Symbolic Fractions Elicit an Analog Magnitude Representation in School-Age Children

Abstract

A fundamental question about fractions is whether they are grounded in an abstract non-symbolic magnitude code, similar to those postulated for whole numbers. Mounting evidence suggests that symbolic fractions could be grounded in mechanisms for perceiving non-symbolic ratio magnitudes. However, systematic examination of such mechanisms in children has been lacking. We asked second and fifth grade children (prior to and after formal instructions with fractions, respectively) to compare pairs of symbolic fractions, non-symbolic ratios and mixed symbolic/non-symbolic pairs. This paradigm allowed us to test three key questions: 1) whether children show an analog magnitude code for rational numbers, 2) whether that code is compatible with mental representations of symbolic fractions, and 3) how formal education with fractions affects the symbolic-non-symbolic relation. We examined distance effects as a marker of analog ratio magnitude processing and notation effects as a marker of converting across numerical codes. Second and fifth grade children's response times and error rates showed classic distance and notation effects. Non-symbolic ratios were processed most efficiently, with mixed and symbolic notations being relatively slower. Children with more formal instruction in symbolic fractions had a significant advantage in comparing symbolic fractions, but a smaller advantage for non-symbolic ratio stimuli. Supplemental analyses showed that second graders relied on numerator distance more than holistic distance, and fifth graders relied on holistic fraction magnitude distance more than numerator distance. These results suggest that children have a non-symbolic ratio magnitude code, and that symbolic fractions can be translated into that magnitude code.

Keywords: numerical cognition, fractions, magnitude processing, rational number processing, triple-code model

Introduction

Psychologists, education researchers, and educators agree that understanding rational numbers, particularly in the form of common fractions (e.g. “ $\frac{3}{4}$ ”), is important for advancement in STEM domains. However, acquiring this understanding is also notoriously difficult for both children and adults (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; Lobato, Ellis, & Zbiek, 2010; Pitkethly & Hunting, 1996; Siegler, Fazio, Bailey, & Zhou, 2013). One proposed explanation for these difficulties is that, although students’ acquisition of symbolic whole numbers is aided by pre-existing neurocognitive architectures (e.g., Feigenson, Spelke & Dehaene, 2004; Piazza, 2010) similar neurocognitive architectures do not exist for rational numbers (e.g., Dehaene, 2011). Despite its wide acceptance, this explanation is contradicted by a growing body of evidence (Bonn & Cantlon, 2012, 2017; Jacob, Vallentin, & Nieder, 2012; Lewis, Matthews, & Hubbard, 2015; McCrink & Wynn, 2007). Here, we provide novel additional evidence that an understanding of symbolic fractions could be linked to pre-existing non-symbolic representations of fraction magnitude that are available to children prior to formal education with symbolic fractions.

Architectures for number processing

Many influential theories of whole number understanding postulate an analog representation of (positive integer) magnitude, which is instantiated in some internal code (Arsalidou & Taylor, 2011; Clark, 1991; Dehaene, 1992; McCloskey, 1992; Pinel et al., 2001; Schmithorst & Brown, 2004; Skagenholt et al., 2018) although the nature of that code is debated. In these theories, different types of symbolic representations of whole numbers (such as Arabic numerals or number words) are thought to have different underlying codes, which must be translated into the internal magnitude code to be understood. Consistent with these models,

infants (Antell & Keating, 1983; Coubart, Izard, Spelke, Marie, & Streri, 2014; Feigenson, Dehaene, & Spelke, 2004; Starkey & Cooper, 1980; Strauss & Curtis, 1981) and a variety of non-human species (Nieder, 2017; Tudusciuc & Nieder, 2007) demonstrate distance effects, the behavioral hallmark of analog magnitude representations. Neuropsychological cases demonstrating dissociations between number word knowledge and quantity suggests that these codes can be selectively impaired and are therefore distinct (Dehaene & Cohen, 1991, 1995; Takayama, Sugishita, Akiguchi, & Kimura, 1994; Warrington, 1982). Finally, behavioral indices of numerical distance effects (decreasing reaction times and error rates with increasing numerical distance) closely align with neuroimaging indices (decreasing parietal-frontal activation with increasing numerical distance), further suggesting that that symbolic number stimuli can elicit an analog internal magnitude representation (Nieder & Dehaene, 2009; Pinel et al., 2001).

Whether the same is true for rational numbers remains an open question. Lewis, Matthews, and Hubbard (2015) have recently argued for the existence of an evolutionarily ancient and developmentally early mechanism for processing non-symbolic ratios, such as comparing the magnitudes of line ratios or dot ratios. They have dubbed this system the ratio processing system (RPS), analogous to the approximate number system (ANS) for whole numbers. For example, the ability to distinguish non-symbolic ratios has been demonstrated in human infants (McCrink & Wynn, 2007), elementary school-age children (Boyer, Levine, & Huttenlocher, 2008; Duffy, Huttenlocher, & Levine, 2005; Meert, Grégoire, Seron, & Noël, 2013; Sophian, 2000; Spinillo & Bryant, 1999), adults (Hollands & Dyre, 2000; Meert, Grégoire, Seron, & Noël, 2011; Stevens & Galanter, 1957), and individuals with limited formal mathematics education (McCrink, Spelke, Dehaene, & Pica, 2013). In fact, these abilities even extend to non-human primates (Vallentin & Nieder, 2008, 2010; Woodruff & Premack, 1981).

The neural systems that support non-symbolic ratio processing have been localized to parietal-frontal networks in non-human primates and in human neuroimaging studies (for reviews, see Vallentin, Jacob & Nieder, 2009; Lewis et al., 2015). However, whether these ratios evoke an analog internal representation of magnitude, and whether *symbolic* fractions can elicit this analog code remain largely unanswered questions.

A growing body of evidence suggests the existence of relations between non-symbolic ratio processing and symbolic fraction knowledge, and also between non-symbolic ratio processing and other math outcomes. Adults are easily able to make magnitude comparisons between symbolic fractions and various types of non-symbolic ratios (Matthews & Chesney, 2015). Furthermore, their performance on symbolic-symbolic magnitude comparisons is impaired when non-symbolic dimensions of printed fractions are incongruent with the symbolic comparison decision (Matthews & Lewis, 2017; Kallai and Tzelgov, 2009). This Stroop-like effect points to a behaviorally compelling automatic sense of magnitude for symbolic fractions. Furthermore, Matthews, Lewis, and Hubbard (2016) found that among undergraduates, non-symbolic ratio comparison accuracy predicted not only symbolic fraction knowledge, but also algebra placement scores, and was even more predictive of these math outcomes than non-symbolic whole number comparison performance (ANS acuity).

In children, other non-symbolic ratio tasks, such as proportional reasoning or fraction number line estimation, predict later conceptual and procedural fraction knowledge (Hansen et al., 2015; Jordan et al., 2013; Möhring, Newcombe, Levine, & Frick, 2015; Ye et al., 2016) as well as other formal mathematical abilities, such as performance on a standardized mathematics assessment (Resnick et al., 2016). Together, these findings support the idea that the neurocognitive architecture that processes non-symbolic ratios is related to formal, symbolic

rational number processing and other forms of mathematical reasoning, but do not provide direct evidence that symbolic fractions elicit an analog internal magnitude representation in children.

Here, we tested three key questions: 1) whether an analog magnitude code exists for rational number in children, 2) whether it can be accessed by symbolic fraction stimuli, and 3) how formal education with fractions affects the symbolic-non-symbolic relation. We examined two well-established behavioral effects: distance effects as a marker for analog ratio magnitude processing and notation effects as a marker for converting across multiple numerical codes.

Distance effects as a marker for magnitude processing

The evidence that symbolic whole number stimuli can elicit the internal analog magnitude representation comes largely from distance effects (Moyer & Landauer, 1967): reaction times (RTs) and error rates (ERs) are inversely related to numerical distance, being low for “far” comparisons (e.g., 3 vs. 9) but higher for “near” comparisons (e.g., 7 vs. 9). Dehaene & Changeux (1993) proposed that such distance effects arise from overlapping tuning curves for quantity, which have been found in both non-human primate electrophysiological (Nieder & Miller 2003, 2004; Sawamura et al. 2002), and human psychophysical and neuroimaging studies (Piazza et al., 2004; Pinel et al., 2001, 2004). Non-symbolic stimuli (such as dot arrays) typically produce distance effects in magnitude comparison or matching tasks (Holloway & Ansari, 2008; Moyer, 1973). When number symbol stimuli (such as Arabic numerals or number words) produce distance effects, the interpretation is that they have activated the underlying analog magnitude code.

Emerging evidence suggests that there is a similar analog magnitude code for ratio stimuli in adults. Overlapping tuning curves for specific proportions have been found in non-human primates using behavioral and electrophysiological methods (Vallentin & Nieder, 2008,

2010) and in humans using psychophysical and neuroimaging methods (Jacobs & Nieder, 2009a, 2009b). Distance effects for non-symbolic ratio stimuli have been reported in adults using estimation tasks (Chesney & Matthews, 2018; Gaëlle Meert, Grégoire, Seron, & Noël, 2012). Moreover, in adults, distance effects for symbolic fractions have been now been reported as well (Hurst & Cordes, 2016; Matthews & Lewis, 2016; Gaëlle Meert, Grégoire, Seron, & Noël, 2013; Schneider & Siegler, 2010).

However, these studies did not examine distance effects for both symbolic fractions and non-symbolic ratios in the same participants. These limitations preclude directly testing for the presence of distance effects across notations. Mock et al. (2018) demonstrated distance effects for both symbolic and several types of non-symbolic ratio stimuli in adults, but the range of distances between stimuli was limited, and the stimulus set had a higher frequency of low-distance items, which limited the possibility of making inferences about distance effects. To address these limitations, Binzak et al. (2019) developed a cross-notation magnitude comparison paradigm (symbolic fractions, non-symbolic line ratios and mixed symbolic-non-symbolic pairs) to measure distance effects within the same group of adults, with numerical distance carefully counterbalanced across notation conditions. Adults showed robust distance effects of similar magnitude when comparing non-symbolic ratio stimuli and symbolic fraction stimuli and were able to compare the holistic magnitude of symbolic fractions with relative ease.

These results, while encouraging, do not yet provide evidence of an analog internal magnitude representation for either symbolic or non-symbolic ratio stimuli in children. It is possible that the observed effects in adults are the result of years of experience and formal instruction with ratios and may require some particular level of psychological (and/or neural) maturity to be present. Meert and colleagues (2013) found that 9-11 year old children were able

to make estimates of both symbolic and non-symbolic ratio magnitudes in a third magnitude format – adjusting a cylinder’s fullness to match the magnitude of the target stimulus. This suggests that children can interpret both symbolic and non-symbolic ratios as magnitudes, but the task (estimation rather than comparison) (1) did not allow for distance effects to be examined, and (2) was presented in such a way that it clearly allowed for the application of algorithmic calculation to translate between formats as opposed to the use of perceived analog magnitudes.

Translating between codes

Evidence for separate internal codes corresponding to different external representations (non-symbolic representations of quantity such as dot arrays, and number symbols such as Arabic numerals and number words) comes from cross-format magnitude comparison tasks, in which the two stimuli to be compared are presented in two different notations. If symbolic number stimuli do not tap into the same magnitude representation that supports non-symbolic number processing, this separation can be seen in the cognitive “cost” of translating between representations. The finding that cross-format magnitude comparisons are more difficult (higher RTs and ERs) than within-format comparisons has been interpreted as evidence that different stimulus formats draw on different underlying magnitude representations that require “translation” into compatible codes before they can be compared (Lyons, Ansari, & Beilock, 2012; Marinova, Sasanguie, & Reynvoet, 2018). This incompatibility is thought to result from a process of “symbolic estrangement”: although symbolic and non-symbolic whole number stimuli may initially access a common sense of magnitude, extensive exposure to symbolic numbers leads them to be processed separately, at times possibly without much recourse to any sense of magnitude.

To date, only one study has examined the cost of translating between symbolic fractions and non-symbolic ratios. Binzak et al. (2019) compared within- and between-format magnitude comparisons for symbolic and non-symbolic ratios in adults. They found that cross-format comparisons were no more difficult than within-format comparisons for symbolic fractions and non-symbolic ratios (instantiated as line ratios). Similarly, reaction times for cross-format magnitude were not significantly different from symbolic-symbolic comparisons: indeed, cross-format comparisons trended toward being faster than purely symbolic comparisons, both of which were slower than purely non-symbolic comparisons. Notably, these results run counter to the predictions of a calculation-based account, in which participants first calculate a symbolic number value for the non-symbolic stimuli and then compare these numerical values. If participants are performing such calculations, then the non-symbolic condition would be expected to have the longest reaction times. However, it is not clear whether this pattern results from a lifetime of training with symbolic fractions or whether it would extend to children before extensive formal instruction with symbolic fractions.

The present study

We investigated performance on symbolic, non-symbolic, and mixed fraction magnitude judgements in two groups of school-aged children: second graders (7-8 years old), who had little to no formal instruction with symbolic rational numbers, and fifth graders (10-11 years old) who have received two years of formal instruction with symbolic rational numbers. We predicted that both age groups would be able to discriminate non-symbolic fractions well, as we posit this ability to be early emerging and non-symbolic in nature. In contrast, we predicted that 5th graders would show more competence with symbolic fractions than 2nd graders would, as symbols systems require instruction for acquisition of competence. If we observe distance effects for non-

symbolic fraction stimuli (line ratios), particularly among second graders, we can interpret this as evidence that these stimuli can elicit an analog internal magnitude representation, even prior to extensive formal instruction with symbolic fractions. We also compared the mixed-format condition to the two within-format conditions for evidence of translation effects. We expected that there would be developmental differences in the cost of translation, and that these differences be largely due to symbolic fraction instruction. We therefore predicted that fifth graders would be faster and more accurate than second graders in the mixed-format and symbolic fraction comparison conditions and would show stronger evidence for a holistic magnitude representation for symbolic fractions.

Methods

Participants

We advertised in local elementary schools to recruit students in second and fifth grades. Participants were drawn from a mix of urban and suburban public school districts, which used a variety of different math curricula. In the U.S., second grade corresponds to 3 years of formal schooling (usually 7-8 years old) and fifth grade to 6 years of formal schooling (usually 10-11 years old). The state in which all the districts are based has adopted Common Core standards, in which formal instruction with symbolic fractions does not begin until 4th grade. In total, 194 second grade students (mean age: 8.03 years; 37.62% female; 80% White, 2% Black, 6% Asian; 9% Hispanic or Latino) and 145 fifth grade students (mean age: 10.95 years; 42% female; 82% White, 12% more than one race or not reported; 4% Black; 15% Hispanic or Latino) participated as part of an ongoing longitudinal study. There was no significant difference between groups (grades) in terms of gender or ethnic background. Participation criteria included fluent English production and comprehension and lack of known developmental or learning atypicalities (by

self-report). As compensation, participants received \$10/hour and a small toy. All recruitment and experimental procedures were approved by the UW-Madison Health Sciences IRB (2016-0665).

Procedure

Participants visited the lab for two sessions, separated by 1-3 weeks on average. During the first visit, we carried out a broad battery of psychometric testing (IQ, working memory, etc.). During the second visit, participants completed additional psychometric testing as well as the experimental tasks described below. Visit length varied from 90-120 minutes, depending on participant factors. Breaks were given in each visit.

Fraction Knowledge Assessment (FKA). The FKA is a pencil-and-paper assessment of both procedural and conceptual fraction knowledge. We constructed this instrument using items taken from key national and international assessments, including the National Assessment of Educational Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS), from *state-based math proficiency tests*, and from instruments developed by psychology and math-education researchers (Carpenter, 1981; Hallett, Nunes, Bryant, & Thorpe, 2012). A separate FKA was constructed for each grade, featuring grade-appropriate items (i.e., there was a second grade version of the FKA and a fifth grade version of the FKA). Each version of the FKA contained 41 items, and each had strong internal consistency (second grade FKA Cronbach's $\alpha = .90$; fifth grade FKA Cronbach's $\alpha = .92$).

Experimental Task. The experimental task was the cross-notation magnitude comparison paradigm developed by Binzak et al. (2019). On each trial, participants were presented with two fractions, instantiated as either common symbolic fractions or non-symbolic fractions (line ratios). They were instructed to choose the larger item (see Figure 1) by pressing the button on the corresponding side of the screen (“j” for the left, “k” for the right) on a standard US QWERTY keyboard using the index and middle fingers of their right hand. There were three types of trials: symbolic (fraction-fraction), non-symbolic (line-line), or mixed-notation (fraction-line). Each participant completed 108 trials, evenly distributed across notations and distances. We presented experimental stimuli on a PC using E-prime 2.0.8.90a (Psychology Software Tools, Shapsburg, PA); all stimuli were presented in light gray on a black background.

Before participants began the task, they were shown a short instructional presentation via Powerpoint illustrating the concept of relative magnitudes with cartoon characters and performed several practice trials by pointing or making a vocal response, receiving feedback from the research assistant. This presentation included 5 practice trials: 2 symbolic, 2 non-symbolic, and 1 mixed. The distances for these trials were in the far and medium bins. Altogether, participants received no more than 15 minutes of total instruction on the fraction comparison task, including symbolic and non-symbolic fraction explanation and practice.

Insert Figure 1 about Here

Symbolic fraction stimuli. Symbolic fraction stimuli in the study included all possible irreducible fractions composed from single-digit numerators and denominators. From all possible paired comparisons, a sample of 36 pairs was selected to balance presentation frequency, numerical distance, and component congruency, since component congruency has been found to affect fraction magnitude judgements (e.g., Meert, Gregoire, & Noel, 2010). A full listing of symbolic stimuli used in the study is available in Supplemental Table 1. See Binzak et al., (2019) for further details.

To avoid confounds with other (non-integer) forms of numerical processing, such as multi-digit processes, we selected only irreducible, single-digit fractions for our stimulus set. However, after such restriction, the resulting stimuli are susceptible to a numerator comparison strategy. That is, in pairs of items, the distance between numerator magnitudes and the distance between holistic fraction magnitudes is correlated ($r = .77, p < .001$), so many trials could be successfully completed by comparing only the numerators. In our analysis, we assessed the possible impact of this correlation by introducing numerator distance as a predictor in a series of linear mixed models. A second approach for assessing the extent to which participants resorted to numerator comparisons would involve examining responses to the items for which a numerator strategy would be counterproductive. However, the number of items in our stimulus set that fit these criteria was prohibitively low for most analytic approaches (6/36 items), so we were not able to use this second method. In future work, this issue could be addressed by strategically increasing the number of items to include more pairs in which a numerator strategy would not be effective. Similarly, these stimuli were designed before the publication of work highlighting the use of some other strategies (such as gap comparisons—see alternative strategy

analyses, below) (Gómez & Dartnell, 2019), but future research could attempt to similarly balance stimulus sets with regard to these alternatives.

Non-symbolic fraction stimuli. Non-symbolic fractions (line ratios) were composed of two vertical lines representing a part-part ratio of the line on the left relative to the line on the right. The non-symbolic fractions were based on the single digit irreducible fractions used for the symbolic fraction stimuli (i.e., the same set of holistic magnitudes), but multiplied by various factors to alter absolute size. The absolute size of the lines was varied independently of the ratio size. Non-symbolic stimuli lengths were jittered by adding variable numbers of pixels less than the amount necessary to alter the ratio to the next closest ratio (for example, a non-symbolic instantiation of $\frac{3}{4}$ might be represented by a line 78 pixels long and a line 106 pixels long).

In all three conditions, the distances between each pair of fractions varied from a minimum of .047 to a maximum of .75. In some contexts (such as the summary statistics in Tables 1 and 2), distances were binned as follows: Near (.047-.233); Medium (.278-.446); Far (.514-.75). However, all statistical analyses for distance effects used continuous distance; the distance bins in Tables 1 & 2 are presented solely to facilitate comparison with previous studies (e.g. Lyons, Ansari, & Beilock, 2012). For a list of the non-symbolic fraction pairs used, please refer to Supplemental Table 1.

Measures

Error Rates. Each trial was scored as correct or incorrect. Error rates were calculated as the percent of responses scored incorrect for each subject, distance, and notation condition.

Participants with an error rate greater than 50% across all conditions (four second graders and one fifth grader, constituting 2% and <1% of participants, respectively) were excluded from analysis.

Reaction Times. The time from stimulus onset to response was collected for each trial. Mean reaction times were calculated for each subject, distance, and notation condition using the correct trials only. Trials in which the RT was less than 600ms were excluded from further analysis (based on a natural cut-point in the histogram for response times), as were trials in which the RT was more than 3 standard deviations from the participant's mean RT. Using these criteria, 2588 trials were excluded (less than 10% of the total of 33,486 trials). The median number of trials excluded per participant was 7 (~7%) and the maximum trials removed for an individual participant was 28 (~26%).

Results

Fraction Knowledge Assessment

Participants varied on their performance on a paper-and-pencil test of fraction knowledge ($Median_{2nd} = 64.5\%$; $Median_{5th} = 68.3\%$). The second grade version of the test included four fraction arithmetic problems, and second graders were near floor on all of these items. The arithmetic item on which second graders did best yielded only 6% correct responses, and the other items were at 2%, 3% and 0%. As a group, second graders also struggled to order three fractions by magnitude (10% correct). In addition, they had difficulties applying fraction magnitudes to objects, such as shading a designated fraction of an object (Shade $\frac{1}{2}$: 50% correct)

or drawing a line length in relation to another line (“draw a line $\frac{1}{4}$ as long as this line” or “draw a line 3 times as long as this line”: 38% and 44% correct, respectively). The second graders’ performance on these items demonstrate that they have not yet mastered fraction concepts and procedures, consistent with them not yet having received much formal instruction with symbolic fractions. The fifth grade fraction knowledge assessment included many fraction arithmetic problems, and the fifth graders displayed a range of performance on these problems (fraction arithmetic median score = 58.8%) that did not suggest floor or ceiling effects.

Magnitude Comparison Task

Tables 1 and 2 display RTs and ERs in each notation for second graders and fifth graders, respectively. As displayed in these tables, both groups performed above chance for all formats and distance bins (highest condition error rate: $M_{second} = 16\%$, $SD = 36.7\%$; $M_{fifth} = 11.1\%$, $SD = 31.4\%$; error rate collapsed across all notation conditions: $M_{second} = 13.4\%$, $SD = 34.1\%$; $M_{fifth} = 9.7\%$, $SD = 29.6\%$).

 Insert Tables 1 and 2 about Here

 Insert Figures 2 and 3 about Here

Reaction Times. Figure 4 shows RTs across pair distances for each notation, by grade. We fit a linear mixed-effects regression model using reaction time as the dependent variable. Predictor variables included the between-participant factor grade (second or fifth) and the within-participant factors distance and notation (non-symbolic, cross-format, or symbolic), and all interactions. In addition, a random intercept was included for participants, and random slopes

were modeled for the participant-level variables distance, notation, and interactions between levels of distance and notation. The cross-format notation condition was used as the reference category for notation effects. The results are shown in Table 3 (Figure 4 and the regression analysis are based on correct trials only).

 Insert Figure 4 about Here

The fixed effects portion of the regression model is given in Equation 1:

$$\widehat{RT} = \hat{\beta}_0 + \hat{\beta}_{Distance}(distance) + \hat{\beta}_{fifth}(fifth) + \hat{\beta}_{disxtanceXfifth}(distance \times fifth) + \hat{\beta}_{Symbolic}(symbolic) + \hat{\beta}_{nonsymbolic}(nonsymbolic) + \hat{\beta}_{symbolicXdistance}(symbolic \times distance) + \hat{\beta}_{nonsymbolicXdistance}(nonsymbolic \times Distance) + \hat{\beta}_{FifthXsymbolicXDistance}(Fifth \times symbolic \times Distance) + \hat{\beta}_{FifthXnonsymbolicXdistance}(Fifth \times nonsymbolic \times Distance)$$

where “fifth” refers to a dummy variable for grade which takes the value 1 for fifth graders and 0 for second graders.

This analysis revealed a significant main effect of holistic distance such that participants were faster for far comparisons than near comparisons ($\beta = -499.5, p < .001$). We also found a main effect of grade such that fifth graders were faster than second graders ($\beta = -241.45, p < .001$) and a main effect of notation such that participants were faster on the non-symbolic condition than the mixed-format condition ($\beta = -272.23, p < .001$), but slower for the symbolic condition than the mixed-format condition ($\beta = +172.54, p < .001$). By computing the model with non-symbolic as the reference category, we obtain the following significant contrast between symbolic and non-symbolic: ($\beta = 448.03, p < .001$). The only significant interaction was the three-way interaction between grade, distance, and symbolic notation, indicating that fifth graders’ distance slope for reaction time was steepest for the symbolic notation condition and steeper than second graders’ RT slope for this notation condition ($\beta = -222.3, p < .001$). The most

noteworthy result was that 2nd graders did indeed exhibit distance effects with symbolic fractions, despite not having experience with curricular instruction on the topic.

 Insert Table 3 about Here

Error Rates. Figure 5 shows ERs across pair distances for each notation by grade. We fit a logistic mixed-effects regression model using the probability of an incorrect response as the dependent variable. Predictor variables included the between-participant factor grade (fifth or second) and the within-participant factors distance and notation (non-symbolic, cross-format, or symbolic), and all interactions. In addition, a random intercept was included for participants, and random slopes were modeled for the participant-level variable distance. (A maximally specified model failed to converge). The cross-format notation condition was used as the reference category for notation effects. The results are shown in Table 4.

 Insert Figure 5 about Here

The fixed effects portion of the regression model is given in Equation 2:

$$ERR = \hat{\beta}_0 + \hat{\beta}_{Distance}(distance) + \hat{\beta}_{fifth}(fifth) + \hat{\beta}_{disxtanceXfifth}(distance \times fifth) + \hat{\beta}_{Symbolic}(symbolic) + \hat{\beta}_{nonsymbolic}(nonsymbolic) + \hat{\beta}_{symbolicXdistance}(symbolic \times distance) + \hat{\beta}_{nonsymbolicXdistance}(nonsymbolic \times Distance) + \hat{\beta}_{FifthXsymbolicXDistance}(Fifth \times symbolic \times Distance) + \hat{\beta}_{FifthXnonsymbolicXdistance}(Fifth \times nonsymbolic \times Distance)$$

where “fifth” refers to a dummy variable for grade which takes the value 1 for fifth graders and 0 for second graders.

For the reference grade category (second graders), this analysis revealed a main effect of distance such that ERs were lower for farther distances ($\beta = -5.41, p < .001$) and a main effect of notation such that participants had lower error rates (i.e., were more accurate) in the non-

symbolic and symbolic conditions than they were in the mixed (non-symbolic vs. cross-format $\beta = -.43, p < .001$; symbolic vs. cross-format $\beta = -.47, p < .001$). By re-computing the model with the non-symbolic condition as the reference category for notation, we observe that error rates were slightly higher in the symbolic than the non-symbolic condition ($\beta = 0.092, p < .001$). However, a significant interaction revealed that the effect of distance was smaller for symbolic trials than either non-symbolic or mixed format trials (which did not differ from each other) ($\beta = 2.05, p < .001$). As shown in Figure 3a, the combination of these effects results in a variable notation effect by distance. Furthermore, as interpreted above, these effects are in relation to the baseline (second grade, mixed format) condition. The net effects of notation and distance were somewhat different for fifth graders, as discussed below.

The differences between second and fifth graders manifested in two-and three-way interactions with grade. There was no main effect of grade collapsed across format and distance, nor was there a difference between second and fifth graders in error rate depending on notation alone. However, the overall distance effect (across notations) was stronger for fifth graders than second graders ($\beta = 1.90, p < .001$). Furthermore, three-way interactions demonstrated that for fifth graders, the difference (from second graders) in the strength of the distance effect also depended on notation. The distance effect was weaker for non-symbolic trials in fifth graders than in second graders ($\beta = 1.55, p < .001$) but stronger for symbolic trials ($\beta = -3.02, p < .001$). In fifth graders, the distance effect was significantly weaker in the non-symbolic condition, but significantly stronger in the symbolic condition, than the distance effect in the mixed-notation condition (see also Figure 3 and the descriptive statistics in Table 2). Post-hoc testing confirms that error rates differ significantly between second and fifth graders for the symbolic condition ($t = 13.41, p < .001$), but not for the non-symbolic condition ($t = 1.38, p = .17$).

Insert Table 4 about Here

Reaction time translation costs. There were significant effects of notation type for both grades in both reaction time. Performance was in non-symbolic notation was the fastest, followed by the cross-format condition, and the symbolic notation condition was the slowest condition. These reaction time findings contradict a model in which non-symbolic fractions are first converted to numerical equivalents and then compared, and they are also incompatible with a model in which symbolic and non-symbolic fractions cannot be converted to a common underlying code. The rank order of notation reaction times suggests instead that the non-symbolic fractions stimuli require the least internal processing, suggesting that symbolic fractions may be converted to an analog magnitude representation before being compared.

Speed-Accuracy Trade-offs. Several patterns in our data argue against a speed-accuracy trade-off. If a speed-accuracy trade-off occurred, we would expect to see an inverse relation between RT and error rates across conditions: that is, conditions with faster RTs would have higher error rates. As seen in Tables 1 and 2 and Figures 2 and 3, this was not the case. Fifth graders were fastest and most accurate for the symbolic notation, suggesting that this condition was consistently easiest. The mixed notation was slower than the non-symbolic notation, but ERs did not differ between these two conditions. For second graders there was a positive relation between RT and ER across conditions, suggesting that ER and RT differences consistently signal differences in difficulty. Neither group performed fastest in their least accurate notation. Taken together, these data argue against the possibility that our notation effects were contaminated by speed-accuracy trade-offs.

Alternative Strategy Analyses.

Gap strategy analysis. The above results suggest that children as young as second grade may be comparing symbolic and non-symbolic fractions using an analog magnitude code corresponding to the holistic value of the fraction stimuli. However, to increase confidence that this was indeed the case, we also must account for the possibility that participants use component-based strategies that might explain for our findings. One alternative strategy that participants might have used to perform the task is a *gap comparison* strategy (Fazio et al., 2016; Gómez & Dartnell, 2019). If participants are using such a strategy, then they would be comparing the numerator-denominator gaps within each fraction to each other, rather than comparing the magnitudes. Here, we use “gap” to refer to the difference between numerator and denominator in one fraction (e.g. the “gap” in the fraction $2/7$ would be $7-2 = 5$). Gap is often correlated with fraction size such that smaller gaps correspond to larger fraction magnitudes simply because gap necessarily decreases as the numerator increases such that when the gap is zero, the fraction is equivalent to $1/1$. We account for this strategy in our analysis by including the difference between gap sizes of each fraction in a comparison pair as an independent variable.

To address the possibility that participants were relying on a gap comparison strategy, we conducted additional linear mixed-model analyses for the symbolic notation condition that added gap difference to the models predicting reaction time and error rate (see Supplemental Tables 1 & 2). We found that holistic distance between fractions remained a significant predictor for both RT and error rate (RT distance $\beta = -609.96$, $p < .001$; error rate distance $\beta = -1.68$; $p < .001$) even when gap difference was included in the model. This was true both for fifth graders and for second graders. In fact, for fifth graders, gap was not a significant predictor of RT or error rate.

Thus our analyses suggest a) that fifth graders were not generally using gap strategies in lieu of holistic distance and b) that holistic strategies were likely at least as prevalent as gap strategies among second graders.

Numerator Strategy Analysis. Another strategy that participants could use involves comparing only specific components across fractions (such as numerators only) rather than the holistic magnitudes of the fractions (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Zhou & Ni, 2005). To address the possibility that participants were relying on a such a componential numerator comparison strategy, we conducted additional linear mixed-model analyses for the symbolic notation condition that added numerator distance to the models predicting reaction time and error rate. If participants were comparing only numerators, then we would expect to see a strong effect of numerator distance, to the exclusion of holistic distance. both second and fifth graders, with increased distance predicting lower RTs and error rates (see Supplemental Tables 3 & 4 for details of this analysis).

To further unpack the effects of the numerator strategy in each grade, we fit separate series of linear mixed models predicting RT for each grade that varied on whether we included holistic or numerator distance. One variant included holistic distance and excluded numerator distance, a second included numerator and excluded holistic, and a third added both numerator and holistic distances. Each variant contained the same original control variables described above. These models can be found in Tables 5 and 6 of the Supplemental Materials. Based on the relative sizes of Aikake Information Criterion (AIC) for these models, numerator distance seems to play a greater role than holistic distance in explaining second graders' RT-based distance effects. That said, there remain effects for holistic distance, consistent with second graders using a combination of strategies. For fifth graders, although numerator distance may contribute to their

reaction time distance effects, holistic distance plays a dominant role. This likely reflects the impact of direction instructional experiences with fractions in fifth graders compared to second graders.

We also examined accuracy for those items in which comparing only the numerators would lead to an incorrect response. Accuracy was well above chance for both groups ($t_{second_grade} = 11.96, p < .001$; $t_{fifth_grade} = 13.64, p < .001$). Error rates for each of these items by grade can be found in the Supplemental Materials, Tables 7& 8. This suggests that although they may rely on a numerator comparison strategy when it is beneficial, children are also able to deploy additional strategies when this strategy fails.

Benchmark strategy analysis. A third possible strategy participants might use rather than directly comparing fraction magnitudes is a “ $\frac{1}{2}$ benchmarking” strategy. In this approach, the fraction $\frac{1}{2}$ is used as benchmark against which other fractions are compared. Thus the participant needs only to determine whether each fraction is greater or less than $\frac{1}{2}$ when fractions are on opposite sides of one half. If one of the fractions is greater than $\frac{1}{2}$ and the other is less, then the greater fraction can be determined logically without consideration of its unique magnitude. If participants were using such a benchmarking strategy, then we would expect them to be slower and less accurate when the two fractions were both less than or both greater than $\frac{1}{2}$. Conversely, we would expect them to be faster and more accurate on trials where one fraction is greater than and one less than $\frac{1}{2}$. We found no mean difference in reaction time between trials (correct response only) with both fractions greater or less than $\frac{1}{2}$ and trials where the two fractions

“spanned” $\frac{1}{2}$ ($t = 1.17, p = .24$) when controlling for distance (Supplemental Tables 5, 6, & 7)¹.

In contrast, participants were more accurate on trials that spanned $\frac{1}{2}$, even when distances were controlled, but the mean accuracies in the *both greater* and *both less* trials were significantly above chance. These results suggest that when a benchmark strategy was not available, participants were still able to successfully compare fraction magnitudes. Furthermore, there was a robust distance effect in both greater/both less trials (in which the benchmarking strategy could not be used) (error rate distance x both greater/less $\beta = -4.28; p < .01$). For full details of this analysis, see Supplemental Material Tables 9, 10, & 11.

Within-experiment learning. Finally, to rule out the possibility that within-task learning took place, we examined the accuracy rates for each notation condition by block (see Tables 5 & 6 below). We also present the mean accuracy for each group over the first 10 trials of the first block to establish a baseline accuracy rate. We then performed a 3 (block) x 3 (notation) x 2 (grade) fully crossed repeated-measures ANOVA for accuracy. There was no main effect of block (i.e. time) on accuracy for either grade ($F(2, 17,928) = 1.25, p = .287$). Both groups started with relatively high mean accuracy and did not improve significantly over the course of the task.

 Insert Tables 5 and 6 about Here

¹ Trials that “span” $\frac{1}{2}$ have a greater range of inter-fraction distances than trials in which both fractions are either greater or less than $\frac{1}{2}$. To test for a benchmark effect, we compared only “span” trials whose distances were in the same range as the “both greater/less” trials (.05 -.32). A list of which fractions were used for this analysis, and details of the analysis can be found in the supplement.

Discussion

We found that school-age children can access an analog magnitude representation for ratios and that this analog representation is available for both non-symbolic and symbolic notation formats, although evidence suggested that fifth graders were more likely than second graders to rely on representations of holistic magnitude when dealing with symbolic fractions. Contrary to the interpretation that a pure symbol-manipulation strategy was used, we found a significant distance effect in the symbolic (fraction-fraction) condition for both grades, suggesting that fifth graders are accessing a sense of magnitude when comparing symbolic fractions. Non-symbolic fraction magnitude comparison was rapid and accurate in both age groups, consistent with a model that posits early-developing perceptual access to fraction magnitudes. In addition, these observed distance effects were measured with greater precision than in previous studies of fractions in children. (see e.g., Duffy, Huttenlocher, & Levine, 2005; McCrink & Wynn). Most notably, both second and fifth graders effectively completed cross-format comparisons and exhibited holistic distance effects when doing so. Because non-symbolic and symbolic processes demonstrate similar signatures and allow fast cross-format comparisons, our findings are consistent with the proposition that a) the two formats share the same analog representation in some instances and b) the frequency with which they share the same code may increase with education and maturation. Ultimately, this leaves open the possibility that processing of fractions symbols – when processed holistically – may recruit the RPS. Thus, the RPS has the potential to serve as an intuitive ground for symbolic forms of rational numbers, such as symbolic fractions and decimals given adequate educational experience.

Evidence for holistic distance representation

We found distance effects for both second graders and fifth graders such that RTs and error rates were inversely related to the holistic numerical distance between stimuli. This effect was present in all notation conditions. We fully expected to find holistic distance effects for non-symbolic fractions for both age cohorts given prior findings of non-symbolic fraction processing capabilities in infants and non-human primates (McCrink & Wynn, 2007; Vallentin & Nieder, 2008, 2010; Woodruff & Premack, 1981). However, we were somewhat surprised to find consistent evidence of for holistic processing of symbolic fractions among second graders, given that prior research has documented a pervasive susceptibility among children and adults to use componential processing (Meert, Gregoire, & Noel, 2009; Zhou & Ni, 2005). Here we note that as recently as a decade ago, Kallai and Tzelgov (2009) doubted that even human adults processed holistic magnitudes, concluding that, from an analog perspective, symbolic fractions were represented as undifferentiated generalized magnitudes smaller than one. Just as work in the intervening period showed that adults could indeed generate analog representations of holistic fraction magnitudes (Matthews & Lewis, 2016; Schneider & Siegler, 2010), our work extends these findings to fifth grade and even to some second grade students.

Furthermore, our findings are inconsistent with a model in which symbolic and non-symbolic fraction stimuli are necessarily represented by incompatible codes in the mind. Both second and fifth graders were able to rapidly and accurately compare fraction magnitudes across notations (i.e., the mixed condition). Our reaction time findings suggest that children do not need to convert non-symbolic fractions to a symbolic representation to compare their magnitudes (compare with Binzak et al. 2019 findings for adults). If anything, our results suggest the opposite. The “cost” of translating between formats increased with the number of symbolic

fractions in the comparison. This pattern of results fits well with an additive-factors model, in which symbolic number stimuli are compared in two discrete, non-interacting stages (first conversion into magnitude values and then comparison of magnitude values) (Dehaene, 1996; Sternberg, 1969). Moreover, the similarity of the RTs and ERs for both grades in the non-symbolic condition suggests a possible preservation of non-symbolic magnitude representation across this range of developmental time.

In many ways, the developmental patterns observed here for fractions mirror previous findings for symbolic and non-symbolic whole numbers (Sekuler & Mierkiewicz, 1977). Fifth graders were faster than second graders overall, more accurate than second graders at small distances, and showed the greatest accuracy advantage in the conditions involving symbolic fractions. We did observe some differences in the slope of the distance effect across grades and notation conditions that do not correspond to findings for whole numbers. Specifically, the pattern of distance effects by notation differed between second and fifth graders, leading to a difference in the rank order error rate by notation between grades (see Figure 5 and Table 4). For whole numbers, distance effect slopes are known to change during the lifespan, becoming smaller (or “shallower”) with age (Duncan & McFarland, 1980; Holloway & Ansari, 2008; Sekuler & Mierkiewicz, 1977). These smaller distance effects in older children and adults are interpreted as evidence of a more precise internal representation of magnitude in older children and adults. Formal instruction in mathematics is suspected to play a role in this increased precision (Lyons, Bugden, Zheng, De Jesus, & Ansari, 2018; Matejko & Ansari, 2016; Piazza, Pica, Izard, Spelke, & Dehaene, 2013). In contrast, we observed a greater (“steeper”) distance effect in fifth graders than second graders, particularly for symbolic fractions, but also greater accuracy and lower reaction times. One possible interpretation is that fifth graders’

representations of symbolic fractions may be more tightly coupled to magnitude than second graders', but may still be more logarithmic than linear. The pattern of developmental changes in fraction magnitude distance slopes is likely to be complex, and we anticipate that a clearer picture of will emerge as we follow these students longitudinally and collect data from additional timepoints.

Evidence of other (not necessarily holistic) representations with symbolic fractions

To be clear, we are not claiming that holistic representation is the whole story, especially with symbolic fractions. Even highly accurate undergraduates appear to use a combination of holistic magnitude and componential processing in symbolic fraction magnitude comparison tasks (Matthews & Lewis, 2016). Likewise, although evidence suggests an increase in the use of holistic magnitude over componential strategies with development and experience, there is also evidence of considerable heterogeneity within age/grade groups (Braithwaite & Siegler, 2017; Meert, Gregoire, & Noel, 2009; Meert, Grégoire, & Noël, 2010; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). Thus, the important question is not whether children exclusively use holistic strategies with symbolic fractions, avoiding all others. It is whether they are capable of generating analog representations for symbolic fractions at all and how these representations might compare to those for fraction represented in non-symbolic formats.

Other than using an internal magnitude code for comparison, we are aware of two classes of possible methods that participants could have used to complete the task. The first is effortful application of algorithms or arithmetic procedures, including conversion to decimal values or determination of least common denominators to facilitate comparison. These approaches are costly in terms of both time and cognitive resources (such as working memory). If the participants were using such algorithmic approaches, we would almost certainly have observed

greater response times, as this technique would require at least six steps: focusing on all four components and minimally engaging in two computations (e.g., division or cross-multiplication). Our mean reaction times are in the range of 1.5- 2 seconds. Our estimate of computation time has a lower bound, given that rapid subvocal counting takes about 240 ms per item (Whalen, Gallistel & Gelman, 1999). In order to complete 6 computations in the observed time, second-grade children would have to complete the 6 algorithmic steps as fast as adults can count! Our observed reaction times are closer to the results of a classic study of *whole number* magnitude comparison in children, which gives a range of about 1.7-1.4 seconds for first graders and about 1.2-.8 seconds for fourth graders (Sekuler & Mierkiewicz, 1977). Moreover, a recent study of fraction and decimal magnitude comparisons in adults supports the notion that although conversion to decimal forms may take place in the absence of time constraints ($> 5s$), symbolic fractions can be directly used to access a magnitude representation without being converted to decimal form when time for conversion is not available ($< 5s$). (Binzak & Hubbard, 2020) .

Furthermore, even with unlimited time, our participants would likely not have been able to use such strategies. On an untimed test of symbolic fraction knowledge, our second grade participants demonstrated a lack of fraction arithmetic knowledge and procedures, clearly ruling out the possibility that they were employing such knowledge in the magnitude comparison task. Even though fifth graders' fraction arithmetic scores were better than those of second graders, they too were far from mastery. Thus, it is highly unlikely that our participants used algorithmic strategies to complete comparisons, especially with the time constraints of the experiment.

The second class of non-magnitude methods for fraction comparison consists of heuristics, such as a gap comparison strategy, a componential isolation strategy, or a benchmarking strategy. As stated above, we fully expect that participants use some combination

of componential and holistic magnitude to make their judgements; our concern is whether they use componential (or other) methods *to the exclusion* of holistic magnitude, and whether reliance on holistic representation changes with age and experience. Our supplemental analyses indicated that, although there is evidence that these heuristics may influence symbolic judgements, there is also clear evidence for the influence of holistic magnitude. In fact, with only one exception – the use of numerators among second graders – holistic magnitude accounts for more variance in response times than any of the tested heuristics.

Importantly, we observed a difference in the explanatory power of heuristic approaches between second graders and fifth graders. For fifth graders, holistic distance explained more variance than any of the tested heuristics. These results suggest that with age and experience, children shift their use of strategies so that they eventually rely more on efficient and accurate strategies, as observed in other areas of mathematical development. This pattern is predicted by Siegler's overlapping waves theory (Siegler, 1996), and we look forward to examining strategy use changes over development in more detail as we follow these children longitudinally.

Conclusion

Although replication and further research are needed before these findings can be translated into practice and policy recommendations, our findings stand to inform future pedagogical theory and investigations. Students performing fraction arithmetic often seem to be engaging in procedures of symbol manipulation divorced from conceptual understanding involving fraction magnitudes (Kerslake, 1986; Ni & Zhou, 2010; Richland, Stigler, & Holyoak, 2012; Siegler, Thompson, & Schneider, 2011; Stafylidou & Vosniadou, 2004). If it were the case that children *could not* intuitively represent fraction magnitudes, or that processing symbolic fractions necessarily required some complex operations to access their magnitudes, then it might

make sense a) to delay instruction with symbolic fractions until students have greater cognitive capacities to support these complex operations and b) to rely on instructional methods that eschew initial emphasis on magnitude when introducing symbolic fractions. However, the current results contradict the notion that children have no holistic intuitions for fraction magnitudes. On average, the elementary school children in our study were indeed able to quickly access and compare fraction magnitudes both in symbolic and non-symbolic format. This is a toolset we should seek to leverage.

Of course, much research needs to be done to develop optimal ways to deploy this tool, as we are not suggesting that holistic processing is all that is needed for a robust understanding of fractions. Clearly, this ability did not automatically translate into facility with aspects of fraction reasoning and fractions arithmetic problems as measured by our fraction knowledge assessment. It seems to be the case that although our participants are able to represent the magnitudes of symbolic fractions for speeded comparison tasks, they do not consistently use magnitude representations when solving fraction reasoning or arithmetic problems. Thus, it is not that students *cannot* represent the magnitude meaning of fractions, but that they may not use this meaning when solving fraction problems; it lies inert. The question of how to solve this inert knowledge problem is key for leveraging the intuitive abilities detailed in this research. Instructional and curricular approaches that stress conceptual understanding of fractions, including fraction magnitude—and that link these to fraction arithmetic procedures—have the potential to improve students' fraction performance.

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Figure 1. Magnitude comparison task stimuli. (a) Symbolic fraction condition (b) mixed condition (c) Non-symbolic fraction condition. Reproduced with permission from Binzak et al. (2019).

Figure 2. Mean reaction times for magnitude comparison by condition and grade (collapsed over all distances). Error bars represent the standard error of the mean.

Figure 3. Mean error rates for magnitude comparison by condition and grade (collapsed over all distances). Error bars represent the standard error of the mean.

Figure 4 Reaction times by distance for second graders (a) and fifth graders (b). Blue = symbolic, magenta = mixed format, red = non-symbolic trials. Shading indicates 95% confidence intervals. Note: For illustrative purposes, points shown represent group mean RTs for each distance and notation and OLS best-fitting lines; however, the mixed-model was calculated using individual (not group) values.

Figure 5. Logistic curves for error by distance for second graders (a) and fifth graders (b). Blue = symbolic, magenta = mixed format, red = non-symbolic trials. Shading indicates 95% confidence intervals. Plotted curves represent the results of the mixed-model analysis.

Table 1. Reaction times by notation, distance bin, and grade.

		<u>Near</u>		<u>Medium</u>		<u>Far</u>	
RTs		M	SD	M	SD	M	SD
Line - Line	second	1786.55	315.14	1630.50	284.62	1501.96	264.37
	fifth	1591.68	285.22	1408.49	246.56	1272.09	219.81
Line - Fraction	second	2043.57	367.01	1977.15	355.98	1804.67	327.39
	fifth	1785.55	336.07	1688.61	307.41	1492.69	258.93
Fraction - Fraction	second	2219.96	389.49	2166.82	392.21	1981.96	378.53
	fifth	1969.50	354.03	1819.15	340.31	1594.23	333.29

Table 2. Error rates by notation, distance bin, and grade.

		<u>Near</u>		<u>Medium</u>		<u>Far</u>	
		M	SD	M	SD	M	SD
Line - Line	second	0.18	0.13	0.08	0.13	0.05	0.12
	fifth	0.18	0.18	0.08	0.16	0.07	0.19
Line - Fraction	second	0.26	0.16	0.13	0.15	0.06	0.13
	fifth	0.21	0.16	0.08	0.14	0.05	0.13
Fraction - Fraction	second	0.21	0.13	0.16	0.16	0.11	0.17
	fifth	0.13	0.12	0.07	0.12	0.02	0.06

Table 3. Linear mixed model results for reaction time (milliseconds)

	β	SE	z	P< z
<i>Fixed effects</i>				
Intercept	2126.32	25.69	82.78	<.001
Distance (log-transformed)	-499.50	37.13	-13.45	<.001
Grade (fifth Grade)	-241.45	39.04	-6.18	<.001
Non-symbolic Notation Condition	-272.23	24.44	-11.14	<.001
Symbolic Notation Condition	172.54	25.67	6.72	<.001
Grade x Distance	-91.74	56.23	-1.63	0.103
Distance x Non-symbolic	-74.09	50.86	-1.46	0.145
Distance x Symbolic	22.01	53.20	0.41	0.679
Grade x Non-symbolic	69.13	37.18	1.86	0.063
Grade x Symbolic	42.02	38.70	1.09	0.278
Grade x Distance x Non-symbolic	2.53	77.57	0.03	0.974
Grade x Distance x Symbolic	-222.30	79.94	-2.78	0.005
<i>Random effects</i>				
Participant (intercept)	261.10	11.75		
Distance	70.30	43.84		
Cross-Format Notation Condition	143.70	12.51		
Symbolic Notation Condition	124.93	10.65		
Distance x Cross-Format	0.13	6.35		
Distance x Symbolic	115.82	53.56		
N(obs)	26,909			
N(participants)	306			
Model χ^2	3028.37, p<.001			
Model df	11			
Snijders/Bosker R-sq Level 1	.153			
Snijders/Bosker R-sq Level 2	.189			

Table 4. Logistic mixed model results for error rate

	B	SE	z	< z
<i>Fixed effects</i>				
Intercept	-0.35	0.08	-4.51	<.001
Distance (log-transformed)	-5.41	0.34	-15.84	<.001
Grade (fifth Grade)	-0.11	0.13	-0.85	0.40
Non-symbolic Notation Condition	-0.43	0.11	-3.89	<.001
Symbolic Notation Condition	-0.47	0.10	-4.54	<.001
Grade x Distance	-1.90	0.57	-3.35	<.001
Distance x Non-symbolic	-0.35	0.35	-1.02	0.31
Distance x Symbolic	2.05	0.31	6.69	<.001
Grade x Non-symbolic	0.07	0.17	0.39	0.70
Grade x Symbolic	0.17	0.18	0.96	0.34
Grade x Distance x Non-symbolic	1.55	0.54	2.85	<.001
Grade x Distance x Symbolic	-3.02	0.57	-5.30	<.001
<i>Random effects</i>				
Participant (intercept)	3.07	0.18		
Distance	0.29	0.06		
N(obs)	30,533			
N(participants)	306			
Model χ^2	756.2, p<.001			
Model df	11			
Snijders/Bosker R-sq Level 1	.043			
Snijders/Bosker R-sq Level 2	.041			

Table 5. Accuracy by block for second graders.

	First 10 trials	Run 1	Run 2	Run 3
Non-symbolic	0.86	0.88	0.89	0.84
Symbolic	0.79	0.75	0.74	0.77
Mixed	0.78	0.79	0.78	0.80

Table 6. Accuracy by block for fifth graders.

	First 10 trials	Run 1	Run 2	Run 3
Symbolic	0.86	0.87	0.90	0.93
Mixed	0.91	0.90	0.87	0.91
Non-symbolic	0.92	0.92	0.95	0.91