = 24.186, p < .00, \eta_p^2 = .801$, group, $F(1, 6) = 14.795, p < .01, \eta_p^2 = .711$, and interaction $F(3, 18) = 4.938, p < .01, \eta_p^2 = .451$, were significant. Post-hoc tests (Newman-Keuls) showed that averaged deviations in Session 1 were significantly higher when compared to Sessions 2, 3, and 4. For RMS, both groups showed a reduction in error across sessions. A significant effect was found for RMS in both session, $F(3, 18) = 18.119, p < .00, \eta_p^2 = .751$ and group, $F(1, 6) = 10.795, p < .01, \eta_p^2 = .643$, but not the interaction ($p = .159$). Post-hoc tests (Newman-Keuls) confirmed that the averaged RMS values for Session 1 were significantly lower than for those in Session 2, 3, and 4. Correlations between responses and correct values significantly increased across sessions, $F(3, 18) = 14.117, p < .00, \eta_p^2 = .702$. There was no significant effect of group and interaction ($ps > .164$). Post-hoc tests (Newman-Keuls) confirmed that the averaged correlations in Session 1 were significantly less than the averages for Sessions 2, 3, and 4.

Using the results of Experiment 2, Table 3 shows the averaged means and standard deviations of sessions 1 and 4 of the four measures. Similar to Experiment 1, both groups showed an improvement for all four measures across the training sessions. It is notable that

these improvements were generally stronger for the difference group in Experiment 2, but they were stronger for the ratio group in Experiment 1.

Figures 9 and 10 show representative scatterplots of the trained values plotted against response for the difference and ratio groups, respectively. Similar to Experiment 1 (Figures 2 & 3), responses were strongly related to the trained values (scaled differences or ratios). Correlations with the trained values appeared to strengthen over sessions. Out of 16 participants, 15 (except Db; Figure 10, centre column) showed greater R^2 in the final session compared to the first session.

Figure 11 shows multiple regression coefficients for trained and untrained relations, averaged across participants. Coefficients were greater for the trained relation for both groups, and across sessions, coefficients for the trained relation increased whereas those for the untrained relation decreased. A repeated-measures ANOVA with session and relation (trained vs untrained) as within subjects factors, and group (difference vs ratio) as between subjects factor found significant effects of session, $F(3, 18) = 5.225, p = .009$, and the session x relation interaction, $F(3, 18) = 3.555, p = .035$, was found to be significant. Additionally, the effect of relation was significant, $F(1, 6) = 68.779, p < .001$. These results are similar to those from Experiment 1 and suggest that control by the trained relation strengthened over sessions, while control by the untrained relation weakened.

Multiple regression results for individual observers are shown in Table 4. Of the 32 sessions in Experiment 2, 21 (65.62%) of them displayed significant control ($p < 0.05$) by both differences and ratios. Furthermore, observers were able to respond well to the trained feedback. All 32 sessions showed strong ($p < 0.001$) control of the trained relation. R^2 also increased over sessions for all observers. Across both experiments, all but one observer (Wa; Figure 5, bottom right panel) showed greater coefficients for the trained relation compared to

the untrained relation in all four sessions. In addition, all observers displayed significant control ($p < 0.05$) by the trained relation in all but one session (Wa in the 4th session).

Response variability was analysed similar to Experiment 1: Responses were grouped in bins by trained value, and standard deviations were estimated for each bin based on half of the response distribution to reduce attenuation by the bounded response bar. Figure 12 shows the estimated standard deviations, averaged across bins, across sessions for both groups. A repeated-measures ANOVA found a significant effect of session, $F(3, 18) = 11.787$, $p < .001$, $\eta_p^2 = .66$, while the effects of group and the interaction were not significant ($ps > .20$).

Pairwise comparisons found that similar to Experiment 1, response variability for both groups decreased from session 1 to 2, but did not significantly change thereafter.

Figure 13 shows estimated standard deviations plotted against trained values for each bin. For the difference group (right panel), the regression slope was close to zero, $R^2 = .00$, $p = .96$. However the slope was significantly positive for the ratio group (left panel), $R^2 = .62$, $p < .01$. The slope was influenced by an outlier (trained value = .80), but when it was excluded the results remained significant, $R^2 = .48$, $p < .01$. Results for the difference group are similar to Experiment 1 and suggest that response variability does not vary with the trained value, as predicted by the OTS. However, the increasing variability for the ratio group is consistent with responding based on the ANS. Possible reasons for this discrepancy are considered below.

General Discussion

Observers were trained in a nonsymbolic, ‘artificial algebra’ task (Grace et al., 2018) to respond based on differences or ratios of stimulus pairs that varied in brightness (Experiment 1) or line length (Experiment 2). On each trial, they viewed a pair of stimuli and

clicked along a horizontal response bar on the screen below, with feedback provided based on the difference or ratio of the nominal stimulus values, linearly scaled along the response bar. Each completed four sessions on separate days ($n = 1,334$ trials in total), so we could evaluate the effects of increased training in this task. In both experiments, results showed that observers learned the task quickly regardless of whether differences or ratios were trained, and accuracy increased over sessions. Regression analyses to predict responding showed that coefficients for the trained operation increased over sessions, while those for the untrained operation (i.e., ratios for the difference groups, differences for the ratio groups) decreased. This suggests that perceptual learning occurred: Observers' ability to discriminate differences and ratios of stimulus magnitudes increased with extended training. Although perceptual learning has been documented for a large number of visual phenomena such as discrimination for single features like orientation, spatial frequency and contrast (see Ahissar & Hochstein, 2004, Doshier & Lu, 2017, Watanabe & Sasaki, 2015, for reviews), to our knowledge the present results are the first demonstration that such learning can occur for a quantitative relation between stimulus values.

A second major aim of our study was to revisit Torgerson's (1961) conjecture that the perceptual system compares stimulus magnitudes with a single operation, either a difference or a ratio. As in Grace et al. (2018), we used multiple regressions to test if the untrained operation contributed significant variance as a predictor in a model with the trained operation. Overall, coefficients for the untrained operation were significant in 67.2% (43/64) of individual sessions (68.8% [22/32] for Experiment 1; 64.3% [21/32] for Experiment 2). These results are similar to those reported by Grace et al. (63.0% [51/81] of individual sessions) and show that observers responded based on two operations, even though only one was trained, contrary to Torgerson's conjecture. The comparability of results across experiments here and in Grace et al. (2018) using both intensive (brightness) and extensive

(line length, area, numbers of dots) stimulus dimensions is notable, and suggests that differences and ratios of subjective magnitudes may be computed in a central system (cf. Walsh, 2003). Our results add to those of previous studies which have begun to identify the computations performed by the perceptual system when making comparisons within and between stimulus modalities (e.g., Ariely, 2001; Balci & Gallistel, 2006).

We also wanted to assess response variability to gain some insight into how performance might depend on systems identified in research on numerical cognition. Prior studies suggest that we have two systems for estimating magnitudes, both shared with nonhumans, an ‘approximate number system’ (ANS) for relatively imprecise estimation of larger magnitudes, and an ‘object tracking system’ (OTS) for more precise identification of smaller quantities (Dehaene, 2011; Feigenson, Dehaene, & Spelke, 2004). The key distinction between these systems is response variability; for the ANS, variability increases with the trained value, but is constant for the OTS. We predicted that if both ratios and differences are automatically computed by the perceptual system (Grace et al., 2018), then variability patterns should be consistent with the OTS.

Results for three of four groups supported this prediction: Standard deviations did not vary systematically with trained values for observers with feedback based on ratios and differences in Experiment 1 (Figure 7) and differences in Experiment 2 (Figure 13). The exception was the ratio group in Experiment 2, where standard deviations increased with trained values. This result suggests that observers’ responding was less consistent for larger ratios of line lengths, but not brightnesses. Although reasons for this discrepancy are unclear, one possibility is that because line length is an extensive dimension and stimuli were limited by screen size, variation among larger ratios was more sensitive to changes in the shorter line length. Consequently, larger ratios of line lengths may have been more difficult to discriminate than shorter ratios. This problem would not apply to brightness (an intensive

dimension), and might have been exacerbated for lines by the positively-skewed distribution of ratios. Thus, the majority of our results found that variability in responding did not change systematically with the trained value, consistent with an automatic process similar to the OTS for computing differences and ratios of stimulus values. An interesting question for future research is whether OTS-like variability patterns can be obtained in our task with ratios of line lengths using a stimulus set designed to avoid potential problems with discriminability of larger ratios (e.g., by using a non-skewed distribution of trained ratios).

As previously noted, results for observer Wa (Experiment 1, ratio group) were anomalous. In the first session, Wa had a weak correlation ($r = .22$) between responses and trained values which was considerably lower than all others (the next lowest was $r = .65$). Regressions for Wa showed that difference coefficients were always strongly positive and greater than those for ratios (which were negative), but decreased over sessions, remaining positive, while ratio coefficients increased (but remaining negative; see Figure 5, lower right panel). If differences and ratios are automatically computed by the perceptual system, as Grace et al. (2018) proposed, selective attention or related effects would be expected to mediate responding, and observers might be expected to ‘confuse’ differences and ratios, at least to some extent. From this perspective, Wa’s anomalous results might indicate that this observer was struggling to respond based on differences, despite the ratio-derived feedback. The negative coefficients for ratios might indicate inhibitory or suppression-like effects.

However, an alternative explanation for Wa’s results might be multicollinearity: Because ratios and differences were highly correlated in the present experiments ($r \sim .90$), multicollinearity might have resulted in unreliable regression coefficients (Kraha, Turner, Nimon, Zientek, & Henson, 2012). Because ratios and differences were also highly correlated in Grace et al.’s (2018) study, to test for multicollinearity they conducted Monte Carlo analyses in which simulated responses were defined as weighted averages of scaled

ratios and differences, plus Gaussian noise. The weights for ratios and differences were varied (between 0 and 0.9) and the degree of noise were varied ($\sigma = .05, .1, .2$), and they found that multiple regression was able to recover accurate estimates of the weights for ratios and differences used to generate the data. Thus Grace et al. concluded that their regression results were not an artefact of multicollinearity. Results of other analyses (ordinal comparisons and non-metric scaling) also agreed with the multiple regressions in terms of identifying the extent to which individual observers relied on differences or ratios, so Grace et al. concluded that multicollinearity was not responsible for their results.

Because Grace et al. (2018) did not include negative weights in their Monte Carlo analyses, we conducted additional simulations to check Observer Wa's results. Specifically, simulated responses were generated for the stimuli used in Experiment 1, assuming weights of .8 for differences, -.6 for ratios, and $\sigma = .05$. A multiple regression was then conducted to predict the simulated responses with the scaled differences and ratios, and the regression coefficients and R^2 were recorded. This procedure was repeated 1000 times. Results showed that the coefficients were successfully recovered by multiple regression: The average difference coefficient $M = .782$ ($SD = .101$), ratio coefficient $M = -.589$ ($SD = .100$), mean $R^2 = .759$ ($SD = .071$), and intercept $M = .005$ ($SD = .019$). This shows that multiple regression can accurately estimate coefficients similar to those found with Wa's data with the stimuli used in the present experiment, and so the anomalous results for this observer are likely not due to multicollinearity. This simulation provides some support for our speculation that the negative ratio coefficient was indicative of Wa 'inhibiting' ratios and trying to respond based on differences. Observer Wa's struggles with brightness ratios are consistent with a recent report by Masin, Brancaccio, and Tomassetti (2019), which suggests that observers may be unable to estimate accurately ratios of intensive stimulus dimensions. However, why the

other observers did not struggle in the same way is unclear. Further research is needed to explore the range of individual differences in our task.

There are some limitations in the present research that should be noted. First, the small sample size in each group ($n=4$ in both experiments) may limit the generality of the results. Small sample sizes also permit outliers to have a much greater impact on group results – for example, the difference between regression results depending on whether observer Wa's results were omitted (Figure 4). Nevertheless, the trend for control by the trained and untrained operations to increase and decrease, respectively, were found regardless of whether Wa's data were included. Although the small sample sizes do not allow us to reach strong conclusions about individual differences in the present task, the evidence for effects of extended training and perceptual learning appear reliable.

Future research should consider alternative designs to assess implicit learning processes with nonsymbolic stimuli. For example, auditory stimuli have not been tested with the 'artificial algebra' task, but this would be an interesting option. In addition, it would be worthwhile to examine performance in the present task with populations other than young adults (i.e., university students), such as children or older adults. Given the extensive numerical cognition research which shows that numerosity discrimination is predictive of real-world mathematics achievement (Chen & Li, 2014; Halberda, Mazocco, & Feigenson, 2008), it would also be interesting to see if performance in our 'artificial algebra' task was related to mathematical ability.

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References

- Ahissar, M., & Hochstein, S. (2004). The reverse hierarchy theory of visual perceptual learning. *Trends in Cognitive Sciences*, 8(10), 457-464.
- Ariely, D. (2001). Seeing sets: Representation by statistical properties. *Psychological Science*, 12(2), 157-162.
- Balci, F. & Gallistel, C. R. (2006). Cross-domain transfer of quantitative discriminations: Is it all a matter of proportion? *Psychonomic Bulletin & Review*, 13(4), 636-642.
- Birnbaum, M. H. (1978). Differences and ratios in psychological measurement. *Cognitive Theory*, 3, 33-74.
- Birnbaum, M. H., Anderson, C. J., & Hynan, L. G. (1989). Two operations for "ratios" and "differences" of distances on the mental map. *Journal of Experimental Psychology: Human Perception and Performance*, 15(4), 785-796.
- Birnbaum, M. H., & Elmasian, R. (1977). Loudness "ratios" and "differences" involve the same psychophysical operation. *Perception & Psychophysics*, 22(4), 383-391.
- Birnbaum, M. H., & Veit, C. T. (1974). Scale convergence as a criterion for rescaling: Information integration with difference, ratio, and averaging tasks. *Perception & Psychophysics*, 15(1), 7-15.
- Chen, Q., & Li, J. (2014). Association between individual differences in non-symbolic number acuity and math performance: A meta-analysis. *Acta Psychologica*, 148, 163-172.
- Dehaene, S. (2003). The neural basis of the Weber–Fechner law: a logarithmic mental number line. *Trends in Cognitive Sciences*, 7(4), 145-147.

- Dehaene, S. (2011). *The Number Sense: How the Mind Creates Mathematics* (2nd Edition). Oxford University Press.
- Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science*, 320(5880), 1217-1220.
- Dosher, B., & Lu, Z.-L. (2017). Visual perceptual learning and models. *Annual Review of Vision Science*, 3, 343-363.
- Etienne, A. S., & Jeffery, K. J. (2004). Path integration in mammals. *Hippocampus*, 14(2), 180-192.
- Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences*, 8(7), 307-314.
- Gibbon, J., & Church, R. M. (1981). Time left: linear versus logarithmic subjective time. *Journal of Experimental Psychology: Animal Behavior Processes*, 7(2), 87-108.
- Grace, R. C., Carvell, G. E., Morton, N. J., Grice, M., Wilson, A. J., & Kemp, S. (2020). On the origins of computationally complex behavior. *Journal of Experimental Psychology: Animal Learning and Cognition*, 46(1), 1-15.
- Grace, R. C., Morton, N. J., Ward, M. D., Wilson, A. J., & Kemp, S. (2018). Ratios and differences in perceptual comparison: A reexamination of Torgerson's conjecture. *Journal of Mathematical Psychology*, 85, 62-75.
- Halberda, J., Mazocco, M. M., & Feigenson, L. (2008). Individual differences in non-verbal number acuity correlate with maths achievement. *Nature*, 455(7213), 665-668.
- Koh, K., & Meyer, D. E. (1991). Function learning: Induction of continuous stimulus-response relations. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 17(5), 811-836.

- Kraha, A., Turner, H., Nimon, K., Zientek, L., & Henson, R. (2012). Tools to support interpreting multiple regression in the face of multicollinearity. *Frontiers in psychology*, 3, 44.
- Krueger, L. E. (1989). Psychophysical law: Keep it simple. *Behavioral and Brain Sciences*, 12(2), 299-320.
- Leibovich, T., Katzin, N., Harel, M., & Henik, A. (2017). From “sense of number” to “sense of magnitude”: The role of continuous magnitudes in numerical cognition. *Behavioral and Brain Sciences*, 40, e164.
- Lucas, C. G., Griffiths, T. L., Williams, J. J., & Kalish, M. L. (2015). A rational model of function learning. *Psychonomic Bulletin & Review*, 22(5), 1193-1215.
- Masin, S. C. (2014). Test of the Ratio Judgment Hypothesis. *The Journal of General Psychology*, 141(2), 130-150.
- Masin, S. C., Brancaccio, A., & Tomassetti, A. (2019). Tests of the abilities to judge ratios of extensive and intensive sensory magnitudes. *Attention, Perception, & Psychophysics*, 81(6), 2102-2119.
- Mazur, J. E. (2002). Evidence against a constant-difference effect in concurrent-chains schedules. *Journal of the Experimental Analysis of Behavior*, 77(2), 147-155.
- McDaniel, M. A., Cahill, M. J., Robbins, M., & Wiener, C. (2014). Individual differences in learning and transfer: Stable tendencies for learning exemplars versus abstracting rules. *Journal of Experimental Psychology: General*, 143(2), 668-693.
- Parker, S., Schneider, B., & Kanow, G. (1975). Ratio scale measurement of the perceived lengths of lines. *Journal of Experimental Psychology: Human Perception and Performance*, 1(2), 195-204.
- Pothos, E. M. (2007). Theories of artificial grammar learning. *Psychological Bulletin*, 133(2), 227-244.

- Reber, A. S. (1989). Implicit learning and tacit knowledge. *Journal of Experimental Psychology: General*, 118(3), 219-235.
- Rule, S. J., Curtis, D. W., & Mullin, L. C. (1981). Subjective ratios and differences in perceived heaviness. *Journal of Experimental Psychology: Human Perception and Performance*, 7(2), 459-486.
- Savastano, H. I., & Fantino, E. (1996). Differences in delay, not ratios, control choice in concurrent chains. *Journal of the Experimental Analysis of Behavior*, 66(1), 97-116.
- Schneider, B., & Bissett, R. (1988). "Ratio" and "difference" judgments for length, area, and volume: are there two classes of sensory continua? *Journal of Experimental Psychology: Human Perception and Performance*, 14(3), 503-512.
- Schneider, B., Parker, S., Farrell, G., & Kanow, G. (1976). The perceptual basis of loudness ratio judgments. *Perception & Psychophysics*, 19(4), 309-320.
- Srinivasan, M. V. (2015). Where paths meet and cross: navigation by path integration in the desert ant and the honeybee. *Journal of Comparative Physiology A*, 201(6), 533-546.
- Tan, L., & Grace, R. C. (2012). Human nonverbal discrimination of relative and absolute number. *Learning & Behavior*, 40(2), 170-179.
- Torgerson, W. S. (1961). Distances and ratios in psychophysical scaling. *Acta Psychologica*, 19, 201-205.
- Walsh, V. (2003). A theory of magnitude: common cortical metrics of time, space and quantity. *Trends in Cognitive Sciences*, 7(11), 483-488.
- Watanabe, T., & Sasaki, Y. (2015). Perceptual learning: toward a comprehensive theory. *Annual Review of Psychology*, 66, 197-221.

Table 1. Results of four measures of accuracy of Experiment 1 (brightness) difference and ratio group. Listed for each measure are the averaged means and standard deviations for the trained relation in sessions 1 and 4. The % improved represents the percentage increase for percentage correct and r , and percentage decrease for absolute deviation and RMS.

Difference Group					Ratio Group				
<u>Measure</u>	<u>Session</u>	<u>Mean</u>	<u>SD</u>	<u>% improved</u>	<u>Measure</u>	<u>Session</u>	<u>Mean</u>	<u>SD</u>	<u>% improved</u>
%Corr	1	0.545	0.111	4.911	%Corr	1	0.584	0.167	14.650
	4	0.572	0.097			4	0.670	0.097	
AbsDev	1	0.093	0.014	16.062	AbsDev	1	0.101	0.064	33.336
	4	0.078	0.021			4	0.067	0.020	
RMS	1	0.115	0.035	10.021	RMS	1	0.145	0.085	33.734
	4	0.104	0.030			4	0.096	0.033	
r	1	0.867	0.075	3.249	r	1	0.705	0.282	22.452
	4	0.895	0.060			4	0.863	0.109	

Table 2. Results of multiple regressions for Experiment 1. Listed for both groups are each observer's regression coefficients (unstandardized) for difference and ratio values predicting responding and R^2 for each session.

Difference Group					Ratio Group				
<u>Observer</u>	<u>Session</u>	<u>Diff b</u>	<u>Ratio b</u>	<u>R²</u>	<u>Observer</u>	<u>Session</u>	<u>Diff b</u>	<u>Ratio b</u>	<u>R²</u>
Db	1	0.423***	0.402***	0.615	Ar	1	0.171*	0.627***	0.569
	2	0.506***	0.157	0.581		2	0.089	0.826***	0.807
	3	0.635***	0.173*	0.697		3	0.076	0.855***	0.838
	4	0.642***	0.194*	0.669		4	0.008	0.904***	0.855
Ja	1	0.817***	0.132	0.753	K	1	0.177***	0.571***	0.775
	2	1.007***	-0.070	0.844		2	0.195***	0.547***	0.810
	3	0.910***	0.087	0.906		3	0.101*	0.690***	0.806
	4	0.818***	0.218***	0.916		4	0.162**	0.627***	0.802
Jo	1	0.533***	0.385***	0.817	L	1	0.197**	0.623***	0.795
	2	0.519***	0.478***	0.856		2	0.130**	0.679***	0.818
	3	0.575***	0.456***	0.881		3	0.089	0.782***	0.843
	4	0.809***	0.035	0.854		4	0.076	0.841***	0.867
Ma	1	0.425***	0.478***	0.684	Wa	1	0.863***	-0.647***	0.086
	2	0.434***	0.387***	0.638		2	0.905***	-0.641***	0.154
	3	0.486***	0.383***	0.737		3	0.818***	-0.474***	0.287
	4	0.593***	0.308***	0.775		4	0.607***	-0.095	0.491

* = $p < 0.05$

** = $p < 0.01$

*** = $p < 0.001$

Table 3. Results of four measures of accuracy of Experiment 2 (line lengths) difference and ratio group. Listed for each measure are the averaged means and standard deviations for the trained relation in sessions 1 and 4. The % improved represents the percentage increase for percentage correct and r , and percentage decrease for absolute deviation and RMS.

Difference Group					Ratio Group				
<u>Measure</u>	<u>Session</u>	<u>Mean</u>	<u>SD</u>	<u>% improved</u>	<u>Measure</u>	<u>Session</u>	<u>Mean</u>	<u>SD</u>	<u>% improved</u>
%Corr	1	0.390	0.040	30.153	%Corr	1	0.68229	0.080	10.251
	4	0.507	0.086			4	0.752	0.112	
AbsDev	1	0.132	0.023	28.891	AbsDev	1	0.071	0.010	20.585
	4	0.094	0.020			4	0.057	0.017	
RMS	1	0.179	0.035	29.166	RMS	1	0.111	0.008	22.019
	4	0.127	0.025			4	0.087	0.020	
r	1	0.681	0.128	23.195	r	1	0.799	0.031	8.090
	4	0.840	0.074			4	0.863	0.057	

Table 4. Results of multiple regressions for Experiment 2. Listed for both groups are each observer's averaged regression coefficients (unstandardized) for difference and ratio values predicting responding and R^2 for each session.

Difference Group					Ratio Group				
Observer	Session	Diff b	Ratio b	R^2	Observer	Session	Diff b	Ratio b	R^2
B	1	0.380***	0.167	0.291	Al	1	0.196***	0.652***	0.653
	2	0.492***	0.302***	0.647		2	0.017	0.892***	0.898
	3	0.577***	0.216*	0.723		3	0.049*	0.863***	0.922
	4	0.573***	0.274***	0.725		4	0.003	0.929***	0.870
Mm	1	0.689***	0.295***	0.713	An	1	0.161***	0.646***	0.704
	2	0.696***	0.271***	0.782		2	0.067	0.731***	0.746
	3	0.850***	0.207**	0.848		3	0.098*	0.678***	0.724
	4	0.889***	0.062	0.834		4	0.056	0.647***	0.718
Mi	1	0.644***	-0.052	0.409	De	1	0.364***	0.370***	0.599
	2	0.554***	0.361***	0.647		2	0.230***	0.596***	0.665
	3	0.500***	0.233*	0.541		3	0.128**	0.678***	0.698
	4	0.458***	0.195*	0.542		4	0.134**	0.548***	0.635
R	1	0.498***	0.190	0.493	Wi	1	0.151**	0.613***	0.599
	2	0.683***	0.152*	0.771		2	0.134**	0.842***	0.795
	3	0.848***	-0.061	0.791		3	0.122**	0.757***	0.755
	4	0.755***	0.060	0.735		4	-0.029	0.887***	0.768

* = $p < 0.05$

** = $p < 0.01$

*** = $p < .001$

Figure 1. Four measures of accuracy for Experiment 1, average %correct, average absolute deviation, RMS, and correlation coefficient, r . The solid line represents the difference group ($n=4$), whereas the dotted line represents the ratio group ($n=4$). Bars indicate one standard error.

Figure 2. Scatterplots for Experiment 1 difference group showing responses as a function of trained values. The solid line indicates perfect performance; the dotted line indicates line of best fit (equation shown). Columns (left to right) show the observers with highest and lowest accuracy, and the averaged results, respectively. Rows (top to bottom) show sessions 1 to 4.

Figure 3. Scatterplots for Experiment 1 ratio group showing responses as a function of trained values. The solid line indicates perfect performance; the dotted line indicates line of best fit (equation shown). Columns (left to right) show the observers with highest and lowest accuracy, and the averaged results, respectively. Rows (top to bottom) show sessions 1 to 4.

Figure 4. Multiple regression coefficients (unstandardized) for trained and untrained operations for difference and ratio groups in Experiment 1 (upper row), and the ratio group excluding Observer Wa (lower right panel). Bars indicate one standard error.

Figure 5. Multiple regression coefficients (unstandardized) by session for individual observers in the difference (left column) and ratio (right column) groups in Experiment 1.

Figure 6. Estimated standard deviations across sessions for Experiment 1. Filled and unfilled circles represent the difference and ratio groups, respectively. Bars indicate one standard error.

Figure 7. Variability analysis of Experiment 1 (brightness). Estimated standard deviations are plotted against the average trained value for each bin, for the ratio (left panel) and difference (right panel) groups. Dashed lines are best-fitting regressions. Bars indicate one standard error.

Figure 8. Four measures of accuracy for Experiment 2, average %correct, average absolute deviation, RMS, and r . The solid line represents the difference group ($N=4$), whereas the dotted line represents the ratio group ($N=4$). Bars indicate one standard error.

Figure 9. Scatterplots for Experiment 2 difference group showing responses as a function of trained values. The solid line indicates perfect performance; the dotted line indicates line of best fit (equation shown). Columns (left to right) show the observers with highest and lowest accuracy, and the averaged results, respectively. Rows (top to bottom) show sessions 1 to 4.

Figure 10. Scatterplots for Experiment 2 ratio group showing responses as a function of trained values. The solid line indicates perfect performance; the dotted line indicates line of best fit (equation shown). Columns (left to right) show the observers with highest and lowest accuracy, and the averaged results, respectively. Rows (top to bottom) show sessions 1 to 4.

Figure 11. Multiple regression coefficients (unstandardized) for trained and untrained operations for difference (left panel) and ratio (right panel) groups in Experiment 2. Bars indicate one standard error.

Figure 12. Estimated standard deviations across sessions for Experiment 1. Filled and unfilled circles represent the difference and ratio groups, respectively. Bars indicate one standard error.

Figure 13. Variability analysis of Experiment 1 (brightness). Estimated standard deviations are plotted against the average trained value for each bin, for the ratio (left panel) and difference (right panel) groups. Dashed lines are best-fitting regressions. Bars indicate one standard error.