Generalized inductive item tree analysis

Ali Ünlü^{a,*}, Martin Schrepp^b

^a*Technical University of Munich (TUM), Munich, Germany*

^b*SAP SE, Walldorf, Germany*

Abstract

Inductive item tree analysis is an established method of Boolean analysis of questionnaires. By exploratory data analysis, from a binary data matrix, the method extracts logical implications between dichotomous test items based on their positive item scores. For example, assume that we have the problems i and j of a test that can be solved or failed by subjects. With inductive item tree analysis, an implication between the items i and j can be uncovered, which has the interpretation “If a subject is able to solve item i , then this subject is also able to solve item j ”. Hence, in the current form of the method, (a) solely dichotomous items are considered, and (b) conclusions are drawn from only positive item scores. In this paper, we provide extensions to these restrictions. First, as remedy for (b), we focus on the dichotomous formulation of the inductive item tree analysis algorithm and describe a procedure of how to extend the dichotomous variant to also include negative item scores. Second, to address (a), we further extend our approach to the general case of polytomous items, when more than two answer categories are possible. Thus, we introduce extensions of inductive item tree analysis that can deal with nominal polytomous and ordinal polytomous answer scales. To show their usefulness, the dichotomous and polytomous extensions proposed in this paper are illustrated with empirical data and in a simulation study.

Keywords: Item tree analysis, Boolean analysis of questionnaires, Exploratory data analysis, Logical implication, Polytomous item, Nominal answer scale, Ordinal answer scale

*Corresponding author. TUM School of Education, Technical University of Munich (TUM), Arcisstrasse 21, 80333 Munich, Germany

Email addresses: aligalibuenlue@gmail.com (Ali Ünlü), martin.schrepp@sap.com (Martin Schrepp)

1. Introduction

Item tree analysis comprises well-established methods of Boolean analysis of questionnaires (e.g., Bart & Krus, 1973; Flament, 1976). The goal is to extract logical implications between dichotomous (i.e., two-valued) items from binary (e.g., 0/1) data matrices by exploratory data analysis. A first algorithm was developed by Leeuwe (1974), which was further refined and optimized in a number of papers (e.g., Sargin & Ünlü, 2009; Schrepp, 1999; Ünlü & Malik, 2008). These latter refinements have been labeled “inductive” item tree analysis (IITA), to differentiate them from the original Leeuwe (1974)’s algorithm. A detailed description of the relation of IITA to other methods of Boolean analysis of questionnaires can be found in Schrepp (2003).

Assume that we have the dichotomous items i and j of a (finite) test I , $i, j \in I$, which can be solved (coded 1) or failed (coded 0) by subjects. If a subject s is able to solve (i.e., master) problem i , we write $s(i) = 1$, and if subject s is not able to solve (i.e., not master) problem i , we write $s(i) = 0$. Implications, $i \rightarrow j$, uncovered by (inductive) item tree analysis have always the form “If a subject is able to solve item i , then this subject is also able to solve item j ”, or put formally, $s(i) = 1$ implies $s(j) = 1$ for all s . Thus, in the current formulation of IITA, conclusions are drawn from only positive (latent) item scores (e.g., $s(i) = 1$, $s(j) = 1$). In this paper, for a subject s and item $i \in I$, we will also denote the manifest response, of 1 or 0, with $s(i) = 1$ or $s(i) = 0$, respectively. From the corresponding context, it will become apparent whether manifest or latent “responses”, “answers”, “outcomes”, and “item scores”, the terms being synonymously used, are considered.

Defining implications based on only positive outcomes is a limitation of IITA. The main reason for this restriction is that IITA was primarily developed for use in knowledge space theory (KST) (Albert & Lukas, 1999; Doignon & Falmagne, 1985, 1999; Falmagne, Albert, Doble, Eppstein & Hu, 2013). The core formulation of KST is based on dichotomous items, and the concept of positive outcome, to solve or master an item, is essential to KST. Take as example, the skills-based extensions of KST in application domains such as mathematics or natural sciences. Content-wise, these domains are highly structured. So, from a substantive viewpoint, one can introduce skills, postulated to underlie the items. In particular, the mastery of an item can be seen to imply the

acquaintance of corresponding skills. Thus, if a problem is mastered by a subject, we can assume that the subject has attained the necessary skills for this problem, and therefore, the subject should also be able to solve all problems that require only a subset of those skills. This underlines the importance of a positive outcome, that is, of solution or mastery (of items and skills), in KST.

However, item tree analysis, or similar techniques of Boolean analysis, can also be applied to analyze data in more diverse application areas. As we will derive and see for the IITA algorithm, those methods are not restricted to special types of items or outcomes, but can operate on any set of dichotomous or polytomous items, with arbitrary item scores. To further motivate this, assume that we want to study questions from a knowledge domain which can be solved solely based on declarative knowledge. An example may be a test in American history that contains the question “Who was the first president of the United States?”. For this task, not much proficiency in complex problem solving is needed; a subject knows or does not know the correct answer. In particular, we can infer from an incorrect answer that the subject does not know a certain fact, and thus, that she or he will also fail on other items requiring the same declarative knowledge. In such applications, implications of the following form do also make sense, “If a subject fails to know the solution to problem i , then this subject also fails to know the solution to problem j ”. That is, additionally, we have $s(i) = 0$ implies $s(j) = 0$ for any subject s .

Things can even get more complicated when we consider applications of IITA to items from social science questionnaires (e.g., Bart & Krus, 1973; Janssens, 1999; Martin & Wiley, 2000; Schrepp, 2002; Wiley & Martin, 1999). Questionnaires of this sort typically contain statements that subjects can agree or disagree with. They can express their level of agreement or disagreement on a Likert scale (Likert, 1932). As an example, for the year 1995, consider the two items from the International Social Survey Programme (ISSP) in cross-cultural research on social science topics (Haller, Jowell & Smith, 2009):

- “Generally speaking, America is a better country than most other countries.”
- “People should support their country even if the country is in the wrong.”

Subjects can express their agreement or disagreement with these statements on a 5-point Likert scale, with the extreme poles “Agree strongly” and “Disagree strongly”. This yields polytomous

items and data. But we could also dichotomize the scale into the two answer categories “Agree” and “Disagree”, to work with dichotomous items and data. In this paper, we offer IITA algorithms for, both, the dichotomous and polytomous instances.

The basic idea behind an application of Boolean analysis of such attitude survey items is that subjects can have certain beliefs, which guide their answers to the questions. Thus, if we record a subject’s specific answer, we can use that information to draw conclusions about her or his beliefs, which explain the answer. From this knowledge, of their beliefs, it may then be possible to infer a subject’s (latent) response to other statements of the questionnaire. In other words, it does make sense to allow for and study implications between any pair of possible answer categories.

To exemplify the discussion, we will focus and elaborate on the dichotomous special case first, before we deal with the polytomous general case. If we take a closer look at such dichotomous statements as the ISSP above (coded 0/1), a method of Boolean data analysis (including IITA) should construct all four possible forms of implications ($i, j \in I$, subject s):

- $s(i) = 1 \rightarrow s(j) = 1$; that is, agreement with i implies agreement with j .
- $s(i) = 1 \rightarrow s(j) = 0$; that is, agreement with i implies disagreement with j .
- $s(i) = 0 \rightarrow s(j) = 1$; that is, disagreement with i implies agreement with j .
- $s(i) = 0 \rightarrow s(j) = 0$; that is, disagreement with i implies disagreement with j .

We will introduce a method that extends the current IITA algorithm, which can detect implications of these more general dichotomous forms.

We will also consider extensions of IITA for the detection of general implications between the answer categories of polytomous (i.e., multi-category) items. As mentioned above, social science attitude questionnaires often comprise ordinal polytomous items. Likert scales are used to capture the subjects’ responses to the items, in the form of statements with which subjects can express their degree of agreement (or disagreement). In such cases, we have a linear or ordinal answer scale that gauges the level of agreement with a statement. Another example of an ordinal polytomous answer scale are achievement tests for which points are used to assess the quality of a solution. The more points are assigned to a solution, the better is the quality of the solution. In addition to

these ordinal polytomous cases, we do also consider multiple qualitative answer categories, that is, nominal polytomous items. This type of answer scale has no (ordinal) structure imposed on it, except for the determination of equality. We will describe extensions of the dichotomous IITA algorithm to (linear) ordinal and nominal polytomous items.

This paper is organized as follows. In Section 2, we review the original, positive outcomes oriented IITA algorithm for dichotomous items (Sargin & Ünlü, 2009; Schrepp, 1999). In Section 3, we describe the dichotomous extension of the algorithm to cases where implications do not only result from positive outcomes, but can have any of the four forms of implications, and we give an empirical example using data from a survey studying the concept of national identity in Germany. In Section 4, we introduce the two most general formulations of the IITA procedure, for nominal polytomous and ordinal polytomous items. In Section 5, we describe how to simulate data, and we report a simulation study which investigates the ability of the generalized algorithms to extract the correct implications. In Section 6, we present two empirical applications with survey data on a national election in the United States and the attitude of people to the environment in Britain. In Section 7, we summarize the main results of this paper and conclude with directions for further research.

2. Description of the inductive item tree analysis algorithm

Let $I = \{i_1, \dots, i_m\}$ be a set of $m \geq 2$ dichotomous items. Assume that $D = \{d_1, \dots, d_n\}$ is a (multi)set (with repetitions) of observed response patterns of $n \geq 2$ subjects who responded to the items in I . Every response pattern $d \in D$ can be considered as a mapping $d : I \rightarrow \{0, 1\}$ that assigns the response $d(i) \in \{0, 1\}$ of a subject to each of the items $i \in I$. Equivalently, D can also be viewed as the data matrix of row-by-row observed response patterns, with the items as their columns, encoded as 0/1 vectors.

IITA constructs a set of logical implications, $i \rightarrow j$, $i, j \in I$, from D , such that the binary relation $\leq$ on I defined by all implications (i.e., $j \leq i \Leftrightarrow i \rightarrow j$) is a quasi-order, that is, reflexive and transitive. Thus, $i \rightarrow i$ for each item $i \in I$ (reflexivity), and $i \rightarrow j$ and $j \rightarrow k$ implies $i \rightarrow k$ for all items $i, j, k \in I$ (transitivity). In KST terminology, this quasi-order defined by all implications is

also called a surmise relation, and we do say that “the implications form a quasi-order or surmise relation” on the item set.

A response pattern $d \in D$ of a subject is called a counterexample for an implication $i \rightarrow j$, $i, j \in I$, if the subject agrees with item i (i.e., $d(i) = 1$) but disagrees with item j (i.e., $d(j) = 0$). Let

$$b_{ij} = |\{d \in D | d(i) = 1 \wedge d(j) = 0\}|$$

be the number of counterexamples observed in the data set D for the implication $i \rightarrow j$. We can define a binary relation R_0 on I by

$$jR_0i :\Leftrightarrow b_{ij} = 0 \quad \text{for all } i, j \in I.$$

Thus, R_0 consists of all implications that are not violated empirically. Leeuwe (1974) showed that this relation is a quasi-order on I . However, this simple approach, to accept only those implications for which no counterexample is observed in the data set, is mere data fitting, generally insufficient, since we must take into account random response errors that occur in real data.

Therefore, IITA allows for varying numbers of observed counterexamples, $L = 0, 1, \dots, n$, and consists of two steps. First, we construct a selection set $\{R_L | L = 0, 1, \dots, n\}$ of quasi-orders on I . Second, we calculate the fit of each constructed quasi-order R_L to the data set D , according to some measure, and we choose that quasi-order of the selection set which best fits the data.

2.1. Construction of the quasi-orders

Starting with the quasi-order R_0 , the construction procedure is inductive in the following sense. Assume that, in the L th step of the procedure, we have constructed a quasi-order R_L . In step $L + 1$, we obtain the surmise relations R_{L+1} by adding specific item pairs to R_L . Among the item pairs not contained in R_L , all those pairs are added to R_L , which have no more than $L + 1$ counterexamples in the data set and that do not cause an intransitivity with the item pairs already contained in R_L or added to R_L . In particular, note that we cannot simply construct the L th step surmise relation by $jR'_L i :\Leftrightarrow b_{ij} \leq L$ for all $i, j \in I$, since the resulting binary relation R'_L , for $L > 0$, is not transitive in general (Leeuwe, 1974).

The procedure to construct R_{L+1} from R_L , $L = 0, 1, \dots, n - 1$, consists of the following sub-steps:

1. First, we determine the set $A_{L+1} = \{(j, i) \in I \times I \mid b_{ij} \leq L + 1 \wedge (j, i) \notin R_L\}$ of item pairs that have no more than $L + 1$ counterexamples in the data set and are not contained in R_L .
2. Second, we repeat the two following operations until the set B_{L+1} is empty:
 - (a) For each element of A_{L+1} , we check if it causes an intransitivity with other elements in $R_L \cup A_{L+1}$. Elements that cause an intransitivity are marked. Let B_{L+1} be the set of all marked elements.
 - (b) We delete all marked elements from A_{L+1} (and reset B_{L+1}).
3. When this process ends (i.e., no element marked, $B_{L+1} = \emptyset$), none of the remaining implications in A_{L+1} causes an intransitivity with elements in $R_L \cup A_{L+1}$. We define $R_{L+1} := R_L \cup A_{L+1}$. The relation R_{L+1} is, by construction, transitive.

Overall, this results in a set $\{R_L \mid L = 0, 1, \dots, n\}$ of quasi-orders on I , with $R_0 \subseteq R_1 \subseteq \dots \subseteq R_n$. In particular, note that some of these quasi-orders may be identical. (If $A_{L+1} = \emptyset$, $R_L = R_{L+1}$.)

2.2. A method to determine the best fitting quasi-order

Among $\{R_L \mid L = 0, 1, \dots, n\}$, we want to determine and select the best fitting quasi-order. We compare each R_L to the data set D . For a level $L \in \{0, 1, \dots, n\}$, if we assume that R_L is the correct quasi-order underlying the data set D , how many counterexamples for an implication $i \rightarrow j$, $i, j \in I$, can we expect under this assumption? For each item $i \in I$, we define

$$p_i = \frac{|\{d \in D \mid d(i) = 1\}|}{n},$$

which is the relative frequency of subjects of the sample who solve item i .

We rely upon implicit assumptions. We assume that (a) there is a reflexive and transitive binary relation of true logical implications between the items underlying the data, (b) violations of true implications are only possible through the impact of random errors, and (c) there is a single error probability for a true implication to be violated due to random errors. Under the assumption that R_L , $L = 0, 1, \dots, n$, is the correct quasi-order, we can estimate the single error probability by γ_L , based on the observed b_{ij} and p_i values:

$$\gamma_L = \frac{\sum \{b_{ij} / (p_i n) \mid j R_L i \wedge i \neq j\}}{|R_L| - m}.$$

182 In this formula, $b_{ij}/(p_i n)$ is the number of observed counterexamples for an implication $i \rightarrow_L j$
 183 ($\Leftrightarrow jR_L i$), $i \neq j$, relative to the number of cases that solve item i , and at all, can violate this
 184 implication. The denominator $|R_L| - m$ is the number of non-reflexive item pairs in R_L , that is,
 185 the number of summands that are added up. Thus, γ_L represents an estimated average amount of
 186 random errors in the data, under the assumption that R_L is the underlying true quasi-order.

187 For any item pair $(j, i) \in I \times I$, $i \neq j$, we have the observed b_{ij} value in the data set. However,
 188 we can also compute a corresponding theoretical value t_{ij} , a value that we can expect and derive
 189 under the assumption that R_L , with its estimated error rate γ_L , is the true surmise relation. In the
 190 case of $(j, i) \in R_L$, $i \neq j$, t_{ij} is the expected number of violations of the implication $i \rightarrow_L j$.

191 More precisely, we distinguish the following three cases:

- 192 1. Assume that $(j, i) \in R_L$, and $i \neq j$. Under the assumption that R_L is the correct quasi-order,
 193 the counterexamples for $i \rightarrow_L j$ do result from random errors. Thus, the expected number of
 194 violations of $i \rightarrow_L j$ is

$$t_{ij} = \gamma_L p_i n.$$

195 The expected value t_{ij} is the number of response patterns, $p_i n$, such that subjects solve item
 196 i , only in this case a counterexample can occur, multiplied with the error probability γ_L .

- 197 2. Assume that $(j, i) \notin R_L$ and $(i, j) \notin R_L$. In this case, under the assumption that R_L is the
 198 true quasi-order, no relational dependencies between the items i and j exist, and we assume
 199 that the items are stochastically independent. Thus, the theoretical value t_{ij} is equal to the
 200 expected number of response patterns d with $d(i) = 1$ and $d(j) = 0$, under independence.
 201 Hence, we use the estimate

$$t_{ij} = (1 - p_j) p_i n.$$

- 202 3. Assume that $(j, i) \notin R_L$ and $(i, j) \in R_L$. The observed number of subjects who solve item j
 203 is $p_j n$. Since $(i, j) \in R_L$, according to the first estimation case above, we can use $t_{ji} = \gamma_L p_j n$
 204 to estimate the number of subjects that solve item j but fail to solve item i . Hence, the
 205 estimated number of people who solve item j and item i is $p_j n - t_{ji} = (p_j - p_j \gamma_L) n$. Therefore,
 206 $t_{ij} = (p_i - (p_j - p_j \gamma_L)) n$. It can happen in rare cases that $p_i - p_j(1 - \gamma_L) < 0$. In these
 207 cases, consistent with interpretation and following common statistical practice, we set $t_{ij} = 0$.

(This is an improvement made to the original publications on IITA, where possible negative estimates have not been treated. We suppose that, with only positive outcomes, a negative t_{ij} value may rather rarely occur, and therefore, may not distort the analysis in general.) Thus, the estimate we use is

$$t_{ij} = \begin{cases} \theta := (p_i - p_j + p_j \gamma_L) n & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}.$$

We can derive a measure to quantify the fit of each R_L , $L = 0, 1, \dots, n$, to the data set D . Based on the observed and expected numbers b_{ij} and t_{ij} for any item pair $(j, i) \in I \times I$, $i \neq j$, respectively, the measure is defined by

$$\text{diff}(R_L, D) = \frac{\sum_{i \neq j} (b_{ij} - t_{ij})^2}{m(m-1)}.$$

We select a quasi-order R_{IITA} in $\mathcal{M} = \{R_L | L = 0, 1, \dots, n\}$ as the solution of the IITA analysis, which yields the minimum diff value. That is,

$$R_{\text{IITA}} = \arg \min_{R_L \in \mathcal{M}} \text{diff}(R_L, D).$$

We conclude this section with remarks. The calculation of the theoretical numbers t_{ij} have been optimized over time. Thus, their definitions are not identical in all publications on IITA. The variant used in this paper corresponds to the version introduced by Sargin & Ünlü (2009). In large simulation studies reported by Ünlü & Schrepp (2015, 2016a, 2017), it was shown that this latter definition of the t_{ij} values outperformed older versions, concerning their ability to reconstruct the correct implications from noisy data. Free software packages to run IITA analyses are available in Schrepp (2006) and Ünlü & Sargin (2010).

3. Description of the dichotomous extension

In this section, based on the dichotomous formulation, we describe how the IITA algorithm can be extended to detect all four possible forms of implications $(i, j \in I, \text{ subject } s)$:

- $s(i) = 1 \rightarrow s(j) = 1$; that is, agreement with i implies agreement with j .
- $s(i) = 1 \rightarrow s(j) = 0$; that is, agreement with i implies disagreement with j .
- $s(i) = 0 \rightarrow s(j) = 1$; that is, disagreement with i implies agreement with j .

230 • $s(i) = 0 \rightarrow s(j) = 0$; that is, disagreement with i implies disagreement with j .

231 We also write these implications as $i = 1 \rightarrow j = 1$, $i = 1 \rightarrow j = 0$, $i = 0 \rightarrow j = 1$, and $i = 0 \rightarrow j = 0$,
 232 respectively. The extensions of IITA to polytomous items will be studied in Section 4.

233 Throughout this section, we use dichotomous items, $i, j \in I$, $|I| = m \geq 2$, with item scores in
 234 $K = \{0, 1\}$. Let D be a binary data set of sample size $n \geq 2$ response patterns. Let $I_e = I \times \{0, 1\} =$
 235 $\{(i, x) | i \in I \wedge x \in \{0, 1\}\}$. Consider the 2×2 table notation

$i \setminus j$	0	1
0	a_{ij}	c_{ij}
1	b_{ij}	d_{ij}

236 where

237 • $a_{ij} = |\{d \in D | d(i) = 0 \wedge d(j) = 0\}|$,

238 • $b_{ij} = |\{d \in D | d(i) = 1 \wedge d(j) = 0\}|$,

239 • $c_{ij} = |\{d \in D | d(i) = 0 \wedge d(j) = 1\}|$,

240 • $d_{ij} = |\{d \in D | d(i) = 1 \wedge d(j) = 1\}|$.

241 There is a connection between the four generalized implications and the expected values under
 242 no random errors in the 2×2 table:

243 • $i = 1 \rightarrow j = 1$: If there are no random response errors, then $b_{ij} = 0$.

244 • $i = 1 \rightarrow j = 0$: If there are no random response errors, then $d_{ij} = 0$.

245 • $i = 0 \rightarrow j = 1$: If there are no random response errors, then $a_{ij} = 0$.

246 • $i = 0 \rightarrow j = 0$: If there are no random response errors, then $c_{ij} = 0$.

247 We denote implications in the form $(i, x) \rightarrow (j, y)$, with $i, j \in I$ and $x, y \in \{0, 1\}$, or alternatively
 248 written as $r(i, j, x, y)$, which represents the ordered pair $((j, y), (i, x))$.

249 3.1. Consistency

250 We work with consistent surmise relations.

251 **Definition 1.** Let $R \subset I_e \times I_e$ be a binary relation. We call R a consistent surmise relation on I_e if
252 and only if (iff)

253 1. R is a surmise relation on I_e , that is, for all $i, j, k \in I$ and $x, y, z \in \{0, 1\}$,

254 (Reflexivity). $r(i, i, x, x) \in R$,

255 (Transitivity). $(r(i, j, x, y) \in R \wedge r(j, k, y, z) \in R) \Rightarrow r(i, k, x, z) \in R$, and

256 2. R satisfies the property of consistency, that is, for all $i, j \in I$ and $x, y \in \{0, 1\}$,

257 (C-I). $(r(i, j, 0, y) \in R \Rightarrow r(i, j, 1, y) \notin R)$ and $(r(i, j, 1, y) \in R \Rightarrow r(i, j, 0, y) \notin R)$,

258 (C-II). $(r(i, j, x, 0) \in R \Rightarrow r(i, j, x, 1) \notin R)$ and $(r(i, j, x, 1) \in R \Rightarrow r(i, j, x, 0) \notin R)$.

259 We want to comment on the consistency conditions (C-I) and (C-II). Subsequently, assume that
260 no random errors occur and that we deal with population quantities. For a choice of $i, j \in I$ and
261 $x, y \in \{0, 1\}$, the conditions, if violated, can be depicted by the following diagrams (“ $\neg$ ” denotes
262 logical negation),

263 and

264 Re (C-I). For any subject, independent of whether item i is mastered ($\neg A$) or not mastered (A),
265 the latent score to item j is always y (B). Thus, in the population of reference, $P(j = y) = 1$. In
266 other words, if for a choice of $i, j \in I$ and $y \in \{0, 1\}$, the consistency condition (C-I) is violated,
267 then necessarily $P(j = y) = 1$ for some $j \in I$ and $y \in \{0, 1\}$. Vice versa, if $P(j = y) = 1$ for some
268 $j \in I$ and $y \in \{0, 1\}$, then for *all* $k \in I$ and $z \in \{0, 1\}$, $(k = z) \rightarrow B$ ($B: j = y$), which includes the
269 hard to interpret implications $\neg A \rightarrow B$ and $A \rightarrow B$ ($A: i = 0$) for some $i, j \in I$ and $y \in \{0, 1\}$, and

other implications that do hold trivially without deeper empirical meaning. (Note that in this case, if $P(j = y) = 1$, things get even more problematic, since $P(j = 1 - y) = 0$ and the issue described in the second condition prevails.) These artificially added implications, they are all of the form $C \rightarrow D$ and $\neg C \rightarrow D$, may carry little or no information about the relational dependencies between the items. To exclude these cases, we assume that in the population $P(j = y) \neq 1$ for all $j \in I$ and $y \in \{0, 1\}$, which is equivalent to saying that, for the true implications, the consistency condition (C-I) is satisfied for all choices of $i, j \in I$ and $y \in \{0, 1\}$. We base the derivation of our method explicitly on the consistency condition (C-I), not on the population proportions. The reason for doing so is, in real data analysis, random errors may occur and the relative frequencies may have their variabilities, in addition to the data-analytic specifics of a particular method.

Re (C-II). An analogous argument holds. For any subject, the latent score to item i must be different from x . If the score would be x , we would have a contradiction. Thus, in the population, $P(i = x) = 0$. That is, if for a choice of $i, j \in I$ and $x \in \{0, 1\}$, the consistency condition (C-II) is violated, $P(i = x) = 0$ for some $i \in I$ and $x \in \{0, 1\}$. On the other hand, if $P(i = x) = 0$ for some $i \in I$ and $x \in \{0, 1\}$, then, again artificially, for all $k \in I$ and $z \in \{0, 1\}$, $A \rightarrow (k = z)$ ($A: i = x$). This includes the implications $A \rightarrow B$ and $A \rightarrow \neg B$ ($B: j = 0$) for some $i, j \in I$ and $x \in \{0, 1\}$, and other implications that do hold by mere logic, without greater empirical significance. (Note that, in this case, $P(i = 1 - x) = 1$, which leads to the problem described in the first condition.) These trivial and even contradictory implications, all of the form $C \rightarrow D$ and $C \rightarrow \neg D$, may carry basically no information about possible relational dependencies. To exclude these cases from further analysis, we assume that in the population $P(i = x) \neq 0$ for all $i \in I$ and $x \in \{0, 1\}$, which, as we have seen, is equivalent to the consistency condition (C-II) being satisfied for all $i, j \in I$ and $x \in \{0, 1\}$. Because of the same afore-mentioned reasons, we base our approach on the consistency condition (C-II), not on population proportions.

Under the assumption that no random errors occur and that we deal with population quantities, we can show that (C-I) and (C-II) are equivalent conditions in the population of reference. (Note that this equivalence does not generalize to polytomous items (Section 4.1). However, in real data analysis with IITA, these conditions can be independently satisfied or violated. This is why, in the data, we do have to consider both conditions at the same time.) We prove that (a) $\neg(\text{C-I})$ implies

299 $\neg(\text{C-II})$, and (b) $\neg(\text{C-II})$ implies $\neg(\text{C-I})$. Re (a): There are $i, j \in I$ and $y \in K$ such that $r(i, j, 0, y) \in R$
300 and $r(i, j, 1, y) \in R$. Thus, $P(j = y) = 1$. In particular, $P(j = 1 - y) = 0$. Then, $r(j, i, 1 - y, 0) \in R$ and
301 $r(j, i, 1 - y, 1) \in R$. Re (b): There are $i, j \in I$ and $x \in K$ such that $r(i, j, x, 0) \in R$ and $r(i, j, x, 1) \in R$.
302 Thus, $P(i = x) = 0$. That is, $P(i = 1 - x) = 1$. Then, $r(j, i, 0, 1 - x) \in R$ and $r(j, i, 1, 1 - x) \in R$. $\square$

303 Note that the consistency condition (C-II) and reflexivity imply that, for any item $i \in I$,
304 $((i, 0), (i, 1)) \notin R$ and $((i, 1), (i, 0)) \notin R$.

305 The diagonal $\Delta = \{(a, a) : a \in I_e\}$ is a consistent surmise relation on I_e . We can also have a
306 non-trivial and important example of a consistent surmise relation, constructed based on the data.
307 Before describing the example, we need to introduce notation.

308 Let $i, j \in I$ and $x, y \in \{0, 1\}$. We define the number of counterexamples for an implication
309 $r(i, j, x, y)$ by

$$\text{viol}(r(i, j, x, y)) = |\{d \in D \mid d(i) = x \wedge d(j) \neq y\}|.$$

310 Clearly,

$$\begin{aligned} \text{viol}(r(i, j, 1, 1)) &= b_{ij}, \\ \text{viol}(r(i, j, 1, 0)) &= d_{ij}, \\ \text{viol}(r(i, j, 0, 1)) &= a_{ij}, \\ \text{viol}(r(i, j, 0, 0)) &= c_{ij}. \end{aligned}$$

315 We define the relative response frequency

$$p(i, x) = \frac{|\{d \in D \mid d(i) = x\}|}{n}.$$

316 Obviously, $0 < p(i, x) < 1$ iff there are $d_1, d_2 \in D$ with $d_1(i) = x$ and $d_2(i) = 1 - x$. In other words,
317 the column vector of the data matrix representing item i is not the constant 0 or 1 vector. That is,
318 not all subjects of the sample agree or disagree with item i .

319 **Definition 2.** The inductive procedure proposed later will be based on the following anchoring
320 relation

$$R_0 := \{r(i, j, x, y) \in I_e \times I_e \mid \text{viol}(r(i, j, x, y)) = 0\}.$$

321 **Lemma 3.** R_0 is a surmise relation on I_e .

322 *Proof.* For all $i \in I$ and $x \in \{0, 1\}$, $\text{viol}(r(i, i, x, x)) = 0$, and $r(i, i, x, x) \in R_0$. Thus, R_0 is reflexive.
 323 Let $i, j, k \in I$ and $x, y, z \in \{0, 1\}$, and assume that $\text{viol}(r(i, j, x, y)) = 0$ and $\text{viol}(r(j, k, y, z)) = 0$.
 324 For all $d \in D$, we have $(d(i) = x \Rightarrow d(j) = y)$ and $(d(j) = y \Rightarrow d(k) = z)$. Thus, for all $d \in D$,
 325 $(d(i) = x \Rightarrow d(k) = z)$, and hence, $\text{viol}(r(i, k, x, z)) = 0$. Thus, R_0 is transitive. $\square$

326 **Lemma 4.** R_0 is consistent iff $0 < p(i, x) < 1$ for all $i \in I$ and $x \in \{0, 1\}$.

327 *Proof.* “ $\Leftarrow$ ”: Let $i, j \in I$ and $x, y \in \{0, 1\}$. (C-I). Let $\text{viol}(r(i, j, 0, y)) = 0$. That is, for all $d \in D$,
 328 $(d(i) = 0 \Rightarrow d(j) = y)$. We have to show that $\text{viol}(r(i, j, 1, y)) \neq 0$. Assume that $\text{viol}(r(i, j, 1, y)) =$
 329 0 . Then, for all $d \in D$, $(d(i) = 1 \Rightarrow d(j) = y)$. Thus, for all $d \in D$, $d(j) = y$. This implies the
 330 contradiction $1 = p(j, y) < 1$. A similar argument holds for the second implication. (C-II). Let
 331 $\text{viol}(r(i, j, x, 0)) = 0$. For all $d \in D$, we have $(d(i) = x \Rightarrow d(j) = 0)$, or $(d(j) = 1 \Rightarrow d(i) \neq x)$.
 332 We have to show that $\text{viol}(r(i, j, x, 1)) \neq 0$. Assume that $\text{viol}(r(i, j, x, 1)) = 0$. Then, for all $d \in D$,
 333 $(d(i) = x \Rightarrow d(j) = 1)$, or $(d(j) = 0 \Rightarrow d(i) \neq x)$. Hence, for all $d \in D$, $d(i) \neq x$. This implies the
 334 contradiction $0 = p(i, x) > 0$. Similarly, the second implication can be shown. This proves that R_0
 335 is consistent.

336 “ $\Rightarrow$ ”: By contraposition, let $i \in I$ and $x \in \{0, 1\}$, with $p(i, x) = 0$ or $p(i, x) = 1$. We show that
 337 this implies that R_0 is not consistent. Let $j \in I$. First case, $p(i, x) = 0$. That is, for all $d \in D$, $d(i) \neq x$.
 338 Thus, $\text{viol}(r(i, j, x, 0)) = 0$ and $\text{viol}(r(i, j, x, 1)) = 0$, and $r(i, j, x, 0) \in R_0$ and $r(i, j, x, 1) \in R_0$. In
 339 particular, (C-II) is violated, and R_0 is not consistent. Second case, $p(i, x) = 1$. That is, for all
 340 $d \in D$, $d(i) = x$. Thus, $\text{viol}(r(j, i, 0, x)) = 0$ and $\text{viol}(r(j, i, 1, x)) = 0$, and $r(j, i, 0, x) \in R_0$ and
 341 $r(j, i, 1, x) \in R_0$. Hence, (C-I) is violated, and R_0 is not consistent. $\square$

342 3.2. Generalization of the inductive construction procedure

343 The modification of the first step of IITA, the construction of an appropriate selection set,
 344 is straightforward (Section 2.1). If for the data set we have $0 < p(i, x) < 1$ for all $i \in I$ and
 345 $x \in \{0, 1\}$, the relation R_0 on I_e is a consistent surmise relation. Thus, we can anchor the inductive
 346 construction with R_0 . We show below that for a general data set, if R_0 is not consistent, we can
 347 always construct and base the inductive construction on a consistent surmise relation $R'_0 \subset R_0$.

348 Assume that for a level $L \in \{0, 1, \dots, n-1\}$ a consistent surmise relation R_L on I_e is defined.
 349 We perform the following steps:

1. Define $A_{L+1} = \{r(i, j, x, y) \in I_e \times I_e \mid \text{viol}(r(i, j, x, y)) \leq L + 1 \wedge r(i, j, x, y) \notin R_L\}$ as the set of generalized implications which have no more than $L + 1$ counterexamples in the data set and are not contained in R_L .

2. Consistency: We remove all generalized implications from A_{L+1} that cause inconsistencies in $R_L \cup A_{L+1}$. That is:

(C-I). For $i, j \in I$ and $y \in \{0, 1\}$,

- mark all $r(i, j, 0, y) \in A_{L+1}$ for which there is an $r(i, j, 1, y) \in R_L \cup A_{L+1}$;
- mark all $r(i, j, 1, y) \in A_{L+1}$ for which there is an $r(i, j, 0, y) \in R_L \cup A_{L+1}$.

(C-II). For $i, j \in I$ and $x \in \{0, 1\}$,

- mark all $r(i, j, x, 0) \in A_{L+1}$ for which there is an $r(i, j, x, 1) \in R_L \cup A_{L+1}$;
- mark all $r(i, j, x, 1) \in A_{L+1}$ for which there is an $r(i, j, x, 0) \in R_L \cup A_{L+1}$.

Remove all marked elements from A_{L+1} (and denote the resulting set again by A_{L+1}). Then, in $R_L \cup A_{L+1}$, consistency is satisfied, and removing further elements from A_{L+1} , as we do in the next step, will not violate consistency.

3. Transitivity: Mark all $r(i, j, x, y) \in A_{L+1}$, $i, j \in I$ and $x, y \in \{0, 1\}$, which cause an intransitivity with elements in $R_L \cup A_{L+1}$. Remove all marked elements from A_{L+1} (and denote the resulting set by A_{L+1}). This step is repeated iteratively until no intransitive element is found.

When this construction process ends, $R_L \cup A_{L+1}$ is a consistent surmise relation on I_e , that is, $R_L \cup A_{L+1} \subset I_e \times I_e$ does not violate any of the conditions in Definition 1. We set $R_{L+1} := R_L \cup A_{L+1}$.

If the surmise relation R_0 is not consistent, that is, if for the data set we have $p(i, x) = 0$ or 1 for some $i \in I$ and $x \in \{0, 1\}$, we can construct a consistent surmise relation $R'_0 \subset R_0$. To do so, we perform the Steps 2 and 3 of the above construction procedure, for $R_L = \Delta$ and $A_{L+1} = R_0 \setminus \Delta$. Then, $R'_0 = R_{L+1} \subset R_0$ is a consistent surmise relation.

This yields the set $\{R_L \mid L = 0, 1, \dots, n\}$ of consistent surmise relations on I_e .

3.3. Generalization of the fit coefficient

We need to modify the second step of IITA and generalize the method to determine the best fitting quasi-order (Section 2.2).

In analogy to the original IITA algorithm, for R_L , $0 \leq L \leq n$, we define γ_L by

$$\gamma_L = \frac{\sum \{\text{viol}(r(i, j, x, y)) / (p(i, x)n) \mid r(i, j, x, y) \in R_L \wedge (i, x) \neq (j, y)\}}{|R_L| - 2m}.$$

As before, the value γ_L can be interpreted as the probability that a true implication, that is, $r(i, j, x, y) \in R_L$, is violated.

We describe the method to determine a fit index for each of the relations R_L , $L = 0, 1, \dots, n$. Analogous to the original algorithm, we define a fit measure based on the comparison between the observed and, under R_L with γ_L , expected values. We have a total of $|(I_e \times I_e) \setminus \Delta| = 2m(2m - 1)$ non-reflexive implications in the generalized case. For $i, j \in I$, we can compute the observed numbers of counterexamples $\text{viol}(r(i, j, 1, 1)) = b_{ij}$, $\text{viol}(r(i, j, 1, 0)) = d_{ij}$, $\text{viol}(r(i, j, 0, 1)) = a_{ij}$, and $\text{viol}(r(i, j, 0, 0)) = c_{ij}$ for the implications $r(i, j, 1, 1)$, $r(i, j, 1, 0)$, $r(i, j, 0, 1)$, and $r(i, j, 0, 0)$, respectively.

Under the true surmise relation R_L , the expected number of counterexamples, $\text{viol}^*(r(i, j, x, y))$, for an implication $r(i, j, x, y) \in I_e \times I_e$ with $(i, x) \neq (j, y)$ depends on whether $r(i, j, x, y)$ is contained or not contained in R_L . Again, we distinguish three cases:

1. If $r(i, j, x, y) \in R_L$, and $(i, x) \neq (j, y)$, we expect the number of violations of this implication

$$\text{viol}^*(r(i, j, x, y)) = \gamma_L p(i, x)n.$$

2. If $r(i, j, x, y) \notin R_L$ and $r(j, i, y, x) \notin R_L$, we expect the number of counterexamples

$$\text{viol}^*(r(i, j, x, y)) = (1 - p(j, y))p(i, x)n.$$

3. If $r(i, j, x, y) \notin R_L$ and $r(j, i, y, x) \in R_L$, we expect the number of counterexamples

$$\text{viol}^*(r(i, j, x, y)) = (p(i, x) - p(j, y) + p(j, y)\gamma_L)n$$

if $p(i, x) - p(j, y) + p(j, y)\gamma_L \geq 0$, and $\text{viol}^*(r(i, j, x, y)) = 0$ else.

Let $S = (I_e \times I_e) \setminus \Delta$. In analogy to the original IITA algorithm, we define the $\text{diff}(R_L, D)$ value

$$\text{diff}(R_L, D) = \frac{\sum_{r(i, j, x, y) \in S} (\text{viol}(r(i, j, x, y)) - \text{viol}^*(r(i, j, x, y)))^2}{2m(2m - 1)}.$$

In the set of consistent surmise relations $\{R_L \mid L = 0, 1, \dots, n\}$, the solution of the generalized IITA algorithm is a surmise relation R_L , which shows the minimum $\text{diff}(R_L, D)$ value.

3.4. Empirical example

We analyze data from the International Social Survey Programme of the year 1995 (ISSP95). This is an annual programme of cross-national collaboration covering important topics of social science research (e.g., Haller et al., 2009). Every year, with this programme, researchers conduct a survey with comparable questions in each of the participating countries. The theme for ISSP95 was national identity.

Exemplarily, for the Germany subsample, we focus on the data for Question 4 of the ISSP95. The formulation of Question 4, which comprises seven items, is:

- “Some people say the following things are important for being truly German. Others say they are not important. How important do you think the following is ...
 1. to have been born in Germany
 2. to have German citizenship
 3. to have lived in Germany for most of one’s life
 4. to be able to speak German
 5. to be a Christian
 6. to respect Germany’s political institutions
 7. to feel German”

To any of these seven items, the subjects could respond with one of the answer categories “Very important”, “Important”, “Not very important”, “Not important at all”, and “Can’t choose”. To derive and work with a dichotomous data set, we code the answer categories “Very important” and “Important” with 1, and the categories “Not very important” and “Not important at all” with 0. Also, all of the response patterns containing the answer “Can’t choose” to one of the seven items are removed from the data set. The reduced data set consists of $n = 1,126$ response patterns.

Fig. 1 shows the implications extracted for this data set.

[Fig. 1 about here]

As we can see, the method detects implications based on agreement, as well as disagreement, with an item. The implications based on negative item scores would not be detectable with the original IITA algorithm.

For this data set, there are no implications of the form “agreement with i implies disagreement with j ”, and vice versa. This is due to the fact that all items have the same polarity. Agreement with an item means a higher level of identification with the concept of national identity, than disagreement. Thus, implications of the form “agreement or disagreement with i implies disagreement or agreement with j , respectively” are semantically implausible for this data set, and such implications are not detected by our algorithm. Moreover, contraposition holds. For all items, if there is an implication “agreement with i implies agreement with j ” (e.g., $3 = 1 \rightarrow 4 = 1$), then we can also find an implication “disagreement with j implies disagreement with i ” (e.g., $4 = 0 \rightarrow 3 = 0$).

The items constitute separate clusters, comprised of Items 1, 3, and 4, of Item 2, of Items 5 and 6, and of Item 7. To have German citizenship (Item 2) or to feel German (Item 7) seem to be entirely independent concepts, not implying or being implied by any other property. These items are incomparable. Items 1, 3, and 5 contain statements that represent an extreme opinion about what is important to be German. People that agree with these statements generally do agree with other less radical statements. To be able to speak German (Item 4) is a prerequisite for to have been born in Germany (Item 1) and to have lived in Germany for most of one’s life (Item 3). People that agree with to be a Christian (Item 5) generally do also agree with to respect Germany’s political institutions (Item 6). However, Item 6, as well as Item 4, are not found to be prerequisites for all items. Hence, their contents, that somebody should respect the political institutions and be able to speak the national language, may not be deemed minimal requirements for national identity. Similar statements formulated in terms of disagreement can also be obtained.

4. Description of the nominal polytomous and ordinal polytomous extensions

We discuss the nominal and ordinal cases in parallel, subsequently abbreviated by “Nominal” and “Ordinal”. Let I be a set of $m \geq 2$ polytomous items i_l with item scores or answer categories in $K_{i_l} = \{0, 1, \dots, k_{i_l}\}$, $k_{i_l} \geq 1$, for $l = 1, \dots, m$, respectively. Let $M = \bigcup_l (\{i_l\} \times K_{i_l})$ be the set of all item-score combinations. Note, the diagonal of reflexive pairs $\Delta = \{(m, m) : m = (i, x) \in M\}$ has size $|\Delta| = \sum_l (k_{i_l} + 1)$. For $i, j \in I$ and $x \in K_i, y \in K_j$, denote the ordered pair $((j, y), (i, x))$ by $r(i, j, x, y)$.

We assume different structures among the item scores. Subsequently, fix $i \in I$.

Nominal: With the determination of equality, we use the standard $=$ among the integers $0, 1, \dots, k_i$, and $0 \neq 1 \neq \dots \neq k_i$.

Ordinal: With the determination of order (and equality), we use the standard $\leq$ (and $=$) among the integers $0, 1, \dots, k_i$, and $0 \leq 1 \leq \dots \leq k_i$ (and $0 \neq 1 \neq \dots \neq k_i$). In particular, we do work with a linear order on the item score set. Although this could be further generalized, for example, to a partially ordered set of item scores, $(K_i, \leq)$ is *the* important use case in applications. For instance, the scales utilized in questionnaires that measure agreement or disagreement with a statement are of this type. See also Sections 3.4 and 6.2.

We posit surmise relations, that is, reflexive and transitive binary relations, R on M , assumed to represent the true implications. Let $i, j \in I$ and $x \in K_i, y \in K_j$.

Nominal: An implication $i = x \rightarrow_R j = y$, also written $(i, x) \rightarrow_R (j, y)$, stands for the item-score pair $((j, y), (i, x)) \in R$, or $(j, y)R(i, x)$. The interpretation of this implication or pair is: “From the (latent) response x to item i ($i = x$), we can surmise the (latent) response y to item j ($j = y$)”.

Ordinal: An implication $i \geq x \rightarrow_R j \geq y$, also written $(i, x) \rightarrow_R (j, y)$, stands for the item-score pair $((j, y), (i, x)) \in R$, or $(j, y)R(i, x)$. The interpretation of this implication or pair is $(r \in K_i, r' \in K_j)$: “From the (latent) response $r \geq x$ to item i ($i \geq x$), we can surmise the (latent) response $r' \geq y$ to item j ($j \geq y$)”.

4.1. Consistency

First, we discuss consistency for nominal polytomous items. In particular, we generalize the discussion of Section 3.1.

Definition 5. Let $R \subset M \times M$ be a surmise relation on M . We call R consistent iff

R satisfies the (nominal) property of consistency, that is, for all $i, j \in I$ and $x \in K_i, y \in K_j$,

(C-I)'. $(r(i, j, x, y) \in R)$ implies $(\exists x' \in K_i, x' \neq x : r(i, j, x', y) \notin R)$,

(C-II)'. $(r(i, j, x, y) \in R)$ implies $(\forall y' \in K_j, y' \neq y : r(i, j, x, y') \notin R)$.

For the dichotomous special case ($K_i = K = \{0, 1\}$), (C-I)' and (C-II)' in Definition 5 generalize (C-I) and (C-II) in Definition 1, respectively. The consistency condition (C-II)' and reflexivity imply that, for any item $i \in I$ and any answer category $x \in K_i$, it holds $((i, x'), (i, x)) \notin R$ for all $x' \in K_i$, $x' \neq x$. Under the assumption that no random errors occur and that we deal with population quantities, for an implication $r(i, j, x, y) \in R$, $i, j \in I$ and $x \in K_i, y \in K_j$, the conclusion of (C-I)' is equivalent to $P(j = y) \neq 1$, and the conclusion of (C-II)' is equivalent to $P(i = x) \neq 0$. Thus, by assuming that the consistency property holds in the population, we exclude artificial or contradictory implications.

In the population of reference, without random errors, we can also show that (C-II)' implies (C-I)'. (Since, in IITA data analyses, these conditions can be independently satisfied or violated, we do have to consider both conditions simultaneously.) We prove that $\neg(\text{C-I})'$ implies $\neg(\text{C-II})'$. There are $i, j \in I$ and $x \in K_i, y \in K_j$ such that $r(i, j, x, y) \in R$ and $r(i, j, x', y) \in R$ for all $x' \in K_i$, $x' \neq x$. Thus, $P(j = y) = 1$. In particular, $P(j = y') = 0$ for all $y' \in K_j, y' \neq y$. Let $y^* \in K_j, y^* \neq y$. Then, $r(j, i, y^*, x) \in R$, and there is $x' \in K_i, x' \neq x$ such that $r(j, i, y^*, x') \in R$. $\square$

Note that, vice versa, (C-I)' may not imply (C-II)', that is, $\neg(\text{C-II})'$ may not imply $\neg(\text{C-I})'$. If $\neg(\text{C-II})'$, there are $i, j \in I$ and $x \in K_i, y \in K_j$ such that $r(i, j, x, y) \in R$ and $r(i, j, x, y') \in R$ for some $y' \in K_j, y' \neq y$. Thus, $P(i = x) = 0$. From here, we can only derive that $P(i \neq x) = 1$, that is, $\sum_{x' \in K_i, x' \neq x} P(i = x') = 1$. In general, we can have $P(i = x') < 1$ for all $x' \in K_i, x' \neq x$. In particular, we cannot conclude with $P(i = x^*) = 1$ for some $x^* \in K_i$, which we would need to show that (C-I)' is violated.

The surmise relation Δ is consistent. Analogous to the dichotomous formulation in Section 3, we can have similar notions in the polytomous case too, and derive a consistent surmise relation from the data D . (D is a data set of $n \geq 2$ polytomous response patterns. A polytomous response pattern $d \in D$, with $d : I \rightarrow \bigcup_i K_i$, assigns the response $d(i) \in K_i$ of a subject to each of the items $i \in I$. Equivalently, D can also be viewed as the matrix of row-by-row $0, 1, \dots, k_i$, $i \in I$, encoded response vectors.) Let $i, j \in I$ and $x \in K_i, y \in K_j$. Similarly, we define the number of counterexamples for an implication $r(i, j, x, y)$

$$v_{jy,ix} = |\{d \in D | d(i) = x \wedge d(j) \neq y\}|,$$

504 and

$$p_{ix} = \frac{|\{d \in D | d(i) = x\}|}{n}.$$

505 Let the entries of the $(k_j + 1) \times (k_i + 1)$ cross-classification table for the items i (column variable)
 506 and j (row variable) be denoted by t_{uv}^{ji} , $u = 0, 1, \dots, k_j, v = 0, 1, \dots, k_i$. We have $v_{jy,ix} = \sum_{u \neq y, v=x} t_{uv}^{ji}$.
 507 Also, $0 < p_{ix} < 1$ for all $i \in I$ and $x \in K_i$ iff in every column of the data set all possible answer
 508 categories in K_i are scored. Then, in an analogous manner,

$$R_0 := \{((j, y), (i, x)) \in M \times M | v_{jy,ix} = 0\}$$

509 is a surmise relation on M .

510 **Lemma 6.** *R_0 is consistent iff in every column of the data set all answer categories occur.*

511 *Proof.* “ $\Leftarrow$ ”: Let $i, j \in I$ and $x \in K_i, y \in K_j$. (C-I)'. Let $v_{jy,ix} = 0$. That is, for all $d \in D$, $(d(i) = x \Rightarrow$
 512 $d(j) = y)$. Assume that $v_{jy,ix'} = 0$ for all $x' \in K_i, x' \neq x$. Then, for all $x' \in K_i, x' \neq x$, for all $d \in D$,
 513 $(d(i) = x' \Rightarrow d(j) = y)$. Thus, for all $d \in D$, $d(j) = y$. This implies the contradiction $1 = p_{jy} < 1$.
 514 (C-II)'. Let $v_{jy,ix} = 0$. For all $d \in D$, we have $(d(i) = x \Rightarrow d(j) = y)$, or $(d(j) \neq y \Rightarrow d(i) \neq x)$.
 515 Assume that $v_{jy',ix} = 0$ for some $y' \in K_j, y' \neq y$. Then, for all $d \in D$, $(d(i) = x \Rightarrow d(j) = y')$, or
 516 $(d(j) \neq y' \Rightarrow d(i) \neq x)$. Hence, for all $d \in D$, $d(i) \neq x$. This implies the contradiction $0 = p_{ix} > 0$.
 517 This proves that R_0 is consistent.

518 “ $\Rightarrow$ ”: Let $i \in I$ and $x \in K_i$, with $p_{ix} = 0$ or $p_{ix} = 1$. Let $j \in I$. First case, $p_{ix} = 0$. That is, for all
 519 $d \in D$, $d(i) \neq x$. Thus, $v_{j0,ix} = 0$ and $v_{j1,ix} = 0$, and $r(i, j, x, 0) \in R_0$ and $r(i, j, x, 1) \in R_0$. In particular,
 520 (C-II)' is violated, and R_0 is not consistent. Second case, $p_{ix} = 1$. That is, for all $d \in D$, $d(i) = x$.
 521 Thus, $v_{ix,jy'} = 0$ for all $y' \in K_j$, and $r(j, i, y', x) \in R_0$ for all $y' \in K_j$. Hence, (C-I)' is violated, and R_0
 522 is not consistent. □

523 Next, we discuss consistency for ordinal polytomous items.

524 **Definition 7.** Let $R \subset M \times M$ be a surmise relation on M . We call R consistent iff

525 R satisfies the (ordinal) property of consistency, that is, for all $i, j \in I$ and $x \in K_i, y \in K_j$,

526 (C-I)'. if $y \neq 0$, $r(i, j, 0, y) \notin R$,

(C-II)''. (a) if $y \neq 0$, $(r(i, j, x, y) \in R)$ implies $(\forall y' \in K_j, y' < y : r(i, j, x, y') \in R)$,

(b) $r(i, j, x, 0) \in R$,

(c) if $i \neq j$ or $x \neq k_j$ (i.e., except for $r(i, i, k_i, k_i) \in R; y = k_j$), $r(i, j, x, k_j) \notin R$.

Re (C-I)''. In the population, under the assumption of no random errors, for $i, j \in I$ and $y \in K_j$, $y \neq 0$, $(r(i, j, 0, y) \notin R) \Leftrightarrow (P(j \geq y) \neq 1)$. Obviously, $r(i, j, 0, y) \notin R$ implies $P(j \geq y) \neq 1$. Assume that $r(i, j, 0, y) \in R$. Then, for every subject s , $s(i) \geq 0$ implies $s(j) \geq y$. Thus, $P(j \geq y) = 1$. In other words, with (C-I)'', for $j \in I$ and $y \in K_j$, $y \neq 0$, we avoid trivial implications of the form $i' \geq x' \rightarrow j \geq y$ for all $i' \in I$ and $x' \in K_i$, which would logically hold if $P(j \geq y) = 1$. Note that, for $i, j \in I$ and $y \in K_j$, $y \neq 0$, we do *not* have $i \geq x \rightarrow_R j \geq y$ for *all* $x \in K_i$.

Re (C-II)''. (a) Note that, for $y' \in K_j, y' < y$, for a response $r' \in K_j$ to item j , $(r' \geq y \wedge y' < y)$ implies $r' \geq y'$. (b) Also, for any response $r' \in K_j$ to item j , $r' \geq 0$. (c) In the population of reference, under the assumption of no random errors, for $i, j \in I$ and $x \in K_i$, $i \neq j$ or $x \neq k_j$, $(r(i, j, x, k_j) \notin R) \Rightarrow (P(i \geq x) \neq 0)$. Thus, with the assumption $r(i, j, x, k_j) \notin R$, we do not a priori include the artificial implications (except for the reflexive pair) $i \geq x \rightarrow j' \geq y'$ for all $j' \in I$ and $y' \in K_j$, which do hold trivially if $P(i \geq x) = 0$. However, this comes at the expense of possibly leaving out true implications of the form $r(i, j, x, k_j) \in R$. For, conversely, $P(i \geq x) \neq 0$ does not imply $r(i, j, x, k_j) \notin R$. It may very well be that $r(i, j, x, k_j) \in R$, if $P(i \geq x) \neq 0$. In this case, Definition 7 of a consistent surmise relation would omit that implication. This is a byproduct of the attempt to eliminate logically artificial implications. Stated differently, if we would remove (c) from Definition 7, we could allow for implications $r(i, j, x, k_j) \in R$, but would also have to include logically trivial implications in case of $P(i \geq x) = 0$. Note that, for $i, j \in I$ and $x \in K_i$, $i \neq j$ or $x \neq k_j$, we do *not* have $i \geq x \rightarrow_R j \geq y$ for *all* $y \in K_j$.

Different equivalent formulations are possible. Reflexivity and (a) imply that, (a)' for any item $i \in I$ and any answer category $x \in K_i$, $r(i, i, x, x') \in R$ for all $x' \in K_i$, $x' < x$. This condition is equivalent to, (a)'' for any $i \in I$ and any $x \in K_i$, $r(i, i, x', x) \in R$ for all $x' \in K_i$, $x' > x$. Under transitivity, each of (a)' and (a)'' imply (a). Given R is a surmise relation, we can have another formulation (a)''', which is equivalent to (a): (a)''' For all $i, j \in I$ and $x \in K_i, y \in K_j$, if $x \neq k_i$, $(r(i, j, x, y) \in R)$ implies $(\forall x' \in K_i, x' > x : r(i, j, x', y) \in R)$. Thus, under the assumption that R is a surmise relation, all of these conditions are equivalent, (a) iff (a)' iff (a)'' iff (a)'''.

The surmise relation Δ is not consistent (e.g., $r(i, j, 1, 0) \notin \Delta$). The binary relation

$$R_\Delta := \Delta \cup \{((i, x'), (i, x)) | i \in I, x, x' \in K_i, x' < x\} \cup \{((j, 0), (i, x)) | i, j \in I, x \in K_i\} \quad (1)$$

is a consistent surmise relation. We can have another example of a consistent surmise relation. We define the number of counterexamples for an implication $r(i, j, x, y)$

$$v_{jy, ix} = |\{d \in D | d(i) \geq x \wedge d(j) \not\geq y\}|,$$

and

$$p_{ix} = \frac{|\{d \in D | d(i) \geq x\}|}{n}.$$

It holds $v_{j0, ix} = 0$ for every pair $r(i, j, x, 0)$, for $i, j \in I, x \in K_i$. Let the entries of the $(k_j + 1) \times (k_i + 1)$ cross-classification table for the items i (column variable) and j (row variable) be denoted by t_{uv}^{ji} , $u = 0, 1, \dots, k_j, v = 0, 1, \dots, k_i$. We have $v_{jy, ix} = \sum_{u \not\geq y, v \geq x} t_{uv}^{ji}$.

Definition 8. With

$$\begin{aligned} A &= \{r(i, j, 0, y) | i, j \in I, y \in K_j, y \neq 0\}, \\ B &= \{r(i, j, x, k_j) | i, j \in I, x \in K_i, i \neq j \text{ or } x \neq k_j\}, \\ C &= \{r(i, j, x, y) \in M \times M | v_{jy, ix} = 0\}, \end{aligned}$$

we define

$$R_0 := C \setminus (A \cup B) = C \cap A^c \cap B^c,$$

where X^c is the complement of a set $X \subset M \times M$, that is, $X^c = (M \times M) \setminus X$.

Lemma 9. R_0 is a consistent surmise relation on M .

Proof. R_0 is reflexive. For all $i \in I$ and $x \in K_i$, $v_{ix, ix} = 0$, $r(i, i, x, x) \notin A$, and $r(i, i, x, x) \notin B$.

R_0 is transitive. For $i, j, l \in I$ and $x \in K_i, y \in K_j, z \in K_l$, let $(l, z)R_0(j, y)$ and $(j, y)R_0(i, x)$. We have to show that $(l, z)R_0(i, x)$. It holds $v_{lz, jy} = 0$ and $v_{jy, ix} = 0$. Thus, for all $d \in D$, $d(j) \geq y$ implies $d(l) \geq z$, and for all $d \in D$, $d(i) \geq x$ implies $d(j) \geq y$. Hence, for all $d \in D$, $d(i) \geq x$ implies $d(l) \geq z$. This proves that $v_{lz, ix} = 0$, and $r(i, l, x, z) \in C$. Assume that $r(i, l, x, z) \in A$. Then, $x = 0$ and $z \neq 0$. Since $r(i, j, x, y) \notin A$ and $x = 0$, we have $y = 0$. This yields the contradiction $r(j, l, y, z) \in A$.

576 This proves that $r(i, l, x, z) \in A^c$. Assume that $r(i, l, x, z) \in B$. Then, $z = k_l$, and $i \neq l$ or $x \neq k_l$. Since
 577 $r(j, l, y, z) \notin B$ and $z = k_l$, we have $j = l$ and $y = k_l$. This gives the contradiction $r(i, j, x, y) \in B$. This
 578 proves that $r(i, l, x, z) \in B^c$.

579 R_0 is consistent. Let $i, j \in I$ and $x \in K_i, y \in K_j, y > 0$ (and $y \leq k_j$). Obviously, $r(i, j, 0, y) \in A$,
 580 and $r(i, j, x, k_j) \in B$ if $i \neq j$ or $x \neq k_j$. Thus, (C-I)'' and (C-II)''-(c) are satisfied. (C-II)''-(a): Let
 581 $(j, y)R_0(i, x)$. Let $y' \in K_j, y' < y$. We have to show that $(j, y')R_0(i, x)$. We have $v_{jy', ix} = 0$. That
 582 is, for all $d \in D$, $d(i) \geq x$ implies $d(j) \geq y$. Assume that $v_{jy', ix} \neq 0$. Then, there is some $d_0 \in D$
 583 such that $d_0(i) \geq x$ and $d_0(j) \not\geq y'$. For this $d_0 \in D$, $d_0(i) \geq x$ implies $d_0(j) \geq y$. This yields the
 584 contradiction $y' > d_0(j) \geq y > y'$. This proves that $r(i, j, x, y') \in C$. Assume that $r(i, j, x, y') \in A$.
 585 Then, $x = 0$ and $y' \neq 0$. Since $r(i, j, x, y) \notin A$ and $x = 0$, we have $y = 0$. This yields the contradiction
 586 $0 < y = 0$. This proves that $r(i, j, x, y') \in A^c$. Assume that $r(i, j, x, y') \in B$. Then, $y' = k_j$. This
 587 gives the contradiction $k_j \geq y > y' = k_j$. This proves that $r(i, j, x, y') \in B^c$. (C-II)''-(b): We have
 588 $r(i, j, x, 0) \in C$, since $v_{j0, ix} = 0$. Obviously, $r(i, j, x, 0) \notin A$ and $r(i, j, x, 0) \notin B$. $\square$

589 4.2. Generalized inductive construction procedures

590 First, we discuss the inductive construction procedure for the nominal case and generalize the
 591 discussion of Section 3.2. The generation is anchored with R_0 , if consistent (Lemma 6). If R_0 is
 592 not consistent, we can base the inductive procedure on and construct a consistent surmise relation
 593 $R'_0 \subset R_0$ (see below). For $0 \leq L \leq n - 1$, assume that R_L is a consistent surmise relation on M . From
 594 R_L , we construct a consistent surmise relation R_{L+1} as follows.

595 1. Define

$$A_{L+1} = \{r(i, j, x, y) \in M \times M \mid v_{jy, ix} \leq L + 1 \wedge r(i, j, x, y) \notin R_L\}.$$

596 This set consists of all item-score pairs that are not contained in R_L and have at most $L + 1$
 597 counterexamples. We successively reduce this set until consistency and transitivity are met.

598 2. Consistency: We remove all implications from A_{L+1} that cause inconsistencies in $R_L \cup A_{L+1}$.

599 (C-I)'. For $i, j \in I$ and $x \in K_i, y \in K_j$, mark all $r(i, j, x, y) \in A_{L+1}$ for which, for all $x' \in K_i$,

$$600 \quad x' \neq x, r(i, j, x', y) \in R_L \cup A_{L+1}.$$

(C-II)'. For $i, j \in I$ and $x \in K_i, y \in K_j$, mark all $r(i, j, x, y) \in A_{L+1}$ for which there is $y' \in K_j$, $y' \neq y$, such that $r(i, j, x, y') \in R_L \cup A_{L+1}$.

Remove all marked elements from A_{L+1} . Denote the resulting set of remaining pairs by A_{L+1} . In Lemma 10, we show that in $R_L \cup A_{L+1}$ consistency is satisfied. Note that when removing further elements from A_{L+1} , as we do in the next step, in $R_L \cup A_{L+1}$ consistency still holds.

3. Transitivity: For $i, j, l \in I$ and $x \in K_i, y \in K_j, z \in K_l$,

- mark all implications $r(i, j, x, y)$ in A_{L+1} for which there is $r(j, l, y, z) \in R_L \cup A_{L+1}$ such that $r(i, l, x, z) \notin R_L \cup A_{L+1}$;
- mark all implications $r(j, l, y, z)$ in A_{L+1} for which there is $r(i, j, x, y) \in R_L \cup A_{L+1}$ such that $r(i, l, x, z) \notin R_L \cup A_{L+1}$.

Remove all marked elements from A_{L+1} . Let the resulting set of remaining pairs be A_{L+1} . This step is repeated iteratively until no intransitive element is found. Then, in $R_L \cup A_{L+1}$, transitivity is satisfied.

When this process terminates, the reflexive binary relation $R_L \cup A_{L+1}$ is transitive and consistent by construction. We define $R_{L+1} := R_L \cup A_{L+1}$.

Lemma 10. Assume that A_{L+1} of Step 1 is reduced according to Step 2 of the above procedure. After removal, in $R_L \cup A_{L+1}$, the consistency conditions (C-I)' and (C-II)' are satisfied.

Proof. For $i, j \in I$ and $x \in K_i, y \in K_j$, let $r(i, j, x, y) \in R_L \cup A_{L+1}$. Without restriction, assume that $r(i, j, x, y) \in R_L$. (C-I)': Since R_L is consistent, there is $x' \in K_i$, $x' \neq x$, such that $r(i, j, x', y) \notin R_L$. Without restriction, let $r(i, j, x', y) \in A_{L+1}$. Then, there is $x'' \in K_i$, $x'' \neq x'$, such that $r(i, j, x'', y) \notin R_L \cup A_{L+1}$. Since $r(i, j, x, y) \in R_L$, we have $x'' \neq x$. (C-II)': Let $y' \in K_j$, $y' \neq y$. We have to show that $r(i, j, x, y') \notin R_L \cup A_{L+1}$. Since R_L is consistent, $r(i, j, x, y') \notin R_L$. Assume that $r(i, j, x, y') \in A_{L+1}$; we show that this leads to a contradiction. Then, for all $y'' \in K_j$, $y'' \neq y'$, $r(i, j, x, y'') \notin R_L \cup A_{L+1}$. In particular, for $y \neq y'$, $r(i, j, x, y) \notin R_L \cup A_{L+1}$. $\square$

If for the data set D , there are $i \in I$ and $x \in K_i$ with $p_{ix} = 0$ or 1 , the surmise relation R_0 is not consistent (Lemma 6). In this situation, we can construct a consistent surmise relation $R'_0 \subset R_0$.

627 To derive $R'_0 = R_{L+1}$, we perform Steps 2 and 3 of the construction procedure, for $R_L = \Delta$ and
 628 $A_{L+1} = R_0 \setminus \Delta$. In the simulation study (Section 5.2), this approach is also used to generate random
 629 consistent surmise relations.

630 Next, we discuss the inductive construction procedure for the ordinal case. The anchoring
 631 relation R_0 (Definition 8) is always consistent (Lemma 9). For $0 \leq L \leq n-1$, let R_L be a consistent
 632 surmise relation on M . The procedure to construct a consistent surmise relation R_{L+1} from R_L is:

633 1. Again, we start with the set of candidate implications

$$A_{L+1} = \{r(i, j, x, y) \in M \times M \mid v_{jy, ix} \leq L+1 \wedge r(i, j, x, y) \notin R_L\}.$$

634 2. (C-I)'': For $i, j \in I$ and $y \in K_j, y \neq 0$, mark all $r(i, j, 0, y)$ in A_{L+1} . Remove all marked
 635 elements from A_{L+1} . Denote the resulting set again with A_{L+1} . Then, since R_L is consistent,
 636 in $R_L \cup A_{L+1}$, (C-I)'' is satisfied. Note that this condition still holds (in $R_L \cup A_{L+1}$), if we
 637 remove further elements (from A_{L+1}).

638 3. (C-II)''-(c): For $i, j \in I$ and $x \in K_i$, with $i \neq j$ or $x \neq k_j$, mark all $r(i, j, x, k_j)$ in A_{L+1} .
 639 Remove all marked elements from A_{L+1} . Let A_{L+1} be the resulting set. Again, since R_L is
 640 consistent, in $R_L \cup A_{L+1}$, (C-II)''-(c) is satisfied. Further removing elements does not violate
 641 this condition.

642 4. (C-II)''-(a) and transitivity:

- 643 • For $i, j \in I$ and $x \in K_i, y \in K_j, y \neq 0$, mark all $r(i, j, x, y) \in A_{L+1}$ for which there is $y' \in K_j$,
 644 $y' < y$, such that $r(i, j, x, y') \notin R_L \cup A_{L+1}$.
- 645 • For $i, j, l \in I$ and $x \in K_i, y \in K_j, z \in K_l$, mark all implications $r(i, j, x, y)$ in A_{L+1} for which
 646 there is $r(j, l, y, z) \in R_L \cup A_{L+1}$ such that $r(i, l, x, z) \notin R_L \cup A_{L+1}$.
- 647 • For $i, j, l \in I$ and $x \in K_i, y \in K_j, z \in K_l$, mark all implications $r(j, l, y, z)$ in A_{L+1} for which
 648 there is $r(i, j, x, y) \in R_L \cup A_{L+1}$ such that $r(i, l, x, z) \notin R_L \cup A_{L+1}$.

649 Remove all marked elements from A_{L+1} . Denote the resulting set of remaining pairs by A_{L+1} .
 650 This Step 4 is repeated iteratively until no element is marked. Then, since R_L is transitive
 651 and consistent, in $R_L \cup A_{L+1}$, transitivity and (C-II)''-(a) are satisfied.

When this process terminates, that is, when in Step 4 of the procedure no element is marked, $R_{L+1} := R_L \cup A_{L+1}$ is a consistent surmise relation. (R_{L+1} is reflexive, since $\Delta \subset R_L \subset R_{L+1}$, and R_{L+1} also satisfies (C-II)''-(b), since $r(i, j, x, 0) \in R_L \subset R_{L+1}$ for all $i, j \in I$ and $x \in K_i$.)

Overall, in both the nominal and ordinal cases, the inductive construction procedures yield the sets $\{R_L | L = 0, 1, \dots, n\}$ of nominal and ordinal consistent surmise relations on M .

4.3. Generalized fit measures

For the nominal and ordinal cases simultaneously, we derive the diff fit measures to determine the best fitting consistent surmise relations among the respective structures R_L , $L = 0, 1, \dots, n$. In particular, we generalize the discussion of Section 3.3.

For a consistent surmise relation R_L , we define the corresponding error rate γ_L .

Nominal: Note that all of the $\sum_i (k_{i_l} + 1)$ reflexive item-score pairs are contained in R_L .

$$\gamma_L = \frac{\sum \{v_{jy,ix} / (p_{ix}n) | r(i, j, x, y) \in R_L, (i, x) \neq (j, y), i, j \in I, x \in K_i, y \in K_j\}}{|R_L| - \sum_{l=1}^m (k_{i_l} + 1)},$$

with

$$v_{jy,ix} = |\{d \in D | d(i) = x \wedge d(j) \neq y\}|.$$

Ordinal: Since $v_{jy,ix} = 0$ for all $((j, y), (i, x)) \in R_\Delta$ (Eq. 1), we exclude these cases from further consideration. Note that $R_\Delta \subset R_L$.

$$\gamma_L = \frac{\sum \{v_{jy,ix} / (p_{ix}n) | r(i, j, x, y) \in R_L \setminus R_\Delta\}}{|R_L \setminus R_\Delta|},$$

with

$$v_{jy,ix} = |\{d \in D | d(i) \geq x \wedge d(j) \not\geq y\}|.$$

A remark is in order. The error probability γ_L , for nominal and ordinal, is assumed to be equal for all items and answer categories, in the sense that γ_L is not defined to be item and/or answer category specific, but is aggregated over all possible combinations.

Subsequently, let $((j, y), (i, x)) \in M \times M$, with $(i, x) \neq (j, y)$, $i, j \in I$ and $x \in K_i, y \in K_j$, in the nominal case. For the ordinal case, we assume that $((j, y), (i, x)) \in M \times M$, with $((j, y), (i, x)) \notin R_\Delta$.

Under R_L , with the corresponding error rate γ_L , we introduce the expected, that is, theoretical, numbers of counterexamples $v_{jy,ix}^*$ (with γ_L , p_{ix} , and p_{jy} defined accordingly, for nominal and ordinal). Except for one estimation instance (in 3), the estimates have the same forms for both the nominal and ordinal cases.

1. If $((j, y), (i, x)) \in R_L$, we set

$$v_{jy,ix}^* = \gamma_L p_{ix} n.$$

This estimate is independent of item $j \in I$ and answer category $y \in K_j$ (ordinal, $y \neq 0$).

2. If $((j, y), (i, x)) \notin R_L$ and $((i, x), (j, y)) \notin R_L$, we set

$$v_{jy,ix}^* = (1 - p_{jy}) p_{ix} n.$$

Nominal: The events $i = x$ and $j = y$ are assumed to be independent. Thus, the events $i = x$ and $\neg(j = y) = (j \neq y)$ are independent.

Ordinal: The events $i \geq x$ and $j \geq y$ are assumed to be independent. Thus, the events $i \geq x$ and $\neg(j \geq y) = (j \not\geq y)$ are independent.

3. If $((j, y), (i, x)) \notin R_L$ and $((i, x), (j, y)) \in R_L$, we set

Nominal:

$$v_{jy,ix}^* = (p_{ix} - p_{jy} + p_{jy} \gamma_L) n \text{ if non-negative, and } v_{jy,ix}^* = 0 \text{ else.}$$

Ordinal:

$$v_{jy,ix}^* = (p_{ix} - p_{jy}) n \text{ if } ((i, x), (j, y)) \in R_\Delta,$$

and if $((i, x), (j, y)) \notin R_\Delta$,

$$v_{jy,ix}^* = (p_{ix} - p_{jy} + p_{jy} \gamma_L) n \text{ if non-negative, and } v_{jy,ix}^* = 0 \text{ else.}$$

Note that if $((i, x), (j, y)) \in R_\Delta$, it holds $(p_{ix} - p_{jy}) n = |\{d \in D | d(i) \geq x \wedge d(j) \not\geq y\}| \geq 0$.

We propose the $\text{diff}(R_L, D)$ fit measure

$$\text{diff}(R_L, D) = \frac{\sum_{((j,y),(i,x)) \in \mathcal{T}} (v_{jy,ix} - v_{jy,ix}^*)^2}{|\mathcal{T}|},$$

where $\mathcal{T} = (M \times M) \setminus \Delta$ (nominal) and $\mathcal{T} = (M \times M) \setminus R_\Delta$ (ordinal).

688 We can decompose the diff coefficient to see how the theoretical numbers of counterexamples
 689 enter into the equation.

690 **Nominal:** We have $\mathcal{T} = A + B + C$ (+ denoting disjoint union), where

$$\begin{aligned} 691 \quad A &= \{((j, y), (i, x)) \in (M \times M) \setminus \Delta \mid ((j, y), (i, x)) \in R_L\}, \\ 692 \quad B &= \{((j, y), (i, x)) \in M \times M \mid ((j, y), (i, x)) \notin R_L \wedge ((i, x), (j, y)) \in R_L\}, \\ 693 \quad C &= \{((j, y), (i, x)) \in M \times M \mid ((j, y), (i, x)) \notin R_L \wedge ((i, x), (j, y)) \notin R_L\}. \end{aligned}$$

694 **Ordinal:** $\mathcal{T} = A + B_1 + B_2 + C$, where

$$\begin{aligned} 695 \quad A &= \{((j, y), (i, x)) \in (M \times M) \setminus R_\Delta \mid ((j, y), (i, x)) \in R_L\}, \\ 696 \quad B_1 &= \{((j, y), (i, x)) \in M \times M \mid ((j, y), (i, x)) \notin R_L \wedge ((i, x), (j, y)) \in R_\Delta\}, \\ 697 \quad B_2 &= \{((j, y), (i, x)) \in M \times M \mid ((j, y), (i, x)) \notin R_L \wedge ((i, x), (j, y)) \in R_L \setminus R_\Delta\}, \\ 698 \quad C &= \{((j, y), (i, x)) \in M \times M \mid ((j, y), (i, x)) \notin R_L \wedge ((i, x), (j, y)) \notin R_L\}. \end{aligned}$$

699 Then, we can recap the diff expression

Nominal:

$$\begin{aligned} 700 \quad \text{diff}(R_L, D) &= \frac{\sum_{((j,y),(i,x)) \in A} (v_{jy,ix} - \gamma_L p_{ix} n)^2}{|\mathcal{T}|} \\ &+ \frac{\sum_{((j,y),(i,x)) \in B} \left(v_{jy,ix} - \begin{cases} \theta := (p_{ix} - p_{jy} + p_{jy} \gamma_L) n & \theta \geq 0 \\ 0 & \theta < 0 \end{cases} \right)^2}{|\mathcal{T}|} \\ 701 &+ \frac{\sum_{((j,y),(i,x)) \in C} (v_{jy,ix} - (1 - p_{jy}) p_{ix} n)^2}{|\mathcal{T}|}. \\ 702 \end{aligned}$$

Ordinal:

$$\begin{aligned}
\text{diff}(R_L, D) = & \frac{\sum_{((j,y),(i,x)) \in A} (v_{jy,ix} - \gamma_L p_{ix} n)^2}{|\mathcal{T}|} \\
& + \frac{\sum_{((j,y),(i,x)) \in B_1} (v_{jy,ix} - (p_{ix} - p_{jy})n)^2}{|\mathcal{T}|} \\
& + \frac{\sum_{((j,y),(i,x)) \in B_2} \left(v_{jy,ix} - \begin{cases} \theta := (p_{ix} - p_{jy} + p_{jy}\gamma_L)n & \theta \geq 0 \\ 0 & \theta < 0 \end{cases} \right)^2}{|\mathcal{T}|} \\
& + \frac{\sum_{((j,y),(i,x)) \in C} (v_{jy,ix} - (1 - p_{jy})p_{ix}n)^2}{|\mathcal{T}|}.
\end{aligned}$$

The decision rule, the same for both the nominal and ordinal generalized IITA algorithms, is to select in $\{R_L | L = 0, 1, \dots, n\}$ a consistent surmise relation, which yields the smallest diff value.

5. Applications to simulated data

We refer to the notation used in Section 4. In particular, let I be the set of m polytomous items i_l , nominal or ordinal, with item scores in K_{i_l} . In Section 5.1, we describe how to simulate data from a posited general surmise relation R on $M = \bigcup_{l=1}^m (\{i_l\} \times K_{i_l})$. We do not assume consistency, but simulate data under a general surmise relation. Based on the derived data generating models, we also report a simulation study in Section 5.2, in which we use the data generating models to create data from the special case, consistent surmise relations. In the simulation study, we examine the polytomous IITA algorithms regarding their ability to extract the correct implications. The computations were performed in R (<https://www.R-project.org>, The R Core Team, 2018).

5.1. Creating generalized response patterns from surmise relations of generalized implications

The approach we use is essentially based on a common construction in KST (e.g., Doignon & Falmagne, 1999). According to original work by Birkhoff (1937), the quasi-order R on M uniquely corresponds to the set $\mathcal{K}_R$ of all those subsets of M that are consistent with the implications in R . That is, for all $K' \subset M$,

$$K' \in \mathcal{K}_R \Leftrightarrow (\forall r(i, j, x, y) \in R : (i, x) \in K' \text{ implies } (j, y) \in K').$$

723 Then, for all $r(i, j, x, y) \in M \times M$,

$$\begin{aligned}
 724 \quad r(i, j, x, y) \in R &\Leftrightarrow (\forall K' \in \mathcal{K}_R : (i, x) \in K' \text{ implies } (j, y) \in K') \\
 725 &\Leftrightarrow \mathcal{K}_R^{(i,x)} \subset \mathcal{K}_R^{(j,y)},
 \end{aligned}$$

726 where $\mathcal{K}_R^{(k,z)} = \{K' \in \mathcal{K}_R | (k, z) \in K'\}$ for all $(k, z) \in M$ (Doignon & Falmagne, 1999, Theorem
727 1.49). This is also referred to as Birkhoff's theorem.

728 In KST, $\mathcal{K}_R$ is called the “quasi-ordinal knowledge space” on M corresponding to the surmise
729 relation R on M . However, in contrast to the common use case, in our context, we have to abstain
730 from the classical interpretation of a “knowledge space”. That is, as we will see below, we cannot
731 interpret $K' \in \mathcal{K}_R$, called a “knowledge state” in KST, as a possible state of knowledge for a subject
732 concerning the “mastery” of the item-score combinations in M . Instead, we utilize $\mathcal{K}_R$ as the
733 mathematically equivalent set-theoretic formulation, alternative to the order-theoretic formulation
734 by R .

735 We can recap the property of consistency in terms of $\mathcal{K}_R$ (Definitions 5 and 7). R is consistent
736 iff:

737 **Nominal:** For all $i, j \in I$ and $x \in K_i, y \in K_j$ such that $(i, x) \in K'$ implies $(j, y) \in K'$ for any $K' \in \mathcal{K}_R$,

738 (C-I)'. there is $x' \in K_i, x' \neq x$ such that $(i, x') \in K'$ and $(j, y) \notin K'$ for some $K' \in \mathcal{K}_R$,

739 (C-II)'. for all $y' \in K_j, y' \neq y$ there is $K' \in \mathcal{K}_R$ such that $(i, x) \in K'$ and $(j, y') \notin K'$.

740 **Ordinal:** For all $i, j \in I$ and $x \in K_i, y \in K_j, y \neq 0$,

741 (C-I)'. there is $K' \in \mathcal{K}_R$ such that $(i, 0) \in K'$ and $(j, y) \notin K'$,

742 (C-II)'. (a) if $(i, x) \in K'$ implies $(j, y) \in K'$ for each $K' \in \mathcal{K}_R$, then, for all $y' \in K_j, y' < y$,

743 $(i, x) \in K'$ implies $(j, y') \in K'$ for any $K' \in \mathcal{K}_R$,

744 (b) $(i, x) \in K'$ implies $(j, 0) \in K'$ for all $K' \in \mathcal{K}_R$,

745 (c) if $i \neq j$ or $x \neq k_j$, there is $K' \in \mathcal{K}_R$ such that $(i, x) \in K'$ and $(j, k_j) \notin K'$.

746 Any subset of M , including the “knowledge states” in $\mathcal{K}_R$, can be represented as 0/1 vector of
747 length $\sum_l |K_{i_l}| = \sum_l (k_{i_l} + 1)$. The two representations, subset and vector, are equivalent. For $W \subset M$,
748 we can have the binary vector

$$(n_{(i_1,0)}, \dots, n_{(i_1,k_{i_1})}, n_{(i_2,0)}, \dots, n_{(i_2,k_{i_2})}, \dots, n_{(i_m,0)}, \dots, n_{(i_m,k_{i_m})}),$$

749 where, for $i_l \in I$ and $x \in K_{i_l}$,

$$n_{(i_l,x)} = \begin{cases} 1 & (i_l, x) \in W \\ 0 & (i_l, x) \notin W \end{cases}.$$

750 Conversely, any $(n_{(i_1,0)}, \dots, n_{(i_1,k_{i_1})}, n_{(i_2,0)}, \dots, n_{(i_2,k_{i_2})}, \dots, n_{(i_m,0)}, \dots, n_{(i_m,k_{i_m})})$, with $n_{(i_l,x)} \in \{0, 1\}$,
751 $i_l \in I$, $x \in K_{i_l}$, defines a subset W of M ,

$$(i_l, x) \in W \Leftrightarrow n_{(i_l,x)} = 1.$$

752 The two mappings, respectively defined, are mutual inverses.

753 A polytomous response pattern $r = (x_1^r, x_2^r, \dots, x_m^r)$ with scores $x_l^r \in K_{i_l}$ to items i_l , $l = 1, \dots, m$,
754 of the data set D can also be represented as a subset of M or as 0/1 vector of length $\sum_l (k_{i_l} + 1)$.
755 However, a reverse equivalent representation, from subset or vector to response pattern, is not
756 possible anymore.

757 The response pattern $r = (x_1^r, x_2^r, \dots, x_m^r)$ can be written as the subset W of M :

758 **Nominal:**

$$W = \{(i_1, x_1^r), (i_2, x_2^r), \dots, (i_m, x_m^r)\}.$$

759 **Ordinal:** Note that for a response $r \in K_{i_l}$ with $r \geq x_l^r$, we also have $r \geq x'$ for all $x' \leq x_l^r$.

$$\begin{aligned} 760 \quad W &= \{(i_1, x_1^r), (i_1, x_1^r - 1), \dots, (i_1, 0), \\ 761 &\quad (i_2, x_2^r), (i_2, x_2^r - 1), \dots, (i_2, 0), \\ 762 &\quad \dots, \\ 763 &\quad (i_m, x_m^r), (i_m, x_m^r - 1), \dots, (i_m, 0)\}. \end{aligned}$$

764 This subset W determines the corresponding 0/1 vector of length $|M| = \sum_l (k_{i_l} + 1)$ in the manner
765 as we described above. That is,

$$w = (n_{(i_1,0)}, \dots, n_{(i_1,k_{i_1})}, n_{(i_2,0)}, \dots, n_{(i_2,k_{i_2})}, \dots, n_{(i_m,0)}, \dots, n_{(i_m,k_{i_m})}),$$

766 where, for $i_l \in I$ und $x \in K_{i_l}$,

767 **Nominal:**

$$n_{(i_l, x)} = \begin{cases} 1 & x = x_l^r \\ 0 & x \neq x_l^r \end{cases}.$$

768 **Ordinal:**

$$n_{(i_l, x)} = \begin{cases} 1 & x \leq x_l^r \\ 0 & x > x_l^r \end{cases}.$$

769 Thus, we can transform the original data matrix D of dimension $n \times m$ of the polytomous item
 770 scores into the 0/1 “data” matrix D' of dimension $n \times |M|$, obtained by substituting all row response
 771 patterns $r = (x_1^r, x_2^r, \dots, x_m^r)$ of length m with their corresponding 0/1 generalized response patterns
 772 w of length $\sum_l (k_{i_l} + 1)$. This matrix D' could be analyzed with the dichotomous IITA methods,
 773 whether classical, positive outcomes oriented (Section 2) or based on generalized implications
 774 (Section 3), to derive a general or consistent surmise relation on M or $M_e = M \times \{0, 1\}$, respectively.
 775 This is an interesting direction for further research.

776 A remark is in order. It is obvious that from subsets W or vectors w of the above forms we can
 777 derive a unique response pattern r . However, from a general subset of M , or equivalently, vector
 778 in $\{0, 1\}^{\sum_l (k_{i_l} + 1)}$, the construction of a unique response pattern in $K_{i_1} \times \dots \times K_{i_m}$ is not possible.
 779 Consider an arbitrary $W \subset M$. (Later, W will be randomly generated and can have any form.)
 780 Then, for an item $i_l \in I$, there are two critical cases that prohibit the unique allocation of an item
 781 score $x_l^r \in K_{i_l}$ to i_l from subset W . If $C = \{(i_l, x) | (i_l, x) \in W, x \in K_{i_l}\}$, the critical cases are $|C| \geq 2$
 782 and $C = \emptyset$. (In case of $|C| = 1$, we could assign the unique answer category to the item.) Note that
 783 this is the reason why we cannot interpret $K' \in \mathcal{K}_R$ as a possible state of knowledge of a subject
 784 concerning the “mastery” of item-score combinations $(i_l, x) \in M$.

785 Instead, we do simulate and analyze generalized response patterns. We utilize the “quasi-
 786 ordinal knowledge space” $\mathcal{K}_R$ corresponding to R . This “knowledge structure” $(M, \mathcal{K}_R)$ can be
 787 equipped with basic probabilistic concepts, to yield the so-called “basic local independence model”
 788 (BLIM), the common data generating model in KST. For a mathematical definition of this model,
 789 see, for example, Doignon & Falmagne (1999, Definition 7.2). We can use this probabilistic model
 790 to create random generalized response patterns.

The BLIM, $(M, \mathcal{K}_R, p, \{\beta_{(i,x)}\}_{(i,x) \in M}, \{\eta_{(i,x)}\}_{(i,x) \in M})$, comprises the following simulation steps:

1. p is a probability distribution on $\mathcal{K}_R$. That is, we draw a “knowledge state” K' from $\mathcal{K}_R$ according to the distribution p . For example, p can be the uniform distribution on $\mathcal{K}_R$, meaning that we are randomly sampling, with equal probability, each of the elements of $\mathcal{K}_R$. The uniform distribution is what we will use in the simulation study.
2. This drawn $K' \in \mathcal{K}_R$ can be represented by its corresponding 0/1 vector

$$k' = (n_{(i_1,0)}, \dots, n_{(i_1,k_{i_1})}, n_{(i_2,0)}, \dots, n_{(i_2,k_{i_2})}, \dots, n_{(i_m,0)}, \dots, n_{(i_m,k_{i_m})}),$$

$$n_{(i_l,x)} = \begin{cases} 1 & (i_l, x) \in K' \\ 0 & (i_l, x) \notin K' \end{cases}.$$

3. For each $(i, x) \in M$, we have two constants (across all elements of $\mathcal{K}_R$), the omission rate $\beta_{(i,x)} \in [0, 1)$ and the intrusion rate $\eta_{(i,x)} \in [0, 1)$, also called the “careless error” probability and “lucky guess” probability in KST, respectively. For example, a special case is a single error rate $\tau = \beta_{(i,x)} = \eta_{(i,x)}$ for all $(i, x) \in M$, which we will use in the simulation study. Their names stem from their applications. For any entry $n_{(i,x)}$, $(i, x) \in M$, of the vector k' , we perform the following random operations:

- (a) If $n_{(i,x)} = 1$, that is, $(i, x) \in K'$, we change this value to 0 with probability $\beta_{(i,x)}$ and leave it unchanged with probability $1 - \beta_{(i,x)}$. Thus, the resulting value for this entry is 0 or 1, respectively. In the former case, we omit the element (i, x) , which is contained in the “knowledge state” $K' \in \mathcal{K}_R$, from the eventually resulting, corresponding subset.
- (b) If $n_{(i,x)} = 0$, that is, $(i, x) \notin K'$, we change the value 0 to 1 with probability $\eta_{(i,x)}$ and leave it unchanged with probability $1 - \eta_{(i,x)}$. Thus, the new value for this entry is 1 or 0, respectively. If the value changes, we intrude the element (i, x) , which is not contained in the “knowledge state” $K' \in \mathcal{K}_R$, into the eventually resulting, corresponding subset.

4. From vector k' , thus, we obtain the new vector

$$w = (n'_{(i_1,0)}, \dots, n'_{(i_1,k_{i_1})}, n'_{(i_2,0)}, \dots, n'_{(i_2,k_{i_2})}, \dots, n'_{(i_m,0)}, \dots, n'_{(i_m,k_{i_m})}),$$

with $n'_{(i_l,x)} \in \{0, 1\}$, $i_l \in I$, $x \in K_{i_l}$, which defines the subset W of M , $(i_l, x) \in W \Leftrightarrow n'_{(i_l,x)} = 1$. In particular, W has been randomly created and can have any form. Overall, this yields the randomly generated, generalized response pattern w .

If we repeat Steps 1–4, n times, we obtain a simulated “data” matrix D' of dimension $n \times \sum_l(k_{i_l} + 1)$, with entries in $\{0, 1\}$. These are the data that we will then analyze in the simulation study.

Why can we analyze such binary “data” matrix D' with our methods, instead of having to use the polytomous response patterns $r = (x_1^r, x_2^r, \dots, x_m^r)$ with scores x_l^r in K_{i_l} ? The reason is that we can compute all components of the IITA methods (i.e., inductive construction, γ , and diff) solely based on the matrix D' . The entities we need are only aggregate values, the values v and p , which can also be readily obtained from D' .

5.2. Simulation study

We have to create random consistent surmise relations (Definitions 5 and 7) underlying the simulation study. We can reduce this problem to simulating random surmise relations, without the consistency property, and then correcting those simulated surmise relations for this property. Thereby, we can apply the methods available for simulating representative random quasi-orders (Schrepp & Ünlü, 2015; Ünlü & Schrepp, 2016b, 2019). Note that, if we want to test the new IITA algorithms in simulation, ideally, we should directly generate and use representative samples of consistent surmise relations. However, currently, techniques for the sampling of representative consistent surmise relations are not available. This is an interesting open issue for future research.

If R is a simulated quasi-order (on $M = \bigcup_{l=1}^m (\{i_l\} \times K_{i_l})$), from it, we can construct a consistent surmise relation R' (in the nominal case, $R' \subset R$). We only have to apply the adequate steps of the inductive construction procedures described in Section 4.2.

Nominal: To derive R' , which we set to be R_{L+1} , we perform Steps 2 and 3 of the (nominal) inductive construction procedure, with $R_L = \Delta$ and $A_{L+1} = R \setminus \Delta$. Note, $\Delta = \{(m, m) : m \in M\}$.

Ordinal: With

$$R_L = R_\Delta = \Delta \cup \{((i, x'), (i, x)) | i \in I, x, x' \in K_i, x' < x\} \cup \{((j, 0), (i, x)) | i, j \in I, x \in K_i\}$$

and $A_{L+1} = R \setminus R_\Delta$, we perform Steps 2, 3, and 4 of the (ordinal) inductive construction procedure, and set $R' = R_{L+1}$.

840 We evaluate the quality of the polytomous extensions of IITA used to reconstruct the true
 841 consistent surmise relations based on a simulation study. The simulation study has a general
 842 scheme.

843 **Step 1:** By a representative sampling process, we create a random, general quasi-order R on M .

844 **Step 2:** Following the inductive construction, from R , we filter a consistent surmise relation R'
 845 (see above). Then, R' is the true nominal or ordinal consistent surmise relation on M under-
 846 lying the simulated data, which we want to uncover by nominal or ordinal polytomous IITA,
 847 respectively.

848 **Step 3:** According to Birkhoff's theorem, the "quasi-ordinal knowledge space" $\mathcal{K}_{R'}$ on M that
 849 corresponds to the consistent surmise relation R' on M is constructed.

850 **Step 4:** Based on $\mathcal{K}_{R'}$, a data set D' of n generalized response patterns is simulated by drawing,
 851 with replacement and with equal probability, n elements from $\mathcal{K}_{R'}$, and for each of these
 852 elements, by applying a specified single error rate τ .

853 **Step 5:** This simulated data set D' is analyzed by one of the polytomous IITA algorithms. As the
 854 solution, the best fitting consistent surmise relation R_{IITA} on M is computed.

855 **Step 6:** We assess the quality of the IITA analysis. Quality of recovery: The true quasi-order R'
 856 and the data-analytic solution R_{IITA} are compared using their symmetric-difference distance,
 857 that is, the number of item-score pairs that differ. Quality of coverage: We also record if the
 858 true consistent surmise relation R' is contained in the inductively constructed selection set
 859 of competing consistent surmise relations $\mathcal{M} = \{R_L | L = 0, 1, \dots, n\}$.

860 The following settings are used in the simulation study. Throughout, three polytomous items
 861 are considered, $I = \{i_1, i_2, i_3\}$, with five, three, and two answer categories, $K_{i_1} = \{0, 1, 2, 3, 4\}$,
 862 $K_{i_2} = \{0, 1, 2\}$, and $K_{i_3} = \{0, 1\}$. Depending on the algorithm, the item scores are treated as
 863 nominal, for example $0 \neq \dots \neq 4$, or ordinal, for example $0 \leq \dots \leq 4$. The set $M = \bigcup_{l=1}^3 (\{i_l\} \times K_{i_l})$,
 864 on which the consistent quasi-orders that we study are defined, consists of 10 elements. Thus,

on M , we have 8,977,053,873,043 (labeled) quasi-orders (Brinkmann & McKay, 2002). Among these, to us, the numbers of the consistent quasi-orders are not known. Obviously, we cannot test all possible quasi-orders, but instead need a representative sample.

To realize a representative sample of surmise relations on M , we apply the uniform extension method (UEM) by Schrepp & Ünlü (2015). We do not review this method, but for a detailed discussion of it, refer the reader to the latter reference, or to the works by Ünlü & Schrepp (2016b, 2017). Under the UEM, for any method, for each combination of error rate and sample size, we simulate a different sample $\mathcal{S}$ of 100 representative random quasi-orders on M .

We specify the parameters τ and n of the general simulation scheme as follows. The error rate τ takes the values 0.010, 0.050, 0.100, 0.150, 0.200, and 0.350. The sample size n (of data set D') varies among 50, 100, 200, 500, 1,000, and 5,000. We have 36 combinations of these settings. For each of the combinations, with any of the nominal polytomous or ordinal polytomous IITA algorithms, Steps 1–6 of the general simulation scheme are repeated 100 times, where the quasi-orders used in Step 1 are taken to be the quasi-orders of the respective sample $\mathcal{S}$. In particular, different quasi-order samples are employed across all combinations or methods.

Consider a combination of τ and n . In any of the 100 runs, for each of the nominal and ordinal IITA algorithms, the data analysis outputs R_{IITA} and $\mathcal{M} = \{R_L | L = 0, 1, \dots, n\}$ are computed and compared with the underlying consistent surmise relation R' . For the comparison of R_{IITA} and R' , we use the dist measure

$$\text{dist}(R_{\text{IITA}}, R') = |R_{\text{IITA}} \ominus R'|,$$

where $R_{\text{IITA}} \ominus R' = (R_{\text{IITA}} \setminus R') \cup (R' \setminus R_{\text{IITA}})$ is the symmetric difference of the sets R_{IITA} and R' . Thus, dist denotes the number of item-score pairs that differ. We report the mean of the 100 dist values between the data analysis solutions and true consistent surmise relations. The smaller this dist value is, the closer are the data-analytic and true surmise relations. For the comparison of $\mathcal{M}$ and R' , we also count if $R' \in \mathcal{M}$, that is, if the selection set covers the true surmise relation. We report the coverage frequency out of the 100 comparisons. The higher this measure is, the more often is a true consistent surmise relation contained in a selection set. Note that a larger misspecification rate for the selection set can imply a larger dist value. The two comparison criteria may be confounded.

892 Also, note that the inductive construction procedures are *not* the same in the nominal and ordinal
893 IITA algorithms.

894 We present the results of the simulation study. In Table 1, for each error probability and
895 sample size combination, the mean dist values between the true consistent surmise relations and
896 data analysis solutions under the nominal and ordinal IITA algorithms are reported, in first and
897 second lines, respectively.

[Table 1 about here]

898 In Table 2, for every combination of τ and n , the coverage counts for the inductive construction
899 procedures under the nominal and ordinal IITA algorithms are shown, in first and second lines,
900 respectively.

[Table 2 about here]

901 Since the coverage counts become low for larger error probabilities or smaller sample sizes, in
902 Table 3, we also present the mean dist values computed for the true consistent surmise relations
903 *not contained* in the selection sets and the data analysis solutions, for the nominal and ordinal IITA
904 algorithms, in first and second lines, respectively.

[Table 3 about here]

905 We discuss a few results of the simulation study. From Tables 1, 2, and 3, we see that the
906 results consistently improve with smaller error probabilities or larger sample sizes. Except for
907 $\tau = 0.350$ under the ordinal method, the dist values are thoroughly low. We recommend not to
908 use the ordinal algorithm with response error rates as high as 0.350. The ordinal algorithm yields
909 comparable dist values for error rates up to $\tau = 0.150$. For the nominal algorithm, throughout, the
910 dist values are better, even low enough for the high error rate of $\tau = 0.350$. This finding suggests
911 that, particularly, the nominal model could also be applied in tests when higher error rates can be
912 expected, for example for multiple choice items. Regarding the inductive construction procedures,
913 we see that the coverage counts get lower, relatively quick, with higher error probabilities or
914 smaller sample sizes. Throughout, we obtain better coverage counts under the ordinal algorithm.
915 For example, to attain a coverage of about 50%, for $\tau = 0.100$, we need a sample size of $n = 100$

under the ordinal approach (50%), and twice as much, $n = 200$, under the nominal model (46%). Even if the true surmise relations may not be covered by the inductively generated selection sets, the computed dist values in these cases, in Table 3, stay low enough, for error rates up to $\tau = 0.200$ for both methods (minimum of 1.000 and maximum of 8.520), and even for the high error rate of $\tau = 0.350$ under the nominal algorithm (minimum of 5.048 and maximum of 8.600).

We can compare our findings with results obtained in publications on the common (positive outcomes oriented) dichotomous IITA algorithm. In Ünlü & Schrepp (2015, Section 5.2), on $m = 6$ items, with $\tau = 0.350$ and $n = 400$, an average dist value of 5.15 (2.63) was reported for the common dichotomous algorithm, compared to 5.790 (2.171) in Table 1 and 5.905 (2.104) in Table 3 under the nominal algorithm, for even $|M| = 10$ elements, $\tau = 0.350$, and $n = 500$. In Ünlü & Schrepp (2017, Table 2), for $m = 10$ items and $n = 100$, for $\tau = 0.100$ a mean dist value of 8.24 and for $\tau = 0.150$ a mean dist of 11.32 were reported for the common dichotomous algorithm, which compares to 2.460 and 3.760 in Table 1 and 3.261 and 4.161 in Table 3 for the nominal polytomous algorithm, and to 2.640 and 3.730 in Table 1 and 3.520 and 4.603 in Table 3 for the ordinal polytomous algorithm, respectively. In Sargin & Ünlü (2009, Table 4), for $m = 9$ items, the following coverage percentages for the inductive construction procedure of the common dichotomous algorithm were reported, 35% for $\tau = 0.050$ and $n = 50$, 34.5% for $\tau = 0.100$ and $n = 100$, 34.2% for $\tau = 0.150$ and $n = 200$, and 29.9% for $\tau = 0.200$ and $n = 400$. The corresponding values, in Table 2, for the nominal polytomous and ordinal polytomous algorithms are 31% and 50%, 31% and 50%, 28% and 41%, and 28% and 38%, respectively.

Thus, the proposed polytomous IITA methods yield simulation results of acceptable accuracy. Overall, we can say that the nominal algorithm is the more reliable and better performing method, albeit the quality of coverage under the ordinal approach is higher.

6. Applications to empirical data

We present two empirical applications. One is survey data from the 2000 American National Election Studies (ANES, <https://electionstudies.org>, Aldrich & McGraw, 2011) for the nominal IITA algorithm, the other comes from the Environment section of the 1990 British Social Attitudes Survey (Brook, Taylor & Prior, 1991) for the ordinal IITA algorithm. Both data sets

are freely available in the R software, accompanying the packages poLCA (<https://CRAN.R-project.org/package=poLCA>) and ltm (<https://CRAN.R-project.org/package=ltm>), respectively.

6.1. Election

The data set we use is slightly different from the data set found in the R package. We removed the incomplete cases, that is, cases with missing values. The entries were recoded, to be numeric, and to start with 0. We also selected 14 out of 17 variables. The resulting data set contains $n = 880$ observations, on $m = 14$ polytomous survey items. Let us describe this data set in more detail.

R package poLCA (adapted accordingly): “Two sets of six questions with four responses each, asking respondents’ opinions of how well various traits (moral, caring, knowledgable, good leader, dishonest, intelligent) describe presidential candidates Al Gore and George W. Bush. The responses are (0) “Extremely well”; (1) “Quite well”; (2) “Not too well”; (3) “Not well at all”.

The data set also includes potential covariates VOTE3, the respondent’s 2000 vote choice (when asked); and PARTY, the respondent’s Democratic-Republican partisan identification.

VOTE3 is coded as (0) “Gore”; (1) “Bush”; (2) “Other”.

PARTY is coded as (0) “Strong Democrat”; (1) “Weak Democrat”; (2) “Independent-Democrat”; (3) “Independent-Independent”; (4) “Independent-Republican”; (5) “Weak Republican”; (6) “Strong Republican”.

We present some descriptive R output. The first six response patterns of the data set are:

	MORALG	CARESG	KNOWG	LEADG	DISHONG	INTELG	MORALB	CARESB	KNOWB	LEADB	DISHONB	INTELB	VOTE3	PARTY
	2	0	1	1	2	1	0	0	1	1	3	1	1	4
	0	1	1	0	2	1	1	1	1	2	2	1	0	0
	1	1	1	1	1	1	1	2	1	1	2	0	0	2
	1	3	1	2	1	1	0	0	1	1	2	1	1	6
	1	1	1	1	2	1	2	3	3	3	2	3	0	0
	1	1	1	2	3	1	1	2	1	1	2	1	0	0

The polytomous survey variables, with their scores and absolute frequencies, are:

\$MORALG

972 0 1 2 3
 973 231 424 155 70
 974
 975 \$CARESG
 976
 977 0 1 2 3
 978 140 375 246 119
 979
 980 \$KNOWG
 981
 982 0 1 2 3
 983 259 482 110 29
 984
 985 \$LEADG
 986
 987 0 1 2 3
 988 112 369 289 110
 989
 990 \$DISHONG
 991
 992 0 1 2 3
 993 78 164 353 285
 994
 995 \$INTELG
 996
 997 0 1 2 3
 998 253 501 95 31
 999
 1000 \$MORALB

```

1001
1002    0    1    2    3
1003  206 455 172  47
1004
1005  $CARESB
1006
1007    0    1    2    3
1008   94 329 293 164
1009
1010  $KNOWB
1011
1012    0    1    2    3
1013  134 475 200  71
1014
1015  $LEADB
1016
1017    0    1    2    3
1018  156 432 217  75
1019
1020  $DISHONB
1021
1022    0    1    2    3
1023   24 142 379 335
1024
1025  $INTELB
1026
1027    0    1    2    3
1028  174 484 167  55
1029

```

1030 \$VOTE3

1031

1032 0 1 2

1033 430 423 27

1034

1035 \$PARTY

1036

1037 0 1 2 3 4 5 6

1038 181 122 127 49 126 113 162

1039 Subsequently, in respective order, these variables are labeled 1–14. Thus, R output such as
1040 1=2 (3L) and 5=1 (18L) stand for MORALG=“Not too well” (3rd item-category combination)
1041 and DISHONG=“Quite well” (18th item-category combination), respectively. Also, for a surmise
1042 relation R on M , and for $i, j \in I$ and $x \in K_i, y \in K_j$, the item-category combinations (i, x) and (j, y)
1043 are called parallel iff $i = x \rightarrow_R j = y$ and $j = y \rightarrow_R i = x$. Parallel item-category combinations are
1044 contained in exactly the same “knowledge states”, if the “quasi-ordinal knowledge space” on M
1045 corresponding to R is considered.

1046 Running the nominal IITA algorithm (with both consistency checks (C-I)' and (C-II)') on this
1047 data set yields the following parallel item-category combinations:

1048 [[1]]

1049 [[1]]\$parallel.items

1050 {3L, 18L}

1051

1052 [[1]]\$labels

1053 [1] "1=2" "5=1"

1054

1055 [[2]]

1056 [[2]]\$parallel.items

1057 {9L, 21L}

```

1058
1059  [[2]]$labels
1060  [1] "3=0" "6=0"
1061
1062  [[3]]
1063  [[3]]$parallel.items
1064  {11L, 23L}
1065
1066  [[3]]$labels
1067  [1] "3=2" "6=2"
1068
1069  [[4]]
1070  [[4]]$parallel.items
1071  {12L, 24L}
1072
1073  [[4]]$labels
1074  [1] "3=3" "6=3"
1075
1076  [[5]]
1077  [[5]]$parallel.items
1078  {28L, 36L, 40L, 48L}
1079
1080  [[5]]$labels
1081  [1] "7=3" "9=3" "10=3" "12=3"
1082
1083  [[6]]
1084  [[6]]$parallel.items
1085  {33L, 37L}
1086

```

```

1087 [[6]]$labels
1088 [1] "9=0" "10=0"
1089
1090 [[7]]
1091 [[7]]$parallel.items
1092 {35L, 47L}
1093
1094 [[7]]$labels
1095 [1] "9=2" "12=2"

```

1096 Fig. 2 depicts the implications extracted by the nominal IITA algorithm for this data set.

[Fig. 2 about here]

1097 A first interesting observation is the dependency between the personal association with a party
1098 (Item 14) and the voting behavior (Item 13). Not surprisingly, $14 = 4 \rightarrow 13 = 1$, $14 = 5 \rightarrow 13 = 1$,
1099 and $14 = 6 \rightarrow 13 = 1$, that is, persons that identify themselves as Republicans vote for Bush. But
1100 on the other hand, we see $14 = 0 \rightarrow 13 = 0$ and $14 = 1 \rightarrow 13 = 0$, but not $14 = 2 \rightarrow 13 = 0$, that
1101 is, only persons that report a strong association with the Democrats vote for Gore. In addition,
1102 $14 = 3 \rightarrow 13 = 1$, that is, independent persons vote for Bush. So, it seems that Bush is preferred by
1103 all Republicans, but Gore is not the preferred candidate for all Democrats. (In one of the closest
1104 elections, Bush defeated Gore and won the election.)

1105 Next, we concentrate on the implications between the association with a party (Item 14) and
1106 the traits attributed to a candidate. We have the six traits (moral, caring, knowledgable, good
1107 leader, dishonest, intelligent), which can be rated as describing the candidate “Extremely well” (0),
1108 “Quite well” (1), “Not too well” (2), and “Not well at all” (3). If we look at the traits in the given
1109 order, we see that $14 = 3$ (Independent-Independent) implies the pattern of traits (1, 1, 1, 1, 2, 1)
1110 for Bush and for Gore, that is, persons without an association with any of the big parties perceive
1111 both candidates equally. For $14 = 1$ (Weak Democrat), the trait patterns (1, 1, 1, 1, 2, 1) for Gore
1112 and (1, 2, 1, 1, 2, 1) for Bush are implied. For $14 = 5$ (Weak Republican), the trait patterns for
1113 Gore and Bush are (1, 2, 1, 2, 2, 1) and (1, 1, 1, 1, 3, 1), respectively. Thus, weak Republicans see

the personal traits of their candidate much more positive compared to his competitor than weak Democrats.

Another interesting question is how the perception of the personal traits of the candidates influenced voting. For Item 5 (perception of a candidate as dishonest), we see a clear picture, for both candidates. We have $5 = 0 \rightarrow 13 = 1$ and $5 = 1 \rightarrow 13 = 1$, and $11 = 0 \rightarrow 13 = 0$ and $11 = 1 \rightarrow 13 = 0$, that is, if a candidate is perceived as dishonest, the person votes for the competitor. For the positive traits and Gore (Items 1, 2, 3, 4, 6), we see the following clear pattern. For an item $i \in \{1, 2, 3, 4, 6\}$, we have $i = 0 \rightarrow 13 = 0$, $i = 2 \rightarrow 13 = 1$, and $i = 3 \rightarrow 13 = 1$, that is, if a positive trait is seen as describing Gore “Extremely well”, then the person votes for Gore. If the personal trait is not associated with Gore (“Not too well” and “Not well at all”), the person votes for Bush. For Bush, we see a similar pattern for Items 7 and 10. For the other positive traits, we have $8 = 0 \rightarrow 13 = 1$, $8 = 1 \rightarrow 13 = 1$, $8 = 3 \rightarrow 13 = 0$, $9 = 0 \rightarrow 13 = 1$, $9 = 3 \rightarrow 13 = 0$, $12 = 0 \rightarrow 13 = 1$, and $12 = 3 \rightarrow 13 = 0$. Thus, a perhaps speculative interpretation is, a candidate for the Democrats has to meet higher moral standards to be voted for, than a candidate for the Republicans.

As the graph of the solution shows, there are a lot more interesting implications that can be investigated. Since our main goal is to demonstrate the possibilities of the approach, and because space is limited, we have only described some of the implications, which are especially interesting from the viewpoint of explaining voting behavior.

6.2. Environment

A sample of $n = 291$ respondents answered $m = 6$ trichotomous items. The answer categories for all items are (0) “Not very concerned”, (1) “Slightly concerned”, and (2) “Very concerned”. The six items are, R package ltm:

“LeadPetrol

Lead from petrol.

RiverSea

River and sea pollution.

RadioWaste

1142 Transport and storage of radioactive waste.

1143 AirPollution

1144 Air pollution.

1145 Chemicals

1146 Transport and disposal of poisonous chemicals.

1147 Nuclear

1148 Risks from nuclear power station.”

1149 The variables of this data set, with their scores and absolute frequencies, are:

1150	Lead	Petrol	River	Sea	Radio	Waste	Air	Pollution	Chemicals	Nuclear
1151	0	17	7		18		9		17	46
1152	1	95	51		56		93		56	95
1153	2	179	233		217		189		218	150

1154 From the data, we can see that, across all variables, people are mostly slightly or very concerned,
1155 only few are not very concerned.

1156 In respective order, these variables are labeled 1–6. In the R output, for example, $1 \geq 0$ (1L)
1157 and $2 \geq 1$ (5L) stand for LeadPetrol $\geq$ “Not very concerned” (1st item-category combination) and
1158 RiverSea $\geq$ “Slightly concerned” (5th item-category combination), respectively. For a surmise
1159 relation R on M , for $i, j \in I$ and $x \in K_i, y \in K_j$, the item-category combinations (i, x) and (j, y)
1160 are called parallel iff $i \geq x \rightarrow_R j \geq y$ and $j \geq y \rightarrow_R i \geq x$. Parallel item-category combinations are
1161 contained in exactly the same “knowledge states”.

1162 Ordinal IITA analysis (without consistency check (C-II)’-(c)) of this data set yields the parallel
1163 item-category combinations:

1164 `[[1]]`

1165 `[[1]]$parallel.items`

1166 `{1L, 4L, 7L, 10L, 13L, 16L}`

1167

1168 `[[1]]$labels`

```

1169 [1] "1>=0" "2>=0" "3>=0" "4>=0" "5>=0" "6>=0"
1170
1171
1172 [[2]]
1173 [[2]]$parallel.items
1174 {2L, 5L, 8L, 11L, 14L}
1175
1176 [[2]]$labels
1177 [1] "1>=1" "2>=1" "3>=1" "4>=1" "5>=1"
1178
1179
1180 [[3]]
1181 [[3]]$parallel.items
1182 {9L, 15L}
1183
1184 [[3]]$labels
1185 [1] "3>=2" "5>=2"

```

1186 Fig. 3 shows the corresponding graph of the ordinal IITA solution.

[Fig. 3 about here]

1187 We want to use the support (supp), confidence (conf), and lift aggregation measures (e.g.,
1188 Zhang & Zhang, 2002) to quantify the strength of an implication $i \geq x \rightarrow j \geq y$,
1189 $i, j \in I, x \in K_i, y \in K_j$, let

$$\begin{aligned}
\text{supp} &= \frac{|\{d \in D : d(i) \geq x \wedge d(j) \geq y\}|}{n}, \\
\text{conf} &= \frac{|\{d \in D : d(i) \geq x \wedge d(j) \geq y\}|}{|\{d \in D : d(i) \geq x\}|}, \\
\text{lift} &= \frac{\text{conf}}{|\{d \in D : d(j) \geq y\}|/n}.
\end{aligned}$$

1193 Especially, confidence and lift are useful measures. Confidence is the conditional probability of

1194 $j \geq y$ given $i \geq x$. Lift is the share of the probability of $j \geq y$ given $i \geq x$ relative to the marginal
 1195 probability of $j \geq y$. Higher value of these measures imply greater implication strength.

1196 The nontrivial implications detected for this data set with the ordinal algorithm, and their
 1197 support, confidence, and lift values, are:

1198	implications	supp	conf	lift
1199	1 >= 1 -> 2 >= 1 :	0.935	0.993	1.017
1200	1 >= 1 -> 3 >= 1 :	0.893	0.949	1.011
1201	1 >= 1 -> 4 >= 1 :	0.928	0.985	1.017
1202	1 >= 1 -> 5 >= 1 :	0.890	0.945	1.004
1203	1 >= 2 -> 2 >= 2 :	0.567	0.922	1.151
1204	1 >= 2 -> 6 >= 1 :	0.564	0.916	1.088
1205	2 >= 1 -> 1 >= 1 :	0.935	0.958	1.017
1206	2 >= 1 -> 3 >= 1 :	0.924	0.947	1.010
1207	2 >= 1 -> 4 >= 1 :	0.962	0.986	1.017
1208	2 >= 1 -> 5 >= 1 :	0.928	0.951	1.010
1209	3 >= 1 -> 1 >= 1 :	0.893	0.952	1.011
1210	3 >= 1 -> 2 >= 1 :	0.924	0.985	1.010
1211	3 >= 1 -> 4 >= 1 :	0.928	0.989	1.021
1212	3 >= 1 -> 5 >= 1 :	0.911	0.971	1.031
1213	3 >= 2 -> 2 >= 2 :	0.667	0.894	1.117
1214	3 >= 2 -> 5 >= 2 :	0.677	0.908	1.212
1215	3 >= 2 -> 6 >= 1 :	0.691	0.926	1.100
1216	4 >= 1 -> 1 >= 1 :	0.928	0.957	1.017
1217	4 >= 1 -> 2 >= 1 :	0.962	0.993	1.017
1218	4 >= 1 -> 3 >= 1 :	0.928	0.957	1.021
1219	4 >= 1 -> 5 >= 1 :	0.921	0.950	1.009
1220	4 >= 2 -> 2 >= 2 :	0.622	0.958	1.196
1221	4 >= 2 -> 3 >= 2 :	0.588	0.905	1.213

1222 4 $\geq$ 2 $\rightarrow$ 5 $\geq$ 2 : 0.591 0.910 1.215
 1223 4 $\geq$ 2 $\rightarrow$ 6 $\geq$ 1 : 0.595 0.915 1.087
 1224 5 $\geq$ 1 $\rightarrow$ 1 $\geq$ 1 : 0.890 0.945 1.004
 1225 5 $\geq$ 1 $\rightarrow$ 2 $\geq$ 1 : 0.928 0.985 1.010
 1226 5 $\geq$ 1 $\rightarrow$ 3 $\geq$ 1 : 0.911 0.967 1.031
 1227 5 $\geq$ 1 $\rightarrow$ 4 $\geq$ 1 : 0.921 0.978 1.009
 1228 5 $\geq$ 2 $\rightarrow$ 2 $\geq$ 2 : 0.670 0.894 1.117
 1229 5 $\geq$ 2 $\rightarrow$ 3 $\geq$ 2 : 0.677 0.904 1.212
 1230 5 $\geq$ 2 $\rightarrow$ 6 $\geq$ 1 : 0.694 0.927 1.101
 1231 6 $\geq$ 1 $\rightarrow$ 1 $\geq$ 1 : 0.801 0.951 1.010
 1232 6 $\geq$ 1 $\rightarrow$ 2 $\geq$ 1 : 0.832 0.988 1.012
 1233 6 $\geq$ 1 $\rightarrow$ 3 $\geq$ 1 : 0.835 0.992 1.057
 1234 6 $\geq$ 1 $\rightarrow$ 4 $\geq$ 1 : 0.835 0.992 1.023
 1235 6 $\geq$ 1 $\rightarrow$ 5 $\geq$ 1 : 0.828 0.984 1.045
 1236 6 $\geq$ 2 $\rightarrow$ 2 $\geq$ 2 : 0.478 0.927 1.157
 1237 6 $\geq$ 2 $\rightarrow$ 3 $\geq$ 2 : 0.481 0.933 1.252
 1238 6 $\geq$ 2 $\rightarrow$ 4 $\geq$ 2 : 0.423 0.820 1.263
 1239 6 $\geq$ 2 $\rightarrow$ 5 $\geq$ 2 : 0.478 0.927 1.237

1240 For all detected implications, we see high confidence values and lift values greater than 1.

1241 We discuss a few results of the empirical application. A slight (or higher) concern on Item 6
 1242 (Risks from nuclear power station) implies a slight concern on all of the other environmental risks,
 1243 Items 1, 2, 3, 4, and 5. A slight concern in one of these other risks implies a slight concern on all
 1244 the others, that is, the corresponding item-score combinations constitute parallel “items”. Thus,
 1245 for people reporting a slight concern on one of these risks, their frequencies are all about equally
 1246 high. For people reporting strong concerns, the picture is more intricate. A strong concern on Item
 1247 6 (nuclear power station) implies a strong concern on Items 2, 3, 4, and 5. This is not surprising,
 1248 since nuclear incidents will most likely cause pollution of water (Item 2) and air (Item 4), and like
 1249 the transport, storage, and disposal of radioactive waste (Item 3) and poisonous chemicals (Item

5) result from unexpected accidents. A strong concern on Item 1 (Lead from petrol) implies a strong concern on Item 2 (River and sea pollution), but to be “Very concerned” with Item 1 is independent from being “Very concerned” with all other risks, Items 3, 4, 5, and 6. A strong concern with regard to Item 2 can be deemed a minimum requirement for strong concerns on all of the other variables.

7. Conclusion

We conclude with a summary and further research.

7.1. Summary

Inductive item tree analysis (IITA) has been steadily developed and has become an established method of relational data analysis (Leeuwe, 1974; Sargin & Ünlü, 2009; Schrepp, 1999, 2002, 2003; Ünlü & Schrepp, 2015, 2016a). The major application field for IITA is knowledge space theory (KST) (Albert & Lukas, 1999; Doignon & Falmagne, 1999; Falmagne et al., 2013). In KST, the primary focus is on dichotomous items, and moreover, the positive answer category, to solve or master an item, is essential. This is the reason why the original, and present, formulation of IITA is restrictive, solely based on dichotomous items, with their positive outcomes.

In this paper, we have generalized this current formulation. Three new IITA algorithms have been proposed: for dichotomous items with generalized implications, for nominal polytomous items, and for ordinal polytomous items. After motivating (Section 1) and a review of the present approach (Section 2), we have extended the dichotomous IITA algorithm to include negative item scores as well, yielding generalized implications (Section 3). We have also introduced the even more general IITA algorithms for nominal polytomous and ordinal polytomous items (Section 4).

The general idea is to work with a surmise relation R defined on the Cartesian product $M = \bigcup_{l=1}^m (\{i_l\} \times K_{i_l})$ of all item-score combinations, where $I = \{i_l : 1 \leq l \leq m\}$ is the item set, and $K_{i_l} = \{0, 1, \dots, k_{i_l}\}$ are the sets of answer categories. Then, the interpretation of an item-score pair, or implication, $((j, y), (i, x)) \in R$ depends on whether the nominal or ordinal models are considered. In the nominal model, the interpretation is “From the (latent) response x to item i , we can surmise the response y to item j ”. In the ordinal model, the interpretation is “From the

(latent) response $r \geq x$ to item i , we can surmise the response $r' \geq y$ to item j ". In particular, we have introduced the pertinent consistency properties for surmise relations (on M), respectively defined in the nominal and ordinal cases. With consistent surmise relations, we have been able to exclude artificial or contradictory implications from the inductive construction, something that has not been considered in the current IITA algorithms so far, and to have the implications be in accordance with their interpretation.

We have tested the dichotomous extension of IITA with empirical data (Section 3.4) and the polytomous extensions in simulated data (Section 5) and two real data sets (Section 6). In the empirical example, on national identity, seven dichotomous items from the ISSP95 were considered, with $n = 1,126$ observations. The simulation study was based on three polytomous items, with five, three, and two answer categories, for various sample sizes. The two empirical data sets, on national election from the 2000 ANES and on environment of the 1990 British Social Attitudes Survey, were based on fourteen and six survey variables, with $n = 880$ and $n = 291$, respectively. The results were promising. In the real data examples, the detected implications were empirically plausible and could be interpreted. In the simulation study, we assessed the quality of recovery and quality of coverage. We demonstrated that these algorithms were capable of detecting the true implications, and that the selection sets included the true surmise relations, to some degree, as more data become available.

Worth mentioning, from a methodological viewpoint, are two issues. In Section 5.1, we have discussed a method for how to simulate generalized response patterns from a surmise relation R on M of generalized implications. Also, in Section 5.2, we have outlined how to reduce the problem of creating random consistent surmise relations to the problem of sampling representative random general quasi-orders.

7.2. Further research

We see the following directions for future research. We have transformed the raw data matrix D of polytomous item scores into the binary matrix D' , by substituting all row vectors r of D with their corresponding binary vectors w of D' (Section 5.1). In follow-up work, this matrix D' can be analyzed with the dichotomous IITA algorithms, both classical, based on only positive outcomes,

or based on generalized implications. Then, the results are a general or consistent surmise relation, on M or $M_e = M \times \{0, 1\}$, respectively.

We have pointed out that, in order to test the new algorithms in simulation, we ideally should base the simulation studies on representative samples of consistent surmise relations (Section 5.2). However, sampling techniques for the representative generation of consistent surmise relations are currently lacking. Techniques of this sort could be developed similar to research reported in Schrepp & Ünlü (2015) and Ünlü & Schrepp (2016b, 2019).

Beside the proposed three new algorithms, we have also contributed two, rather inconspicuous improvements on the common dichotomous IITA algorithms. The first improvement corrects for possible negative estimates during the estimation of the theoretical value t_{ij} (Section 2.2). The second improvement relates to the consistency checks during the inductive construction of the selection set, also aimed at eliminating trivial or artificial implications (Section 3.2). So far, in IITA publications, these issues have not been studied. Thus, further research could explore the impact of these improvements on the performances of the classical IITA algorithms.

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Tables

Table 1

Average dist values between the true consistent surmise relations and the data analysis solutions under the nominal polytomous IITA algorithm (first line) and ordinal polytomous IITA algorithm (second line), for three polytomous items, with five, three, and two answer categories. The means are taken over 100 trials; for each combination and method, a different representative random sample of 100 quasi-orders on ten items were drawn from the UEM. Standard errors are in parentheses.

	<i>n</i>					
	50	100	200	500	1,000	5,000
τ						
0.010	1.000 (1.206)	0.460 (0.937)	0.150 (0.575)	0.060 (0.422)	0.020 (0.141)	0.000 (0.000)
	2.260 (1.993)	2.210 (2.459)	1.370 (1.802)	1.220 (1.593)	1.120 (1.765)	1.050 (1.648)
0.050	2.320 (2.069)	1.220 (1.447)	0.770 (1.171)	0.500 (1.000)	0.200 (0.603)	0.080 (0.307)
	2.810 (2.246)	2.270 (2.160)	1.710 (2.076)	1.530 (1.856)	1.380 (1.704)	1.040 (1.576)
0.100	3.550 (2.245)	2.460 (1.966)	1.640 (1.738)	0.820 (1.274)	0.730 (1.153)	0.580 (1.027)
	3.350 (2.945)	2.640 (2.754)	2.040 (2.428)	1.670 (2.040)	1.270 (1.734)	1.170 (1.798)
0.150	4.800 (2.399)	3.760 (2.142)	2.480 (1.823)	1.680 (1.847)	1.250 (1.553)	1.140 (1.436)
	4.640 (3.603)	3.730 (3.578)	3.640 (3.915)	3.320 (3.895)	2.850 (3.896)	2.630 (3.961)
0.200	5.760 (1.859)	4.840 (2.489)	3.860 (1.970)	2.740 (2.111)	2.460 (2.037)	1.160 (1.354)
	7.350 (4.573)	7.220 (4.943)	7.100 (5.454)	7.070 (5.629)	7.140 (5.703)	7.000 (5.726)
0.350	8.600 (2.947)	7.740 (2.533)	7.000 (2.774)	5.790 (2.171)	5.360 (2.484)	4.560 (2.244)
	10.940 (3.578)	11.190 (2.950)	12.240 (2.778)	12.490 (2.525)	12.550 (2.276)	12.440 (2.622)

Table 2

Coverage counts, out of 100, that the true consistent surmise relations are contained in the inductively constructed selection sets under the nominal polytomous IITA algorithm (first line) and ordinal polytomous IITA algorithm (second line), for three polytomous items, with five, three, and two answer categories.

	<i>n</i>					
	50	100	200	500	1,000	5,000
<i>τ</i>						
0.010	63	85	97	98	100	100
	73	94	100	100	100	100
0.050	31	56	69	76	90	94
	50	71	91	97	99	100
0.100	18	31	46	68	71	73
	33	50	60	84	86	95
0.150	5	13	28	42	55	56
	20	37	41	58	64	68
0.200	0	7	19	28	30	52
	14	24	33	38	39	50
0.350	0	0	2	5	10	16
	10	12	15	16	21	25

Table 3

Average dist values between the true consistent surmise relations, when these were not contained in the selection sets, and the data analysis solutions under the nominal polytomous IITA algorithm (first line) and ordinal polytomous IITA algorithm (second line), for three polytomous items, with five, three, and two answer categories. Standard errors are in parentheses. NA marks cases for which no value could be computed (mean or standard deviation of no or one measurement).

τ	n					
	50	100	200	500	1,000	5,000
0.010	2.054 (1.026)	2.200 (1.207)	2.000 (1.000)	2.500 (2.121)	NA (NA)	NA (NA)
	3.185 (1.642)	4.667 (3.724)	NA (NA)	NA (NA)	NA (NA)	NA (NA)
0.050	3.087 (1.931)	2.341 (1.397)	2.097 (1.221)	2.042 (0.999)	1.700 (0.949)	1.167 (0.408)
	3.580 (2.195)	3.034 (2.061)	2.556 (1.590)	1.333 (0.577)	1.000 (NA)	NA (NA)
0.100	4.134 (1.986)	3.261 (1.804)	2.796 (1.534)	2.344 (1.208)	2.276 (0.996)	2.074 (0.874)
	3.940 (3.015)	3.520 (2.697)	2.950 (2.819)	2.125 (1.586)	2.000 (1.617)	1.200 (0.447)
0.150	5.000 (2.274)	4.161 (1.976)	3.194 (1.562)	2.672 (1.690)	2.556 (1.423)	2.455 (1.190)
	5.150 (3.551)	4.603 (3.765)	4.763 (3.897)	5.024 (4.257)	4.028 (4.462)	4.125 (4.263)
0.200	5.760 (1.859)	5.022 (2.471)	4.395 (1.579)	3.556 (1.891)	3.286 (1.843)	2.271 (1.086)
	7.372 (4.456)	7.355 (4.658)	8.119 (5.201)	8.194 (5.244)	8.049 (5.258)	8.520 (5.369)
0.350	8.600 (2.947)	7.740 (2.533)	7.122 (2.660)	5.905 (2.104)	5.567 (2.427)	5.048 (1.981)
	11.022 (3.422)	11.216 (2.723)	12.106 (2.440)	12.512 (2.315)	12.418 (2.257)	12.147 (2.624)

Figure captions

Fig. 1. Detected implications for Question 4 of the ISSP95, containing the dichotomous items 1, 2, ..., 7. A circle represents agreement with an item (first row plot) and a rectangle disagreement (second row plot). Sample size is $n = 1,126$.

Fig. 2. Implications detected by nominal IITA of the 2000 ANES data set. The data set contains twelve candidate evaluation items, each with four answer categories, and two covariate variables, with three and seven answer categories. Sample size is $n = 880$. Note that “transitive implications” are not depicted and are assumed to hold implicitly.

Fig. 3. Implications detected by ordinal IITA of the Environment data set. The data set contains six trichotomous items. Sample size is $n = 291$. “Transitive implications” are assumed implicitly.

1406 **Figures**

1407 **Fig. 1**

1408

1411 **Fig. 3**

1412