**Direct Estimation of Diagnostic Classification Model Attribute Mastery Profiles via a
Collapsed Gibbs Sampling Algorithm**

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Abstract

This paper proposes a novel collapsed Gibbs sampling algorithm that marginalizes model parameters and directly samples latent attribute mastery patterns in diagnostic classification models. This estimation method makes it possible to avoid boundary problems in the estimation of model item parameters by eliminating the need to estimate such parameters. A simulation study showed the collapsed Gibbs sampling algorithm can accurately recover the true attribute mastery status in various conditions. A second simulation showed the collapsed Gibbs sampling algorithm was computationally more efficient than another MCMC sampling algorithm, implemented by JAGS. In an analysis of real data, the collapsed Gibbs sampling algorithm indicated good classification agreement with results from a previous study.

Keywords: diagnostic classification models, Markov chain Monte Carlo method, Gibbs sampling, Bayesian inference

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1. Introduction

Diagnostic classification models (DCMs; e.g., Rupp et al., 2010) are psychometric models that classify examinees into latent attribute profiles where each latent attribute is a binary indicator of an examinee being a master of a skill or trait. DCMs are restricted latent class models (e.g., Rupp & Templin, 2008) which are also special cases of general finite mixture models (e.g., Bishop, 2006; McLachlan et al., 2019; McLachlan & Peel, 2000). In educational contexts, DCMs can reveal examinees' strengths and weaknesses and make it possible to personalize instruction.

DCMs have three parameter sets: one set for examinees (the latent attribute profiles), one set to describe the proportion of examinees with each attribute profile (the structural model parameters), and one set describing the probability of an examinee answering an item correctly conditional on their attribute profile (the measurement model or item parameters). Estimation of DCM item and structural model parameters can be difficult. As we show in our simulation study, estimates of item parameters can “stick” to their parameter boundaries, an issue referred to as the boundary problem (e.g., DeCarlo, 2011; Galindo-Garre & Vermunt, 2006; Maris, 1999). In the boundary problem, correct item response probability parameters are estimated at values of 0 or 1 even if the true correct item response probabilities are not 0 or 1. Similarly, structural model parameters, representing the proportion of examinees with a given attribute profile, have a similar problem, with estimates sometimes becoming 0 or 1.

The boundary problem can happen when small sample sizes are present (Philipp, Strobl, de la Torre, & Zeileis, 2018) and may lead to inaccurate estimates of person parameters. Further, estimates of structural parameters also can be unstable under small sample sizes, particularly for classes with small numbers of observations. Additional problems can occur with large sample sizes if the number of attributes and number of items are also large. The number of attribute profiles increases exponentially for each additional attribute measured in an analysis.

For example, Lee et al. (2011) employed 15 attributes to analyze the TIMMS 2007 fourth grade mathematics data. In addition, correlations among attributes can be high in real world settings (e.g., von Davier, 2008). These facts can cause empty attribute mastery profiles and estimation instability even with the sample sizes in the thousands.

Moreover, boundary problems can become problematic in estimation of individual attribute profiles. Estimation of attribute profiles is based on the joint likelihood of item responses conditional on attribute mastery. If some items have boundary problems, especially with estimates close to 0, calculation of the likelihood becomes 0 or, equally, the log likelihood approaches negative infinity. In such situations some maximum likelihood estimation-based software packages, such as the GDINA package (Ma, & de la Torre, 2020) and Mplus (Muthén & Muthén, 1998-2017) arbitrarily restrict parameter estimates to have a boundary close to (but not at) 0 or 1, without theoretical justification. Both point estimates and standard errors of parameters are affected by the boundary problem (e.g., Philipp, et al. 2018; Yamaguchi & Okada, 2020). As such, our primary motivation in proposing the CGS algorithm is to avoid the boundary problem in estimation of DCMS.

One solution to improve the stability of parameter estimates is the use of maximum-a-posteriori (MAP) estimates (DeCarlo, 2011; Galindo-Garre & Vermunt, 2006; Maris, 1999; Ma & Jiang, 2021), however, such parameters are point estimates leaving the posterior of item parameters to be estimated. In addition, numerous Markov chain Monte Carlo (MCMC) methods exist for DCMS (e.g., Chen et al., 2018; Chen et al., 2020; Culpepper, 2015, 2019a, 2019b; DeCarlo, 2012; Culpepper & Hudson, 2018; Chung, 2019; de la Torre & Douglas, 2004; Henson et al., 2009; Jiang & Carter, 2019; Li et al., 2016; Zhan et al., 2019; Yamaguchi & Templin, 2022) and they are employed to estimate item and structural model parameters and attribute profiles. However, these estimation methods still must estimate item and structural model parameters. Models that can marginalize across item and structural parameters hold the potential to offer greater estimation efficiency and limit the impact of estimation instability.

In this paper, we propose a new estimation method to avoid direct estimation of DCM item and structural model parameters while directly estimating examinee latent attribute profiles: Collapsed Gibbs Sampling (CGS; e.g., Blei & Jordan, 2006). CGS was developed for and used in machine learning methods such as latent Dirichlet allocation (e.g., Blei & Jordan, 2006; Porteous et al., 2008). In the CGS algorithm, sets of parameters are marginalized out of estimation, making it possible to avoid some problems related to the estimation of model parameters. The central idea of CGS is marginalizing parameters in the MCMC sampling procedure. In related research, Liu (1994) proposed a collapsed Gibbs sampling method for estimating missing data to marginalize unrelated parameters, not for individuals' latent traits. Culpepper (2019a) and Chen et al. (2020) marginalized out structural parameters in sampling step of attribute mastery patterns, but they did not marginalize across all model parameters. On the other hand, CGS methods marginalizing across all item and structural model parameters in DCMS have not yet been developed and evaluated and, thus, are the focus of this study.

The primary benefit of CGS is that the algorithm does not have to use computational time estimating item and structural model parameters because they are marginalized. In other words, we can estimate the probability a person has an attribute profile in a short time. Similar to other Bayesian methods, the CGS algorithm can incorporate previously estimated parameter values into prior distributions, unlike maximum likelihood algorithms. Furthermore, typical MCMC methods incorporate past estimation results as parameter prior distributions but then sample across the posterior distributions of these parameters in their MCMC iterations. The CGS algorithm, however, does not estimate such parameters, which may provide a viable option as an estimation method. Another potential benefit of the CGS algorithm is faster convergence of the MCMC chain because the dimensionality of the parameter space is reduced by collapsing across parameters (e.g., Liu, 1994). Moreover, autocorrelations of MCMC samples from previously implemented collapsed methods were lower than non-collapsed MCMC methods (e.g., Figure 1 in Liu, 1994). In addition, computer memory cost in MCMC samples sometimes tends to be high

when the number of parameters is large. Collapsed Gibbs sampling needs smaller amounts of memory because the number of parameters in CGS is much fewer than MCMC methods.

In the remainder of this manuscript, we develop a novel estimation method for DCMs that can directly estimate examinee latent attribute profiles using a CGS algorithm. We have two main objectives for this study: (1) to show that the CGS estimation algorithm can effectively classify examinees under what we believe to be realistic simulation conditions and (2) to examine the effectiveness of the CGS algorithm when compared to other estimation methods that provide direct estimates of examinee attribute profiles. In the next section, we introduce the mathematical formulation of DCMs along with the prior distributions used for the CGS algorithm. Following that, we outline our DCM CGS algorithm. In the fourth section, we evaluate attribute mastery recovery via a simulation study to compare a non-parametric DCM as a base-line method. In section five, we compare the newly developed CGS algorithm to a previously employed MCMC algorithm and a maximum-likelihood algorithm. Although the comparison among estimation methods is not the main purpose of this study, such comparisons can help to understand the benefits and weaknesses of the CGS algorithm. In section six, we apply our CGS algorithm to real data and compare it with other, non-collapsed MCMC sampling methods to show that the CGS algorithm can provide results of comparable quality when compared to other estimation methods. We conclude the article by discussing our CGS algorithm and indicating future directions.

Model and Prior Specification for Saturated DCMs

2.1. The mixture model formulation of saturated DCMs

DCMs can be expressed as finite mixture models (e.g., Templin & Henson, 2006; Yamaguchi, 2020; Yamaguchi & Okada, 2020, 2021; Yamaguchi & Templin, 2022), a formulation which we use for deriving the specifics of our algorithm as the class-based probabilities metric of the parameter set naturally extends to the CGS algorithm. There is a one-to-one mapping between the parameters in the DCM mixture-based formulation and more commonly used DCMs such as the log-linear cognitive diagnosis model (Henson et al., 2009),

generalized deterministic noisy “AND” gate model (de la Torre, 2011), and general diagnostic model (e.g., von Davier, 2008).

We begin our discussion of general DCMS by noting that the DCMS we consider are constrained versions of latent class models which are special cases of general finite mixture models (e.g., Bishop, 2006; McLachlan et al., 2019; McLachlan & Peel, 2000). We restrict our discussion to cases where items are binary. Let a random variable $X_{ei} \in \{0,1\}$ be an item response of examinee e ($e = 1, \dots, E$) for item i ($i = 1, \dots, I$) with its realization x_{ei} . In addition, we assume a membership indicator vector $\mathbf{z}_e = [z_{e1}, \dots, z_{eC}]^\top$ for a specific latent class c ($c = 1, \dots, C$) (e.g., Bishop, 2006) where only one element is 1, representing that examinee e is a member of class c , and the other elements are 0. For an examinee e , the general latent class model gives the complete data likelihood of a vector of item responses $\mathbf{x}_e = [x_{e1}, \dots, x_{eI}]^\top$ and $\mathbf{z}_e$:

$$\begin{aligned} P(\mathbf{X}_e = \mathbf{x}_e, \mathbf{z}_e | \boldsymbol{\theta}_i, \boldsymbol{\pi}) &= P(\mathbf{X}_e = \mathbf{x}_e | \boldsymbol{\theta}_i, \mathbf{z}_e) P(\mathbf{z}_e | \boldsymbol{\pi}) \\ &= \prod_{c=1}^C [\pi_c P(\mathbf{X}_e = \mathbf{x}_e | \boldsymbol{\theta}_i, z_{ec})]^{z_{ec}}, \\ &= \prod_{c=1}^C \left[\pi_c \prod_{i=1}^I \theta_{ic}^{x_{ei}} (1 - \theta_{ic})^{1-x_{ei}} \right]^{z_{ec}}, \end{aligned} \tag{1}$$

where θ_{ic} is the probability of a correct response to item i from an examinee in class c . In the general latent class model, each item has one item response probability per class, which we denote by the $C \times 1$ vector $\boldsymbol{\theta}_i$. The π_c is a population mixing rate for class c and $\boldsymbol{\pi}$ is the $C \times 1$ vector $[\pi_1, \dots, \pi_C]^\top$. In addition, $P(\mathbf{X}_e = \mathbf{x}_e | \boldsymbol{\theta}_i, z_{ec} = 1)$ is the (conditional) joint probability for a specific latent class c : $\prod_{i=1}^I \theta_{ic}^{x_{ei}} (1 - \theta_{ic})^{1-x_{ei}}$. The complete data likelihood provides the marginal probability of the general latent class model:

$$P(\mathbf{X}_e = \mathbf{x}_e | \boldsymbol{\theta}_i, \boldsymbol{\pi}) = \sum_{c=1}^C P(\mathbf{X}_e = \mathbf{x}_e | \boldsymbol{\theta}_i, \mathbf{z}_e) P(\mathbf{z}_e | \boldsymbol{\pi}), \tag{2}$$

$$= \sum_{c=1}^C P(\mathbf{X}_e = \mathbf{x}_e | \boldsymbol{\theta}_i, \boldsymbol{\pi}, \mathbf{z}_e) = \sum_{c=1}^C \pi_c \prod_{i=1}^I \theta_{ic}^{x_{ei}} (1 - \theta_{ic})^{1-x_{ei}},$$

where $P(\mathbf{z}_e | \boldsymbol{\pi})$ is a one-trial multinomial distribution or categorical prior distribution for $\mathbf{z}_e$ represented as:

$$P(\mathbf{z}_e | \boldsymbol{\pi}) = \prod_{c=1}^C \pi_c^{z_{ec}}, \quad (3)$$

where the probability parameter π_c satisfies $\pi_c \geq 0$ and $\sum_{c=1}^C \pi_c = 1$. Note that multinomial distributions are often integer-valued, corresponding to a latent class number. Here, the latent class number is converted into binary vector $\mathbf{z}_e$.

DCMs place a set of equality constraints on the parameters of the latent class model via the model parameterization and via the Q-matrix, an item-by-attribute-sized matrix with binary elements q_{ia} with values of one indicating that an item i measures a latent attribute a ($a = 1, \dots, A$) otherwise zero. In a DCM, each latent class is a specific permutation of attribute mastery statuses, where the total number of latent classes is $C = 2^A$. Using the latent class indicator vector $\mathbf{z}_e$, an attribute mastery pattern is a permutation of mastery (= 1) and non-mastery (= 0) for an examinee e is $\boldsymbol{\alpha}_e = \boldsymbol{\alpha}^\top \mathbf{z}_e$, where $\boldsymbol{\alpha}$ is a $2^A \times A$ binary matrix representing all possible attribute mastery patterns. For example, $\boldsymbol{\alpha}$ is $[[0,0,0]^\top, [1,0,0]^\top, \dots, [1,1,1]^\top]^\top$ for a case where the number of attributes is $A = 3$. The indicator vector $\mathbf{z}_e$ selects one of the possible attribute mastery patterns and $\boldsymbol{\alpha}_e$ indicates that $\alpha_{ea} = 1$ if examinee e is a master of attribute a and is zero otherwise.

General DCMs have similar forms in the mapping of the Q-matrix to item parameter constraints. For example, under the LCDM, the latent class model probability of a correct response θ_{ic} , to an item i from an examinee e in class c (and with attribute profile $\boldsymbol{\alpha}_e$) is:

$$P(X_{ei} = 1 | \boldsymbol{\alpha}_e, \mathbf{q}_i) = \theta_{ic} = \frac{\exp(h(\boldsymbol{\alpha}_e, \mathbf{q}_i))}{1 + \exp(h(\boldsymbol{\alpha}_e, \mathbf{q}_i))}, \quad (4)$$

where $h(\boldsymbol{\alpha}_e, \mathbf{q}_i)$ is a so-called helper function that provides the mapping from a latent class/attribute profile to an item response. The result of the helper function provides the linear predictor of the item response function:

$$h(\boldsymbol{\alpha}_e, \mathbf{q}_i) = \lambda_{i,0} + \sum_{a=1}^A \lambda_{i,1,(a)} (\alpha_{ea} q_{ia}) + \sum_{a=1}^{A-1} \sum_{b=a+1}^a \lambda_{i,2,(a,b)} (\alpha_{ea} q_{ia}) (\alpha_{eb} q_{ib}) + \dots \quad (5)$$

We point the reader to other sources (e.g., Rupp et al., 2010) for the interpretation of these as our motivation in this context is to note that DCMS put constraints on latent class model item parameters across classes, thereby reducing the total number of parameters for an item from C (one for every class) to $2^{q_{i+}}$ (where $q_{i+} = \sum_{a=1}^A q_{ia}$).

The key distinction in DCMS is that an item can distinguish only q_{i+} patterns—often far less than the number of total latent classes (2^A) in a general latent class model. This distinction limits the set of parameters for each item to $\boldsymbol{\theta}_i = [\theta_{i1}, \dots, \theta_{i2^{q_{i+}}}]^T$. Here after, we represent restricted item parameters using the subscript p ($= 1, \dots, 2^{q_{i+}}$) for item i and item specific parameter as θ_{ip} . To convert the latent class indicator to an item specific class, we introduce a $2^{q_{i+}} \times 2^A$ binary matrix $\mathbf{G}_i$ for each item i (Yamaguchi & Okada, 2021; Yamaguchi & Templin, 2022). The elements of the G_i matrix for item i , $g_{pc}^{(i)}$, come from the result of an indicator function $I(\boldsymbol{\alpha}_p^{(i)} = \boldsymbol{\alpha}_c)$ for the element of $\{a; q_{ia} \neq 0\}$, where $\boldsymbol{\alpha}_p^{(i)}$ is the p -th item-specific attribute profile vector $[\alpha_{pa}^{(i)}, \dots, \alpha_{pA}^{(i)}]^T$ whose superscript (i) indicates this pattern is for item i . In addition, $\boldsymbol{\alpha}_p^{(i)}$ has elements “*” for $\{a; q_{ia} = 0\}$ -th element and this means that the attribute should be ignored as its Q-matrix value is set to zero, indicating the attribute is not measured by the item. The G_i indicates which original latent class (or attribute mastery pattern) should have the same correct item response probability based on the $\mathbf{q}_i$ vector. In other words, the elements of G_i indicate which of the original set of 2^A latent classes are equivalent with respect to the item.

For example, assume a test measuring two attributes ($A = 2$) and an item i measuring only the first attribute: $\mathbf{q}_i = [1, 0]^T$. Then, the total number of latent classes/attribute mastery

profiles is $2^A = 2^2 = 4$: $\alpha_1 = [0,0]^\top$, $\alpha_2 = [0,1]^\top$, $\alpha_3 = [1,0]^\top$, and $\alpha_4 = [1,1]^\top$. As item i can only distinguish $2^{q_{i+}} = 2^1 = 2$ attribute patterns that are referred as item-specific classes: $\alpha_1^{(i)} = [\alpha_{11}^{(i)} = 0, \alpha_{12}^{(i)} = *]^\top$ and $\alpha_2^{(i)} = [\alpha_{21}^{(i)} = 1, \alpha_{22}^{(i)} = *]^\top$. As shown by this notation, under a general DCM, examinees with attribute patterns 1 and 2 have equal probabilities of a correct response and examinees with attribute patterns 3 and 4 have equal probabilities of a correct response. This means that attribute patterns 1 and 2 were mapped to the first item specific class $\alpha_1^{(i)} = [0, *]^\top$ and attribute patterns 3 and 4 were mapped to the second item specific class $\alpha_2^{(i)} = [1, *]^\top$. Then, the size of $\mathbf{G}_i$ is $2^1 \times 2^2$, and its elements are the shaded part in Table 1. The $\mathbf{G}_i$ for other q_i is derived in the same manner.

Using the previous notation, and, under the assumption of local independence of items and independence of examinees, the complete data likelihood function is:

$$P(\mathbf{X}, \mathbf{Z} | \boldsymbol{\Theta}, \boldsymbol{\pi}) = \prod_{e=1}^E \left(\prod_{c=1}^{2^A} \left\{ \pi_c \prod_{i=1}^I \prod_{p=1}^{2^{q_{i+}}} \left[\left\{ \theta_{ip}^{x_{ei}} (1 - \theta_{ip})^{1-x_{ei}} \right\}^{g_{pc}^{(i)}} \right] \right\}^{z_{ec}} \right), \quad (6)$$

where we use $\mathbf{X}$, $\mathbf{Z}$, and $\boldsymbol{\Theta}$ for sets of all examinees' item response vector and latent class indicators and set of all item parameters. Next, we introduce the priors for $\boldsymbol{\Theta}$ and $\boldsymbol{\pi}$.

2.3. Prior specification and posterior representation

As with previous research, we choose similar prior distributions for the parameters of the model (e.g., Yamaguchi & Okada, 2021). As $\boldsymbol{\pi}$ is a probability vector with, for $A > 1$, more than three elements, an appropriate prior for $\boldsymbol{\pi}$ is the Dirichlet distribution with parameters $\boldsymbol{\delta}^0 = [\delta_1^0, \dots, \delta_{2^A}^0]^\top$:

$$P(\boldsymbol{\pi} | \boldsymbol{\delta}^0) \propto \prod_{c=1}^{2^A} \pi_c^{\delta_c^0 - 1}. \quad (7)$$

The item-level correct response probability parameter θ_{ip} has a prior distribution that follows a Beta distribution with positive-valued hyper parameters and b_{ip}^0 :

$$P(\theta_{ip} | a_{ip}^0, b_{ip}^0) \propto \theta_{ip}^{a_{ip}^0 - 1} (1 - \theta_{ip})^{b_{ip}^0 - 1}. \quad (8)$$

Under conditional independence of each parameter, the joint distribution of θ_{ip} is

$$P(\boldsymbol{\Theta}|\mathbf{A}^0, \mathbf{B}^0) \propto \prod_{i=1}^I \prod_{p=1}^{2^{q_{i+}}} \theta_{ip}^{a_{ip}^0-1} (1 - \theta_{ip})^{b_{ip}^0-1}, \quad (9)$$

where the $\mathbf{A}^0$ and $\mathbf{B}^0$ are the sets of hyper parameters a_{ip}^0 and b_{ip}^0 . Note that monotonicity constraints on correct item response probability parameters (e.g., Henson et al., 2009) are sometimes imposed in DCMs for model identifiability (e.g., Gu & Xu, 2020; Xu & Shang, 2018). The derivation of the normalizing constant of the truncated multivariate Beta distribution, needed to incorporate monotonicity constraints directly, is beyond the scope of this paper, which seeks to investigate the properties of the CGS algorithm.

As introduced before, $\mathbf{z}_e$ follows a one-trial multinomial distribution and the joint distribution of $\mathbf{Z}$ under independence of examinees is

$$P(\mathbf{Z}|\boldsymbol{\pi}) = \prod_{e=1}^E P(\mathbf{z}_e|\boldsymbol{\pi}) = \prod_{e=1}^E \prod_{c=1}^{2^A} \pi_c^{z_{ec}}. \quad (10)$$

Finally, using Equation 6, 7, 9 and 10, via Bayes theorem the joint distribution of $\mathbf{X}, \mathbf{Z}, \boldsymbol{\Theta}$, and $\boldsymbol{\pi}$ is:

$$\begin{aligned} P(\mathbf{X}, \boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{Z}|\mathbf{A}^0, \mathbf{B}^0, \boldsymbol{\delta}^0) &\propto P(\mathbf{X}|\boldsymbol{\Theta}, \boldsymbol{\pi}, \mathbf{Z})P(\mathbf{Z}|\boldsymbol{\pi})P(\boldsymbol{\pi}|\boldsymbol{\delta}^0)P(\boldsymbol{\Theta}|\mathbf{A}^0, \mathbf{B}^0), \\ &= \left(\prod_{e=1}^E \left(\prod_{c=1}^{2^A} \left\{ \pi_c \prod_{i=1}^I \prod_{p=1}^{2^{q_{i+}}} \left[\theta_{ip}^{x_{ei}} (1 - \theta_{ip})^{1-x_{ei}} \right]^{g_{pc}^{(i)}} \right\}^{z_{ec}} \right) \right) \left(\prod_{e=1}^E \prod_{c=1}^{2^A} \pi_c^{z_{ec}} \right) \\ &\quad \times \left(\prod_{c=1}^{2^A} \pi_c^{\delta_c^0-1} \right) \left(\prod_{i=1}^I \prod_{p=1}^{2^{q_{i+}}} \theta_{ip}^{a_{ip}^0-1} (1 - \theta_{ip})^{b_{ip}^0-1} \right). \end{aligned} \quad (11)$$

In this study, all random variables and their parameters are explicitly included in equations to clarify the dependency among them.

Before deriving the CGS algorithm, we mention the difference of our current model formulation and other general DCMs such as LCDM or GDINA frameworks. Notably, our

model formulation has priors for the correct item response probability and mixing parameters as this is a Bayesian model. Moreover, the current model formulation is based on the general latent class model. More importantly, the current formulation includes all variables appearing in the model and their dependencies are explicitly noted, which are represented as conditional distributions. This Bayesian formulation makes it possible to marginalize out certain sets of parameters as shown in the next section. In other words, this estimation procedure uses the uncertainty of parameters in the estimation step even if the posterior distributions are not concentrated in around the peak of the likelihood function, which can help provide stable examinee attribute profile probability estimation.

On the other hand, the GDINA and LCDM model formulations focus on the main effects and interactions of attributes, which are similar to a linear model. In addition, these models often are estimated with marginal maximum likelihood methods such as the EM algorithm, providing a set of point estimates for the parameters and asymptotically derived standard errors. Finally, both frameworks use the point estimates of the item and structural model parameters in their estimates of examinee attribute profile probabilities. Therefore, any estimation error in the item and structural model can result in attribute profile misclassification.

2. A Collapsed Gibbs Sampling Algorithm for Saturated DCMS

The collapsed Gibbs sampling (CGS) algorithm only samples examinee latent attribute patterns and does not sample correct item response probability parameters or latent class mixing parameters. To do so, marginalization of the parameters in the model is used. The derivation process follows Sato (2016) and Suyama and Sugiyama (2017).

First, the joint distribution of the observed variables $\mathbf{X}$ and the examinee latent class indicator vector $\mathbf{Z}$ can be represented as:

$$P(\mathbf{X}, \mathbf{Z}) = \iint P(\mathbf{X}, \mathbf{Z}, \boldsymbol{\Theta}, \boldsymbol{\pi}) d\boldsymbol{\Theta} d\boldsymbol{\pi}. \quad (12)$$

We omit the hyper parameters from Equation (11) to simplify the notation. If we calculate Equation (12), we can also calculate the distribution $P(\mathbf{Z}|\mathbf{X})$ but it is not obvious and

computationally difficult. Therefore, we aim to generate a random sample of $\mathbf{z}_e$ from the conditional distribution $P(\mathbf{z}_e|\mathbf{X}, \mathbf{Z}_{/e})$ where $\mathbf{Z}_{/e}$ is a set from $\mathbf{z}_e$ is removed.

From Equation (12) and from Bayes theorem, the following conditional distribution of $P(\mathbf{z}_e|\mathbf{X}, \mathbf{Z}_{/e})$ can be derived:

$$P(\mathbf{z}_e|\mathbf{X}, \mathbf{Z}_{/e}) \propto P(\mathbf{x}_e|\mathbf{X}_{/e}, \mathbf{z}_e, \mathbf{Z}_{/e})P(\mathbf{z}_e|\mathbf{Z}_{/e}), \quad (13)$$

where $\mathbf{X}_{/e}$ is a data set that does not contain $\mathbf{x}_e$, which is the item response vector for examinee e . From Equation (13), the functional forms of $P(\mathbf{x}_e|\mathbf{X}_{/e}, \mathbf{z}_e, \mathbf{Z}_{/e})$ and $P(\mathbf{z}_e|\mathbf{Z}_{/e})$ should be determined to sample $\mathbf{z}_e$ from its distribution $P(\mathbf{z}_e|\mathbf{Z}_{/e})$.

With respect to $P(\mathbf{z}_e|\mathbf{Z}_{/e})$:

$$P(\mathbf{z}_e|\mathbf{Z}_{/e}) = \int P(\mathbf{z}_e|\boldsymbol{\pi})P(\boldsymbol{\pi}|\mathbf{Z}_{/e}) d\boldsymbol{\pi}. \quad (14)$$

Previously, we noted that $P(\mathbf{z}_e|\boldsymbol{\pi})$ follows a one-trial multinomial distribution. Furthermore, $P(\boldsymbol{\pi}|\mathbf{Z}_{/e})$ follows a Dirichlet distribution with parameters $\delta_{/e,c}^* = \sum_{e' \neq e} z_{e',c} + \delta_c^0, \forall c$, where the summation is taken over subscript e' . Using a conjugate prior between the one-trial multinomial and Dirichlet distributions (e.g., Nakajima et al., 2019; Suyama & Sugiyama, 2017), $P(\mathbf{z}_e|\mathbf{Z}_{/e})$ follows a one-trial multinomial distribution with parameters $r_{ec}^* = \delta_{/e,c}^* / (\sum_c \delta_{/e,c}^*), \forall c$.

Second, $P(\mathbf{x}_e|\mathbf{X}_{/e}, \mathbf{z}_e, \mathbf{Z}_{/e})$ can be represented in an integral form:

$$\begin{aligned} P(\mathbf{x}_e|\mathbf{X}_{/e}, \mathbf{z}_e, \mathbf{Z}_{/e}) &= \int P(\mathbf{x}_e|\mathbf{z}_e, \boldsymbol{\Theta})P(\boldsymbol{\Theta}|\mathbf{X}_{/e}, \mathbf{Z}_{/e}) d\boldsymbol{\Theta}, \\ &= \int \prod_{i=1}^I P(x_{ei}|\mathbf{z}_e, \boldsymbol{\theta}_i)P(\boldsymbol{\theta}_i|\mathbf{x}_{/e,i}, \mathbf{Z}_{/e}) d\boldsymbol{\Theta}, \end{aligned} \quad (15)$$

where the $\mathbf{x}_{/e,i}$ is a vector from which x_{ei} is omitted and whose length is $I - 1$. The item response function $P(x_{ei}|\mathbf{z}_e, \boldsymbol{\theta}_i)$ follows a Bernoulli distribution, and $P(\boldsymbol{\theta}_i|\mathbf{x}_{/e,i}, \mathbf{z}_e)$ follows a Beta distribution with parameters $a_{/e,ip}^* = \sum_{e' \neq e} \sum_c z_{e',c} g_{pc}^{(i)} x_{e',i} + a_{ip}^0$ and $b_{/e,ip}^* = \sum_{e' \neq e} \sum_c z_{e',c} g_{pc}^{(i)} (1 - x_{e',i}) + b_{ip}^0$. Item responses are assumed conditionally independent and posterior distributions of correct response probability parameters are independent via the

decomposition of the complete conditional distribution. Therefore, the integration and product can be exchanged in Equation (15) which is represented as:

$$\begin{aligned}
 P(\mathbf{x}_e | \mathbf{X}_{/e}, \mathbf{z}_e, \mathbf{Z}_{/e}) &= \prod_{i=1}^I \int P(x_{ei} | \mathbf{z}_e, \boldsymbol{\theta}_i) P(\boldsymbol{\theta}_i | \mathbf{x}_{/e,i}, \mathbf{Z}_{/e}) d\boldsymbol{\theta}_i, \\
 &= \prod_{i=1}^I \int \prod_{p=1}^{2^{q_i}+} C_{Beta}(a_{/e,ip}^*, b_{/e,ip}^*) \left\{ \prod_{c=1}^{2^A} \left[\left\{ \theta_{ip}^{x_{ei}} (1 - \theta_{ip})^{1-x_{ei}} \right\}^{g_{pc}^{(i)}} \right]^{z_{ec}} \right\} \\
 &\quad \times \left[\theta_{ip}^{a_{/e,ip}^*-1} (1 - \theta_{ip})^{b_{/e,ip}^*-1} \right] d\boldsymbol{\theta}_i,
 \end{aligned} \tag{16}$$

where $C_{Beta}(a_{/e,ip}^*, b_{/e,ip}^*)$ is a normalizing constant of Beta distribution parameters $a_{/e,ip}^*$ and $b_{/e,ip}^*$. The full conditional posterior item distributions are independent of each other and, as such, we separately assess the parameter elements of each. In addition, we consider the situation where $z_{ec} = 1$ for simplicity and then Equation (16) can be represented as:

$$\begin{aligned}
 P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}) \\
 &= C_{Beta}(a_{/e,ip}^*, b_{/e,ip}^*) \int \theta_{ip}^{g_{pc}^{(i)} x_{ei} + a_{/e,ip}^* - 1} (1 - \theta_{ip})^{g_{pc}^{(i)} (1 - x_{ei}) + b_{/e,ip}^* - 1} d\theta_{ip}.
 \end{aligned} \tag{17}$$

The integrand of Equation (17) is the beta function, which provides the inverse of the normalization constant of the beta distribution:

$$P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}) = \frac{C_{Beta}(a_{/e,ip}^*, b_{/e,ip}^*)}{C_{Beta}(g_{pc}^{(i)} x_{ei} + a_{/e,ip}^*, g_{pc}^{(i)} (1 - x_{ei}) + b_{/e,ip}^*)}. \tag{18}$$

More explicitly, $C_{Beta}(a, b)$ can be written using the gamma function $\Gamma(w)$: $C_{Beta}(a, b) = \Gamma(a + b) / (\Gamma(a)\Gamma(b))$. With this relationship, Equation (18) is decomposed as:

$$\begin{aligned}
 P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}) \\
 &= \frac{\Gamma(a_{/e,ip}^* + b_{/e,ip}^*) \Gamma(g_{pc}^{(i)} x_{ei} + a_{/e,ip}^*) \Gamma(g_{pc}^{(i)} (1 - x_{ei}) + b_{/e,ip}^*)}{\Gamma(a_{/e,ip}^*) \Gamma(b_{/e,ip}^*) \Gamma(g_{pc}^{(i)} + a_{/e,ip}^* + b_{/e,ip}^*)},
 \end{aligned} \tag{19}$$

Recall $g_{pc}^{(i)}$ is an element of the $\mathbf{G}_i$ matrix for item i . Equation (19) depends on the value of $g_{pc}^{(i)}$. In the case where $g_{pc}^{(i)} = 1$, using the formula $\Gamma(1+w) = w\Gamma(w)$ for $w > 0$, then $\Gamma(1 + a_{/e,ip}^* + b_{/e,ip}^*) = (a_{/e,ip}^* + b_{/e,ip}^*)\Gamma(a_{/e,ip}^* + b_{/e,ip}^*)$ and Equation (19) becomes

$$P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}) = \frac{1}{(a_{/e,ip}^* + b_{/e,ip}^*)} \frac{\Gamma(x_{ei} + a_{/e,ip}^*)\Gamma(1 - x_{ei} + b_{/e,ip}^*)}{\Gamma(a_{/e,ip}^*)\Gamma(b_{/e,ip}^*)}. \quad (20)$$

For an item parameter, in the case of $x_{ei} = 1$ and with $\Gamma(1 + a_{/e,ip}^*) = a_{/e,ip}^*\Gamma(a_{/e,ip}^*)$, the numerators and denominators in Equation (18) cancel, leaving

$$P(x_{ei} = 1 | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{c}_{/e}) = \frac{a_{/e,ip}^*}{a_{/e,ip}^* + b_{/e,ip}^*}. \quad (21)$$

Similarly, in the case of $x_{ei} = 0$, Equation (18) becomes

$$P(x_{ei} = 0 | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{c}_{/e}) = \frac{b_{/e,ip}^*}{a_{/e,ip}^* + b_{/e,ip}^*}. \quad (22)$$

This means that $P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e})$ follows a Bernoulli distribution with probability

parameter equal to $\frac{a_{/e,ip}^*}{a_{/e,ip}^* + b_{/e,ip}^*}$ when $g_{pc}^{(i)} = 1$. In addition, if $g_{pc}^{(i)} = 0$, then

$P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}) = 1$. Combining these results, the following conditional correct item response probability is derived:

$$\begin{aligned} P(\mathbf{x}_e | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}, \mathbf{g}_{pc}) \\ = \prod_{i=1}^I \left(\left(\frac{a_{/e,ip}^*}{a_{/e,ip}^* + b_{/e,ip}^*} \right)^{x_{ei}} \left(\frac{b_{/e,ip}^*}{a_{/e,ip}^* + b_{/e,ip}^*} \right)^{1-x_{ei}} \right)^{g_{pc}^{(i)}}, \end{aligned} \quad (23)$$

where $\mathbf{g}_{pc}$ is an $I \times 1$ vector $[g_{pc}^{(1)}, \dots, g_{pc}^{(I)}]^\top$.

From the result of $P(\mathbf{z}_e | \mathbf{Z}_{/e})$ and Equation (23), we can calculate

$$\eta_{/e,c} = \log \delta_{/e,c}^* + \log P(\mathbf{x}_e | \mathbf{X}_{/e}, \mathbf{z}_e, \mathbf{Z}_{/e}), \quad (24)$$

and finally, we gain the conditional distribution of $\mathbf{z}_e$ given $\mathbf{X}$ and $\mathbf{Z}_{/e}$ which follows a one-trial multinomial distribution:

$$P(\mathbf{z}_e | \mathbf{X}, \mathbf{Z}_{/e}) = \prod_{c=1}^C \left(\frac{\exp(\eta_{/e,c})}{\sum_c \exp(\eta_{/e,c})} \right)^{z_{ec}}. \quad (25)$$

After sampling $\mathbf{z}_e$, to sample the latent class indicator vector of examinee e' , the parameters will be updated as:

$$\delta_{/e',c}^* = \delta_{/e,c}^* + z_{ec} - z_{e'c}, \quad (26)$$

$$a_{/e',ip}^* = a_{/e,ip}^* + z_{ec}g_{pc}^{(i)}x_{ei} - z_{e'c}g_{pc}^{(i)}x_{e'i}, \quad (27)$$

$$b_{/e',ip}^* = b_{/e,ip}^* + z_{ec}g_{pc}^{(i)}(1 - x_{ei}) - z_{e'c}g_{pc}^{(i)}(1 - x_{e'i}). \quad (28)$$

In summary, a pseudo code version of the CGS algorithm is shown below and is summarized as follows: First, initialize $\mathbf{Z}$, set hyper parameters and the number of iterations, and calculate δ^* , $\mathbf{A}^*$, and $\mathbf{B}^*$ with $\delta_c^* = \sum_e z_{ec} + \delta_{c,}^0$, $a_{ip}^* = \sum_e \sum_c g_{pc}^{(i)} z_{ec} x_{ei} + a_{ip}^0$, and $b_{ip}^* = \sum_e \sum_c g_{pc}^{(i)} z_{ec} (1 - x_{ei}) + b_{ip}^0, \forall c, i, p$. Set all parameters using $\delta_{/1,c}^* = \delta_c^* - z_{1c}$, $a_{/1,ip}^* = a_{ip}^* - z_{1c}g_{pc}^{(i)}x_{1i}$ and $b_{/1,ip}^* = b_{ip}^* - z_{1c}g_{pc}^{(i)}(1 - x_{1i})$.

For iteration in 1 to the number of iterations, continue following:

For e in 1 to E , continue following:

For c in 1 to C :

Calculate $\sum_i \log P(x_{ei} | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e}, g_{pc}^{(i)})$ using Equation (23)

Calculate $\eta_{/e,c} = \log \delta_{/e,c}^* + \log P(\mathbf{x}_e | \mathbf{X}_{/e}, z_{ec} = 1, \mathbf{Z}_{/e})$

End for c .

Sample $\mathbf{z}_e \sim P(\mathbf{z}_e | \mathbf{X}, \mathbf{Z}_{/e}) = \prod_c \left(\frac{\exp(\eta_{/e,c})}{\sum_e \exp(\eta_{/e,c})} \right)^{z_{ec}}$

Update parameters:

$$\delta_{/e',c}^* = \delta_{/e,c}^* + z_{ec} - z_{e'c},$$

$$a_{/e',ip}^* = a_{/e,ip}^* + z_{ec}g_{pc}^{(i)}x_{ei} - z_{e'c}g_{pc}^{(i)}x_{e'i},$$

$$b_{/e',ip}^* = b_{/e,ip}^* + z_{ec}g_{pc}^{(i)}(1 - x_{ei}) - z_{e'c}g_{pc}^{(i)}(1 - x_{e'i}).$$

End for e .

End for an iteration.

After posterior sampling is conducted, the examinee latent attribute profiles are then converted, for each attribute, to a marginal probability of attribute mastery.

We note that our CGS algorithm may suffer from the finite mixture model label switching problem (e.g., Papastamoulis, 2015; Stephens, 2000), where attribute levels could switch, especially without the monotonicity constraints. Previous methods finding the optimal

permutation of class labels such Stephens (2000) may be difficult to implement in DCMS as the number of latent classes (attribute mastery profiles) is relatively large, making the number of class label permutations exponentially larger than in general latent class models. Therefore, we employ an ad-hoc approach: After a long run of a single MCMC chain, latent class/attribute profile probabilities are calculated. From anecdotal experience in building our algorithm, we noticed that the profile probability is sometimes negatively correlated with the sum-score $\sum_i x_{ei}$, which not appropriate as a high probability of mastering an attribute should result in an increase in a test score (following a similar rationale as the monotonicity constraints in certain DCMS, e.g., Henson et al., 2009). Therefore, to fix this problem, attribute mastery probabilities for profiles with negative correlations with the sum score are reversed by taking one minus the probability. This correlation check was done following MCMC sampling of attribute profiles. This ad-hoc method also appears to fix the label switching problem. As this procedure worked well in small-scale tests, it was added to our simulation study.

3. Simulation Study 1

In our simulation study, we investigate the use of a single MCMC chain, our choice of parameter starting values, and our ad-hoc label switching correction strategy.

3.1. Simulation Settings

The goal of the simulation study was to examine the classification accuracy of examinee latent class/attribute profiles. We use, as a benchmark, a comparison of the classification accuracy of our algorithm result versus that of the non-parametric methods by Chiu and Douglas (2013). Wang and Douglas (2015) established the asymptotic consistency of non-parametric methods in DCMS under some regularity conditions, so in large sample size settings, the results of the two methods should be similar. However, in situations with small sample sizes, differences may occur. There were 16 simulation conditions: the data generating model (a general DCM or DINA), sample size (100 and 1000), quality of items (low or high, see below), the number of attributes (four and eight), and number of items on a test (30 and 60).

Two data generation models were used: general DCM or the DINA model. The estimation model underlying the CGS is a general DCM and underlying one of the non-parametric methods is the DINA model. Therefore, employing the two types of data generation models makes it possible to check the effects of model misspecification on correct classification rates. Under the DINA model, the slipping and guessing item parameters were set to $s_i = g_i = 0.1$ and 0.3 for all items, which are named the high- and low-quality items conditions. For the general DCMs, the correct item response probability parameters were determined based on the number of attributes. For an item-specific pattern p , the probability θ_{ip} is set as:

$$\theta_{ip} = g_i + \left(\sum_{\alpha_{pa'}^{(i)} \neq *} \alpha_{pa'}^{(i)} \right) \frac{(1 - s_i - g_i)}{\sum_a (q_{ia})}, \quad (29)$$

where $\alpha_{pa'}^{(i)}$ is an element of the item-specific profile vector $\alpha_p^{(i)}$. Therefore, for a given item, more measured attributes being mastered results in a higher item response probability.

The Q-matrix for the four- and eight-attribute conditions is shown in Tables 2 and 3, respectively. The four-attribute Q-matrix in Table 2 has eight items measuring one attribute, twelve items measuring two attributes, four items measuring three attributes, and two items measuring four attributes. The eight-attribute Q-matrix in Table 3 has sixteen items measuring one attribute, eight items measuring two attributes, and four items measuring four attributes. There were two test length conditions: short (30 items) and long (60 items). In the 30-item condition, the Q-matrix in Table 2 or 3 with 30 items was employed. In the 60-item condition, same two Q-matrices were stacked so a total of 60 items were simulated.

We fixed the tetrachoric correlation among attributes to 0.7, representing correlations often found in real data analyses. In the data generating process, the attributes were generated in a similar manner to Chiu and Douglas (2013). First, continuous variables underlying the attributes were randomly generated from multivariate normal distribution whose means were zero and covariance matrix with diagonal elements set to one and off-diagonal elements set to 0.7. Then, the continuous underlying variables were converted to binary attributes with a value

of one if the continuous underlying variable values were greater than the thresholds that were defined as $\Phi^{-1}(a/(1 + A))$, $\forall a$, where $\Phi^{-1}(\cdot)$ is the inverse cumulative normal distribution function. These thresholds create a difference in the proportion of examinees mastering each of the attributes. The procedure was repeated 30 times for each condition. The average correct classification rate (CCR) was employed for parameter recovery indicators. CCR was calculated for each attribute (CCRA) and for the whole pattern (CCRP). Moreover, standard errors for CCRA and CCRP were also calculated across simulation to evaluate amount of variability of estimation methods.

For each condition, CGS and non-parametric estimation with joint maximum likelihood (JMLE) for the DINA model procedures were applied and the examinee latent attribute profiles were estimated. The non-parametric estimation method was conducted by JMLE function in the NPCD package (Zheng et al., 2019). The convergence criteria of the JMLE algorithm for item parameters was set to 0.0001, for class parameters was set 0.0001, and the maximum number of iterations was set to 1,000. For the CGS algorithm, the hyper parameters were all set to one, the number of MCMC iterations was 2,000, and the first 1,000 iterations were discarded as a burn-in period. Note that it is difficult to diagnose convergence in the CGS algorithm because the algorithm eliminates item and mixing parameters. In fact, previous research with other CGS algorithms do not report convergence information (e.g., Blei & Jordan, 2006; Porteous et al., 2008). Similarly, we report our results without convergence checks, a point we revisit in the discussion. All simulation procedures were coded in *R* (R Core Team, 2019).

3.2. Results

Figure 1 shows the CCRA and CCRP and their SEs for data generated with the general model and four attributes. For large sample sizes, the CGS algorithm had higher CCRA and CCRP than the JMLE algorithm in large sample size conditions. More variability was found in the small sample size conditions. For high quality items with small sample sizes, the CGS algorithm had better classification accuracy and smaller SEs than the JMLE algorithm. For low quality items with small sample sizes, the CCRs of the CGS algorithm were slightly better than

ones of the JMLE algorithm. For attributes with a higher proportion of masters, JMLE showed better classification rates. With lower proportions of masters such as attribute tree or four, CGS had better classification rates. Longer tests slightly improve CCRs, as shown in Figure 1.

Figure 2 indicates the CCRAs and CCRPs and their SEs for data generated with the DINA model data with four attributes. In the large sample size condition, the CCRs of the CGS algorithm were better than those from JMLE. However, in the small sample size and low-quality items conditions, the CCRAs of attribute 1 and 2 and CCRP with JMLE were better than the CGS. The CCRAs and CCRPs in the small sample size and high-quality items conditions were high and only slight differences between the two methods were found. Test length in Figure 2 showed same tendency in Figure 1: long tests were better than short test in terms of CCRs.

Figure 3 shows the results for data generated with the general model and eight attributes and Figure 4 shows the results for data generated with the DINA model and eight attributes. Both results are similar. Specifically, the CGS algorithm was generally better than JMLE, especially under large sample sizes. Again, however, in small sample size and low-quality items conditions, results were mixed, with JMLE having higher CCRs for attributes with larger proportions of masters such as attribute 1 to 3. The differences between the two methods disappeared in the small sample size and high-quality items conditions in Figure 3 and 4. The test length improved CCRs slightly for both methods.

We also checked the average ratio of frequency occurrence for the ad-hoc monotonicity correlation check for the CGS algorithm. Figures S1 and S2, from the four and eight attribute conditions, are provided in Supplemental Material A. The posterior attribute mastery probabilities and sum-scores were frequently negatively correlated, but no systematic trends were observed. Moreover, we calculated a point-biserial correlation between the correct classification rate and occurrence of a negative mastery/sum-score correlation for each attribute. The results were shown in Figures S3 and S4. Again, these correlations did follow any systematic trends. From these results, we conclude that the attribute recovery results of the CGS algorithm shown in Figure 1 to 4 were not affected by mastery and non-mastery switching.

In summary, the results show that the CGS algorithm can provide a high rate of classification recovery of true attribute mastery statuses across various simulation conditions, even when data were generated from models subsumed by the general DCMS. Moreover, in large sample conditions, the CGS algorithm had higher classification rates than JMLE. In addition, classification rates improved for conditions indicating greater information about the latent traits: when the number of items was large or the quality of test items was high. However, in conditions where data contained less information about the latent trait (i.e., the small sample size and low-quality items conditions), JMLE and CGS had mixed results, with JMLE having better classification rates for attributes with a lower proportion of masters.

4. Simulation Study 2

The purpose of simulation study 2 is to compare the GCS algorithm to two different estimation methods: an MCMC sampling method implemented with the JAGS program (Plummer, 2003) and marginal ML with the GDINA package (Ma & de la Torre, 2020; where we checked how often the boundary problem occurred).

4.1. *Simulation Settings*

Data generation settings, CGS settings, evaluation criteria, and the number of replications were the same as simulation 1. However, we omitted the DINA data generation condition because it was for comparison with the JMLE algorithm. As comparative estimation methods, we employed the LCDM with positivity main effect constraints that was conducted using MCMC with JAGS and the ML method with GDINA package (Ma & de la Torre, 2020).

The model expression in the JAGS language was shown in Zhang et al. (2019). The prior settings were also followed by Zhang et al. (2019). Convergence was checked with the Gelman-Rubin $\hat{R}$ index (Brooks & Gelman, 1998) that was calculated with the CODA package (Plummer et al., 2006). The convergence criterion set was $\hat{R} < 1.10$ for all parameters. Each MCMC analysis consisted of four MCMC chains, each with 20,000 iterations (of which the first 10,000 were discarded as a burn-in period). We note that we initially set the number of iterations to 2,000 to keep comparability between CGS and MCMC with JAGS but no JAGS analyses

achieved convergence. Therefore, the number of iterations was increased in JAGS to improve convergence. Note that the JAGS sometimes selects Metropolis-Hastings (MH) algorithm type algorithm based on model description. In our simulation, however, inefficient MH based samplers were not selected for all parameters. Specifically, JAGS selected “base::RealSlicer” for the lambda parameters, “bugs::ConjugateDirichlet” for the mixing parameters, and “base::Finite” for individual attribute profiles.

For checking the boundary problem in ML estimation with the GDINA we operationally defined the boundary problem if an estimated correct item response parameter was greater than 0.9999 or less than 0.0001, which were the arbitrary parameter limits employed in the GDINA package. The item parameter estimation was conducted with default setting of the GDINA function with GDINA model. Note that the GDINA package can estimate correct item response probabilities that are essentially same as the θ parameters in our latent class model formulation. The proportion of times boundary problem occurred in the item parameter set was calculated for each simulation replication, which was averaged over 30 simulation replications.

4.2. Results

Less than 1% of replications using the JAGS MCMC algorithm failed to converge. The highest non-convergence rate was 2.2% in the low-quality items condition with four attributes, a long test, and sample size of 100. In real data analysis situations, the number of iterations will likely need to be increased to satisfy convergence for all parameters. However, the non-convergence rates were low for our study.

Table 4 shows the average rate of occurrence of the boundary problem. The 100 sample size, high-quality items, eight attributes, and short test condition showed largest rate of the boundary problem with 0.398. The 1,000 sample size, low-quality items, four attribute, and long test condition had the smallest value boundary problem rate with 0.095. In the 100 sample size and high quality items condition encountered boundary problems in greater than 30% of the replications. Boundary problem rates were more than 10% in all other conditions. The eight-attribute Q-matrices generally had larger boundary problem rates. In addition, the smaller sample

size conditions tended to have more boundary problems. Test length did influence boundary problem rates in the high-quality items conditions but the longer test length showed slightly lower boundary problem rates in low-test quality conditions. In small sample size conditions, high-quality item conditions showed more boundary problems than low-quality item conditions. However, the difference between item quality conditions was small in the large sample size setting. In summary, boundary problems occurred frequently in some conditions.

Table 5 showed the average estimation time (minutes) both CGS and JAGS results. Note that the number of iterations between CGS and JAGS MCMC estimation methods was 10 times different, and that these algorithms were written in different languages (CGS in R, JAGS in C++). In the four attribute conditions, shown in the upper part of Table 5, the CGS algorithm was about five to ten times faster than JAGS. Time differences increased with increases in test length and sample size. The quality of items did not affect estimation time. Estimation times in the eight attribute conditions, shown in lower part of Table 5, were markedly different between two methods. In many conditions, the CGS algorithm was more than 20 times faster than JAGS. In particular, the CGS algorithm was about 30 times faster than JAGS in the 100 sample size, high-quality items, and long test condition. These results indicate CGS is an effective Gibbs sampling method in large attribute situations.

Figures 5 and 6 show correct attribute recovery results of the three estimation methods in the four and eight attribute conditions. Figure 5 indicated that the CGS algorithm and JAGS had almost same recovery rates in the high-quality items or large sample size conditions. The CCRs in these cases were enough high even with boundary problems occurring. The ML method in high-quality conditions showed almost as same CCRs as the two MCMC methods. However, in large sample size and low-quality items conditions, the dotted line of the CCR that represent ML method results in Figure 5 were lower than the old and new MCMC methods.

A divergence among the three methods was found in the small sample size and low-quality items conditions. In both short and long test conditions, the CCRA1, CCRA2 and CCRP for the CGS algorithm were higher with JAGS. The CCRA3 showed a reversed result.

However, the ML results were similar to CGS results in these conditions. Figure 6 indicated that CCRA_s and CCRP of the two MCMC sampling methods and ML were roughly equal in most cases. However, in the four attribute conditions, the large sample and low-quality items conditions showed that the MCMC methods were slightly better in CCRA₃ to CCRA₇.

Simulation study 2 showed the CGS algorithm provided at least the same estimation quality to JAGS and ML methods in many cases. Moreover, the CGS algorithm needed fewer iterations than JAGS. The small number of iterations improves estimation time. In addition, the CGS method had results that were roughly the same or slightly better than the ML methods with the GDINA package across various conditions. This result indicated the efficacy of attribute mastery estimation with the CGS method. Of note were the higher CCR_s results of the CGS method seen in large sample size settings, a point we will return to later.

5. Analysis of ECPE Data

5.1. *Data description and analysis settings*

The main objective of the real data example was to compare the CGS algorithm with JMLE. Therefore, we selected the grammar section of Examination for the Certificate of Proficiency in English (ECPE) assessment that has been used in previous DCMS studies (e.g., Templin & Bradshaw, 2014; Templin & Hoffman, 2013). The ECPE Q-matrix is shown in Table 6 and contained 28 items and 3 attributes: morphosyntactic rules, cohesive rules, and lexical rules. Templin and Hoffman (2013) provide a description of the ECPE. The total sample size was 2,922. The data was obtained from CDM package in R (George et al., 2016).

For the CGS algorithm, a single MCMC chain was used. We set the total number of iterations to 10,000 with the first 5,000 iterations discarded as a burn-in period. Hyperparameters were all set to one, which resulted in uniform prior distributions. We used the JMLE algorithm as a reference for comparing the results of the CGS algorithm. The estimation settings of the JMLE algorithm were the same used in the simulation study. We also estimated attribute mastery profiles with the LCDM using Mplus version 8.1 for Windows (Muthén & Muthén, 1998-2017) to compare the CGS and JMLE algorithms to results from a general DCM. As a

measure of agreement of estimated attribute profiles, we employed Cohen's κ index (Cohen, 1960) using the psych package (Revelle, 2020) in R. The data analysis code of the CGS and JMLE estimation algorithms are available in Open Science Framework page:

<https://osf.io/bk75q/>.

We note that in an actual data analysis, we may first explore the appropriate statistical model and then select estimation algorithm. However, in this example, we did not intend to search the best model for the ECPE data. Rather, the purpose of this analysis was to compare various types of estimation methods in a real data situation to better understand how the CGS method works. Therefore, JMLE was also employed in the real data example. This information should be helpful for the reader who wishes to see the similarities and differences between the proposed CGS, JMLE, ML methods (with the LCDM) in a real data analysis. Moreover, the ECPE data has been used frequently in DCM studies and provides a relatively simple analysis setting. If the CGS were to not provide a consistent result with previous estimation methods, the CGS algorithm would be seen as not being reliable. The comparison of the three estimation methods revealed that the CGS algorithm gave results similar to the LCDM with the ML algorithm in this data set. We believe this result shows the CGS provides estimates similar to other methods, at least in this example.

5.2 Result

Tables 7 to 9 show contingency tables of attribute mastery classifications to demonstrate the consistency of the two methods where examinees with a probability of mastery greater or equal to .5 were classified as masters and those with a probability of mastery less than .5 were classified as non-masters. Table 7 shows that 757 examinees that were classified as non-masters with the CGS algorithm were classified as masters for morphosyntactic rules attribute with the JMLE algorithm. However, there was no examinee who was classified as a non-master with JMLE but classified as master with CGS. This difference could be due to the differences between the DINA model of the JMLE and general DCM. This tendency was also shown in Table 8 and 9 which show the results of the cohesive rules and lexical rules attributes.

The right upper off-diagonal elements of Table 8 and 9 are larger than the left lower off-diagonal elements.

We also examined mastery pattern-wise consistency, which showed similar tendencies in the attribute-level analysis. Table 10 shows a contingency table of the pattern-wise result. Cohen's κ from Table 10 was .382 (95%CI = [.364, .401]). The left lower off-diagonal elements were sparse, but the right upper off-diagonal elements were not. In addition, the row-marginal sum of the CGS attribute mastery pattern estimation results showed specific patterns with large numbers of examinees. For example, pattern "000" had 907 examinees and "111" had 1050 examinees. On the other hand, the result of JMLE (column-marginal sum) showed varied patterns. The largest pattern was "111" which was the same as CGS but the number of other patterns were different. Notably, examinees in pattern "000" with CGS were scattered to other patterns in JMLE. We checked the item response patterns for eleven respondents who the CGS algorithm classified as having no mastered attributes and those who the JMLE algorithm classified as having all attributes mastered. The item responses are shown in Table S1 in the Supplemental Material B. The range of the sum-scores was 18 to 20. Intuitively, they could have mastered some attributes. However, the attribute mastery classification with Mplus had results similar to the CGS algorithm: all attributes were not mastered. It was difficult to find any patterns from these item responses.

In addition to the comparison between CGS and JMLE algorithms, the contingency tables of estimated attribute mastery patterns with Mplus and the other two methods were shown in Tables S2 and S3 in the Supplemental Material B. The results of attribute mastery pattern classification with Mplus and CGS shown in Table S2 indicated that the estimation results between Mplus and CGS were very similar by Cohen's κ ($\kappa = .894$, 95%CI = [.881, .907]). In addition, the contingency table between Mplus and JMLE shown in Table S3 indicated a similar tendency to Table 10 that showed estimation agreement between the CGS and JMLE results. The Cohen's κ value between Mplus and JMLE was 0.401 (95%CI = [.382, .420]). From these results, the CGS result was more similar to Mplus rather than JMLE, and JMLE was different

from both CGS and Mplus. Main cause of this result is likely the model behind the estimation method. The CGS and Mplus assumed general DCM or LCDM and the JMLE assumed the DINA model. The Mplus result was equivalent to that reported by Templin and Bradshaw (2014), and whole pattern estimation result with CGS was very similar to Templin and Bradshaw (2014). From these results, we concluded that the CGS algorithm provided results consistent with other estimation methods.

6. Discussion

This study proposed a novel CGS algorithm for general DCMs. The CGS algorithm is a Bayesian estimation procedure that avoids sampling DCM item and structural model parameters. Comparisons between the CGS algorithm and non-parametric methods showed that the CGS algorithm could more precisely recover attribute mastery patterns in many cases and that the CGS algorithm provided classification results very close to the non-parametric methods in the remaining cases. Further, the CGS algorithm accurately estimated examinee attribute mastery patterns in conditions where the data generation model was subsumed by a general DCM, implying that the CGS algorithm may be useful in a wide range of situations. The second simulation showed the CGS algorithm provided better or similar classification accuracy as other MCMC sampling or ML methods. A real data analysis demonstrated that the CGS algorithm showed different classification results from JMLE that were, however, consistent with results from previous studies.

A clear benefit of the CGS algorithm is that it does not require DCM item and structural model parameters to be sampled. Moreover, as the CGS algorithm is Bayesian, we can use prior knowledge of an assessment to improve estimation accuracy. Such strategies may be appropriate in small sample size settings. One drawback of the CGS algorithm for DCMs may be the computational cost. This is because the algorithm must sequentially update each examinee's class and each iteration includes for-loops for classes and items to update parameters, procedures that make the algorithm difficult to vectorize. Even with such disadvantages, the second

simulation study showed that the CGS algorithm provided better estimates of attribute mastery in fewer iterations than JAGS.

The CGS algorithm was different not only from previous MCMC methods but also from the method proposed by Ma and Jiang (2021). Ma and Jiang's (2021) method assumed priors for item parameters and estimated the MAP parameters within the GDINA framework. As such, the Ma and Jiang (2021) method was intended to gain point estimation of model item and structural parameters. The CGS algorithm does not estimate item or structural model parameters as they are marginalized across. Therefore, point estimation-oriented approaches and the CGS algorithm have different purposes and they are not directly comparable. The CGS algorithm, however, provides a new data analysis method within the context of DCMS. Especially, the derivation of the CGS algorithm demonstrated that estimation of item and structural model parameters were not necessary to for estimation of attribute mastery statuses, a new perspective that has not been considered in a DCMS context. This perspective may lead to the development of more new estimation methods in DCMS. For example, a method to partially collapse and estimate parameters should also be able to be developed. This study also started to solve the boundary problem by not estimating the parameters that caused the issue. Therefore, a main contribution of this paper was to provide a novel estimation method that reduced the dimensional complexity needed in Gibbs sampling algorithms by omitting item and structural model parameters.

In the first simulation study, performance of the CGS algorithm was highly stable among attributes, especially when data were generated from a general DCM. In the derivation of the CGS algorithm, general DCMS were assumed. In the CGS algorithm, the item and structural parameters are not randomly sampled but their working parameters ($a_{/e',ip}^*$, $b_{/e',ip}^*$ and $\delta_{/e',c}^*$) are similar to the parameters of conditional distribution for a Gibbs sampling algorithm in the general DCMS (Yamaguchi & Templin, 2022). These parameters are summaries of item and structural parameters based on current attribute mastery patterns, so the working parameters help to distinguish item responses with multiple attributes among assumed attribute profiles.

On the other hand, the JMLE algorithm assumed the DINA model for ideal responses, which distinguishes only two classes for each item. Therefore, the JMLE can employ less item information than the CGS method. In addition, item responses far from the DINA ideal responses may increase when the number of examinees mastering some attribute profiles is small because attribute patterns with partial mastery were classified into all non-mastering patterns under the DINA model. Under conditions with data generated from general DCMs, examinees may correctly answer items even if they partially master the attributes required for the items. This mismatch between ideal response and actual response can cause over estimation of attribute mastery in JMLE, which assumes the DINA model. Moreover, this result is not due to estimation method but model. In other words, this mismatch causes some examinees who are not masters of any attributes to be mistakenly classified as mastering of all attributes. This over-estimation phenomenon was also shown in the real data example: the JMLE algorithm tended to classify more into a mastering status than the CGS algorithm.

Inversely, when the data generating process is the DINA model, the CGS algorithm sometimes showed underperformance to the JMLE algorithm. Because the CGS algorithm assumed an overly parameterized model, this may have caused overfitting of the item responses with sampling error in the small sample size and low-quality items conditions. On the contrary, the JMLE algorithm could employ just enough information and ignore random error just like implicit regularization methods in such conditions. Therefore, these parameters might help to improve the accuracy of attribute profile probabilities in the CGS algorithm if the data generating process follows a general DCM. These may be why the CGS algorithm outperformed the JMLE algorithm when the data generating procedure follows general DCMs.

The primary motivation to develop the CGS algorithm was to deal with the boundary problem that occurs in unstable parameter estimation situations such as small sample sizes. The CGS algorithm, by not estimating item or structural model parameters, avoids the boundary problem entirely. As shown by our simulation study, however, the CGS algorithm sometimes gave underperforming results, particularly when the data generating process did not match the

assumed model of the CGS. As mentioned, the CGS algorithm used working parameters summarizing item responses to generate random samples of attribute profiles. Therefore, the instability of item and structural parameters could affect the CGS algorithm. In other words, the CGS algorithm does not completely overcome problems encountered in small sample size situations. However, the second simulation indicated the accuracy of attribute classification of the CGS algorithm was similar to or better than the other estimation methods across various conditions. From this evidence, we believe the CGS method can be considered superior to other estimation methods with respect to its performance in small sample size situations.

In the simulation studies, the proposed CGS method sometimes outperformed the other estimation methods not only in the small sample size conditions but also in the large sample size conditions. In the CGS method, a conditional distribution of the individual attribute mastery profile was given Equation 13. Equation 13 is conditioning on all individual item responses $\mathbf{X}$ and the other individual latent attribute profiles $\mathbf{Z}_{/e}$. This means that an individual attribute profile is predicted not only by the individual item response pattern but also by the other individuals item response patterns and latent attribute profile status. This may one reason that the CGS method showed high performance in large sample size settings.

Another feature of the CGS method is that it is a fully Bayesian method so priors can be employed to consider previous information to improve estimation accuracy and stability even if the CGS method does not estimate item and structural parameters. The ML method does not implement such priors (although MAP methods do). In addition, to use priors in other MCMC methods, we must generate random samples of item and structural parameters. The CGS algorithm can use estimated parameters without fixing them. In other words, we can keep the uncertainty of item and structural parameters in the estimation procedure without estimating them. Neither ML nor other MCMC methods have this property. In the case of the real data analysis with small sample sizes, it may be important to include prior information.

We also note that the modeling framework of CGS can be expressed using other DCMS, such as DINA model, by modifying the G-matrix. Therefore, if the model was already

known (such as in an operational large-scale assessment), CGS estimation can be used with minimal modifications. From our simulation study, misspecification of the analysis model can result in inaccurate and/or unstable attribute mastery estimates so the model checking procedure is important to improve attribute mastery estimation. One direction for future research is how such mismatching models affect attribute mastery estimation with the CGS method. Our estimation code is available on OSF so it can be modified and employed for future studies.

The second simulation investigated how frequently the boundary problem happens in various conditions and found this issue occurred relatively often. Small sample sizes or eight attribute conditions tended to show higher boundary problem rates than larger sample size or four attribute conditions. However, attribute recovery rates in small sample size or eight attribute conditions was not reduced if the quality of items was high. The item quality conditions affected attribute recovery rates but did not cooccur with the boundary problem. The relationship between the boundary problem and attribute recovery was not clear in this study. Previous studies also did not clearly mention the relationship between boundary problems and estimation of individual parameters. This might be because if estimation of item parameters estimation, the person parameter estimates were not able to be obtained. In addition, previous studies mainly focused on the item parameter recovery and SE estimation. The item parameters are important to conduct statistical tests for the item parameters and Q-matrix validation (Ma & Jiang, 2021) so the person parameters have not been considered in this context. This study provided new estimation procedure to skip estimate of item parameters so a detailed investigation of the relationship between boundary problem and person parameters was not conducted. However, future research should investigate the impact of the boundary problem on the estimation of attribute mastery profiles with both theoretical and empirical ways.

Galindo-Garre and Vermunt (2006) mentioned that the boundary problems generate numerical computational problems in the calculation of asymptotical SEs for model parameters. Some software for DCMS have hidden tuning parameters to avoid boundary problems so the parameter estimates do not stick to their bound but stick to the newly defined bounds that were

defined by the tuning parameters. Therefore, the users may not realize that their parameter estimates have boundary problems. In such cases, the results are based on problematic item parameter estimates, drawing into question the person parameter estimates. Moreover, if the parameters strictly stick to their parameter bounds, the optimization procedure for parameter estimation will fail. For example, Mplus stops estimation and does not provide the person parameters (attribute mastery patterns) if the optimization procedure failed.

Further complicating the issue, we do not know whether true parameters are *close to* or *exactly* on their boundaries or if their estimates stick to the boundaries because of sampling error, however, we showed the boundary issue is reduced as sample sizes increase. The situation that the parameters are on the boundaries of their parameter space may occur when all students who do not master any attributes measured by the item incorrectly answer the item (or all correctly answer the item). If a test has items with boundary problems, it may indicate the quality of items may be questionable or that estimation failed for some reason. Therefore, it may indicate that the diagnostic information from the test is not trustworthy.

The second simulation study compared the CGS method not only to the JAGS MCMC algorithm but also to standard ML methods. We mention one additional note from the results of the second simulation. The ML method showed high CCRs even when boundary problems occurred. However, the ML method was conducted using an extensively developed R package. The GDINA package implements an arbitrary limit to prevent item parameter estimates from getting close to their boundaries. This may be one reason for such stable estimates. If we employed an ML method without this limit, the log likelihood function would likely approach negative infinity when the item or structural parameters were close to zero. Therefore, the performance of the ML method in the second simulation is specific to the GDINA package and not all ML algorithms. This notion is also applied to the JMLE results in the first simulation.

JAGS was selected as a reference method in the second simulation because the CGS algorithm was an MCMC method and a comparison between MCMC methods was required. Moreover, JAGS code of the LCDM has already been provided in a previous study (Zhan et al.,

2019) so the code can be easily assessed. MAP estimation is based on the same line of ML methods because it optimizes the product of likelihood and prior distributions. On the other hand, MCMC based methods, sometimes called fully Bayesian methods, approximate posterior distributions using random samples. Therefore, MCMC-based methods are generally slower than optimization-based estimation methods which only seek to find a global maximum of a likelihood or loss function. Therefore, MAP methods are faster than the CGS algorithm. However, this fact does not decrease the importance of MCMC methods because MCMC can correctly estimate posterior distributions if the number of MCMC samples increase but optimization-based method cannot. Therefore, Ma & Jiang's (2021) MAP method should be compared to similar optimization-based Bayesian methods such as variational inference (e.g., Yamaguchi & Okada, 2021) rather than MCMC-based methods. In addition, the MAP estimates are sensitive to the prior (Garriand-Garre & Vermunt, 2006). However, the CGS marginalizes out these parameters, so we do not evaluate their parameter estimates. The simulation study provided supported that the CGS algorithm can recover true attribute mastery statuses. The CGS algorithm has already been compared against a set of established estimations methods, so additional detailed comparisons or speed tests are left for future research.

As a general note on the Bayesian methods, marginalization is not unique to the CGS algorithm but the CGS algorithm explicitly distinguishes nuisance parameters and parameters of interest. Of note here, however, is that while averaging over item parameters is done for, or a part of, some parameter summaries in other Bayesian algorithms, the dimensionality of these algorithms—the number parameters sampled—is different when compared to the CGS algorithm. As such, the CGS algorithm needs less time for convergence for the parameters it samples from, a benefit that is achieved at a trade-off: Less information from the parameters that are not sampled. On the other hand, usual Gibbs sampling algorithms sample hidden parameters even if they are not interest. It is sometimes said that all Bayesian methods should provide the same results. However, this is not always true. For example, if posterior distributions are not symmetric, the MAP and EAP estimates take different values so the result of traditional MAP

and MCMC methods can be incongruous. Moreover, different Bayesian estimation methods such as variational Bayesian methods (e.g., Yamaguchi & Okada, 2021) provide similar point estimates but the posterior SDs are generally a little smaller than MCMC methods. If the data generating function is simple and a large amount of data is available, posteriors may be able to be approximated by simple normal distributions (i.e., where they converge in distribution, asymptotically), then the parameter estimates are not as different. However, such cases are not always found in analyses of real data. Furthermore, considering constraints among parameters (e.g., monotonicity constraints) also affects parameter estimates. In sum, various factors have an effect on the parameter estimates.

In the real data analysis, the CGS algorithm and the JMLE algorithm overall had somewhat diverging levels of attribute classification consistency, as measured by Cohen's κ . The results may, however, be similar in some ways. This is because the number of attributes in the ECPE data is small. Moreover, the Q-matrix was not overly complex, so differences in the models underlying the two algorithms might be small. We find it interesting that the attribute mastery patterns with CGS almost always underestimated the number of attributes mastered when compared to JMLE. In other words, the combination of non-mastery in the CGS algorithm and mastery in the JMLE algorithm frequently happened more than the combination of mastery in the CGS algorithm and non-mastery in the JMLE algorithm (see the off-diagonal cells in Table 5-8). This might indicate that CGS judged mastery relatively more conservatively than JMLE. One reason for this result is that JMLE assumes DINA-type ideal responses so it may estimate extreme results for cases where DINA does not fit the data. On the other hand, the CGS algorithm could better distinguish more attribute mastery profiles than the JMLE algorithm, as shown by the numbers classified across all profiles.

The real data we used was well known and with relatively stable results from many previous studies. Using a data set where boundary problems happened and comparing various estimation methods might be appropriate for future study. However, in real data analysis settings, not only is the appropriate model not obvious but also the number of attributes is not obvious.

The estimation results are affected by both model specification and estimation methods. Therefore, if we had observed different estimation results between the CGS algorithm and conventional estimation methods, it might be difficult to conclude the results of CGS algorithm are always better. In the simulation study, when the data generating model was different, we observed some attribute mastery recovery rates with the CGS algorithm were slightly worse than the rates for the JMLE algorithm (see Figure 4). As such, we used a well-known data set that had been analyzed many times so we could compare our findings to those already established in the research literature. The ECPE data was employed as a data analysis example of LCDM and its Q-matrix structure is relatively simple. In short, we thought the ECPE data is reliable for the purpose of comparing estimation methods. If the CGS algorithm *did not* provide similar results with the conventional ML estimation methods in the well-known simple real data analysis case, we would not be able to rely on the estimates of the CGS algorithm. However, as we expected, the results of the CGS and conventional ML estimates were congruent, so the CGS provided valid estimates, at least in the ECPE data, from a conservative perspective. The CGS algorithm proposed in this paper is based on a new perspective so further application examples of the CGS are needed to understand the utility and efficiency of the new estimation method. Gathering empirical evidence of the utility of CGS method is future research topic.

We also need to mention the situations where the CGS algorithm can be used. The CGS algorithm can be employed in situations where the estimates of the item and structural model parameters are not important. That is, situations where attribute mastery estimation is the only part needed. An example use case could be when data come from a well-known test that uses similar items to previously-developed tests, we may be able to avoid the estimation procedure of the item and structural model parameters. Another example is the case where we have available previous point estimates of item and structural model parameters, but large SEs exist (i.e., when sample size is small). In this case, we may hesitate to fix the pre-estimated parameters with past item response data because their SEs are large, which implies that the estimates are unstable. Moreover, current available data is small so the point estimates based on the data may not be

stable. In such cases, the CGS algorithm uses priors but does not estimate the parameters. Using such priors can contribute to improve estimates attribute mastery patterns. In summary, the CGS algorithm can be employed cases where detailed information of items or structural model parameters is not important or cases where we can use prior estimates but do not need to re-estimate the parameters.

The CGS algorithm does not explicitly provide of model fit indices. However, the item and structural model parameters can be recovered by the output of the CGS, through a process using the Markov chains of attribute mastery patterns. With such reconstructed parameters, various types of fit indices can be calculated. In addition, modifying the G_i matrix employs different DCMS so the item response functions are also testable.

One limitation of the CGS algorithm is a lack of clear convergence diagnostics. Most convergence evaluation methods available for MCMC algorithms were developed for continuous parameters (e.g., Brooks & Gelman, 1998; Cowles & Carlin, 1996; Geweke, 1992). However, in the CGS algorithm for DCMS, the MCMC samples are discrete—latent classes. Therefore, it is difficult to check the convergence of the algorithm. In this paper, following previous research (e.g., Blei & Jordan, 2006; Porteous et al., 2008), we did not evaluate the convergence of the CGS algorithm in either our simulation study or our analysis of the ECPE data. We suspect that a lack of convergence would result in more discrepant results, particularly in the simulation study. One possible remedy would be to run the CGS algorithm multiple times, different chain lengths and to check the similarity of the results from different MCMC samples. Another solution could be to use traditional convergence diagnostics on a moving average across a Markov chain for probabilities of latent class membership for each examinee. We leave this issue to future studies.

Our ad-hoc method to deal with the label switching problem has no mathematical verification (which is why we label it as ad-hoc). However, this procedure provided good classification rates in our simulation study, where we found results which indirectly indicated label switching was not an issue. As we mentioned previously, the label switching problem in DCMS is difficult to solve because the large number of classes in DCMS resulting in many

possible permutations of classes—features that make it difficult to use conventional remedies for label switching. We leave the discovery of label switching remedies for the CGS algorithm to the future research. Finally, one related direction is to develop the CGS algorithm with monotonicity constraints.

Another future direction of this CGS algorithm is an extension for unknown Q-matrix situations. Current CGS algorithm requires a known Q-matrix. However, the appropriate Q-matrix is sometimes unknown in real data analysis settings. If our CGS algorithm will be applicable for an unknown Q-matrix or in exploratory DCMS settings, the availability and usefulness of the DCMS will be increased.

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Table 1.

*Example of $\mathbf{G}_i$ matrix for $\mathbf{q}_i = [1,0]^\top$
(shaded part)*

			Attribute profile			
			$\alpha_1^\top$	$\alpha_2^\top$	$\alpha_2^\top$	$\alpha_3^\top$
Item specific profile			0	0	1	1
			0	1	0	1
$\alpha_1^{(i)}$	0	*	1	1	0	0
$\alpha_2^{(i)}$	1	*	0	0	1	1

Table 2.

Four-attribute Q matrix (short test length condition)

Item	Attribute				Item	Attribute			
	1	2	3	4		1	2	3	4
1	1	0	0	0	16	1	0	1	0
2	0	1	0	0	17	1	0	0	1
3	0	0	1	0	18	0	1	1	0
4	0	0	0	1	19	0	1	0	1
5	1	0	0	0	20	0	0	1	1
6	0	1	0	0	21	1	1	1	0
7	0	0	1	0	22	1	1	0	1
8	0	0	0	1	23	1	0	1	1
9	1	1	0	0	24	0	1	1	1
10	1	0	1	0	25	1	1	1	0
11	1	0	0	1	26	1	1	0	1
12	0	1	1	0	27	1	0	1	1
13	0	1	0	1	28	0	1	1	1
14	0	0	1	1	29	1	1	1	1
15	1	1	0	0	30	1	1	1	1

Table 3.

Eight-attribute Q matrix (short test length condition)

Item	Attribute								Item	Attribute							
	1	2	3	4	5	6	7	8		1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0	16	0	0	0	0	0	0	0	1
2	0	1	0	0	0	0	0	0	17	1	1	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	18	0	0	1	1	0	0	0	0
4	0	0	0	1	0	0	0	0	19	0	0	0	0	1	1	0	0
5	0	0	0	0	1	0	0	0	20	0	0	0	0	0	0	1	1
6	0	0	0	0	0	1	0	0	21	1	0	0	0	1	0	0	0
7	0	0	0	0	0	0	1	0	22	0	1	0	0	0	1	0	0
8	0	0	0	0	0	0	0	1	23	0	0	1	0	0	0	1	0
9	1	0	0	0	0	0	0	0	24	0	0	0	1	0	0	0	1
10	0	1	0	0	0	0	0	0	25	1	1	1	1	0	0	0	0
11	0	0	1	0	0	0	0	0	26	0	0	1	1	1	1	0	0
12	0	0	0	1	0	0	0	0	27	0	0	0	0	1	1	1	1
13	0	0	0	0	1	0	0	0	28	1	0	1	0	1	0	1	0
14	0	0	0	0	0	1	0	0	29	0	1	0	1	0	1	0	1
15	0	0	0	0	0	0	1	0	30	1	1	0	0	0	0	1	1

Table 4.

Average boundary problem rates with maximum likelihood estimation method with GDINA package in simulation 2

Sample Size	Item Type	Four Attributes		Eight Attributes	
		Short Test	Long Test	Short Test	Long Test
100	High-quality items	.319	.337	.398	.392
	Low-quality items	.216	.152	.366	.280
1000	High-quality items	.123	.112	.173	.169
	Low-quality items	.141	.095	.253	.172

Table 5.

Average and SD of estimation times (minutes) of collapsed Gibbs sampling (CGS) and Gibbs sampling with JAGS language (JAGS) in simulation 2

# Attribute	Sample Size	Item Type	Test Length	CGS	(SD)	JAGS	(SD)
4	100	High-quality	Short	1.949	(0.055)	9.225	(1.187)
		items	Long	3.627	(0.185)	24.377	(2.67)
		Low-quality	Short	1.981	(0.106)	12.205	(0.304)
		items	Long	3.493	(0.306)	26.696	(2.49)
	1000	High-quality	Short	17.279	(1.683)	181.383	(12.788)
		items	Long	36.416	(2.685)	369.601	(49.26)
		Low-quality	Short	20.152	(1.085)	167.401	(13.247)
		items	Long	38.223	(0.842)	368.968	(41.761)
8	100	High-quality	Short	3.810	(0.147)	94.028	(3.282)
		items	Long	7.077	(0.667)	211.093	(44.575)
		Low-quality	Short	3.841	(0.314)	105.476	(9.858)
		items	Long	7.707	(0.393)	188.945	(58.942)
	1000	High-quality	Short	39.383	(4.312)	835.594	(204.549)
		items	Long	76.705	(6.438)	2170.985	(440.507)
		Low-quality	Short	40.777	(0.639)	764.524	(219.266)
		items	Long	80.247	(1.524)	1298.675	(250.775)

Note. The number of iterations of GCS was 2000 and JAGS was 2,0000.

Table 6.

Q-matrix of ECPE data

Item	Attribute			Item	Attribute		
	Morphosyntactic rules	Cohesive rules	Lexical rules		Morphosyntactic rules	Cohesive rules	Lexical rules
1	1	1	0	15	0	0	1
2	0	1	0	16	1	0	1
3	1	0	1	17	0	1	1
4	0	0	1	18	0	0	1
5	0	0	1	19	0	0	1
6	0	0	1	20	1	0	1
7	1	0	1	21	1	0	1
8	0	1	0	22	0	0	1
9	0	0	1	23	0	1	0
10	1	0	0	24	0	1	0
11	1	0	1	25	1	0	0
12	1	0	1	26	0	0	1
13	1	0	0	27	1	0	0
14	1	0	0	28	0	0	1

Table 7.

Contingency table of morphosyntactic rules mastery with collapsed Gibbs sampling (CGS) and joint maximum likelihood estimate (JMLE)

CGS \ JMLE	Non-master	Master	Sum
Non-master	1087	757	1844
Master	0	1078	1078
Sum	1087	1835	2922

Table 8.

Contingency table of cohesive rules mastery with collapsed Gibbs sampling (CGS) and joint maximum likelihood estimate (JMLE)

CGS \ JMLE	Non-master	Master	Sum
Non-master	707	570	1277
Master	182	1463	1645
Sum	889	2033	2922

Table 9.

Contingency table of lexical rules mastery with collapsed Gibbs sampling (CGS) and joint maximum likelihood estimate (JMLE)

CGS \ JMLE	Non-master	Master	Sum
	Non-master	Master	Sum
Non-master	656	271	927
Master	9	1986	1995
Sum	665	2257	2922

Table 10.

Contingency table of all attribute mastery pattern with collapsed Gibbs sampling (CGS) and joint maximum likelihood estimate (JMLE)

CGS \ JMLE	Pattern								Sum
	000	001	010	011	100	101	110	111	
000	200	122	194	101	98	30	151	11	907
001	1	112	0	100	0	122	0	12	347
010	0	0	2	3	0	0	6	2	13
011	0	1	1	250	0	44	5	276	577
100	0	0	0	0	0	1	1	0	2
101	0	0	0	0	0	21	0	0	21
110	0	0	0	0	0	0	4	1	5
111	0	0	0	0	0	137	2	911	1050
Sum	201	235	197	454	098	355	169	1213	2922

Note: Cohen's $\kappa = .382$ (95% CI = [.364, .401])

Figure 1

Simulation 1 results for general DCM data generation with four attributes. CCRA = correct classification rate, CCRP = correct classification rate for whole pattern, CGS = collapsed Gibbs sampling algorithm, JMLE = joint maximum likelihood method with NPCD package, and error bars representing SEs.

Figure 2

Simulation 1 results for DINA model data generation with four attributes. CCRA = correct classification rate, CCRP = correct classification rate for whole pattern, CGS = collapsed Gibbs sampling algorithm, JMLE = joint maximum likelihood method with NPCD package, and error bars representing SEs.

Simulation 1 results for general DCM data generation with eight attributes. CCRA = correct classification rate, CCRP = correct classification rate for whole pattern, CGS = collapsed Gibbs sampling algorithm, JMLE = joint maximum likelihood method with NPCD package, and error bars representing SEs.

Simulation 1 results for DINA model data generation with eight attributes. CCRA = correct classification rate, CCRP = correct classification rate for whole pattern, CGS = collapsed Gibbs sampling algorithm, JMLE = joint maximum likelihood method with NPCD package, and error bars representing SEs.

Figure 5

Simulation 2 results for general DCM data generation with four attributes. CCRA = correct classification rate, CCRP = correct classification rate for whole pattern, CGS = collapsed Gibbs sampling algorithm, JAGS = Gibbs sampling with JAGS language, ML = maximum likelihood estimates with GDINA package, and error bars representing SEs.

Figure 6

Simulation 2 results for general DCM data generation with eight attributes. CCRA = correct classification rate, CCRP = correct classification rate for whole pattern, CGS = collapsed Gibbs sampling algorithm, JAGS = Gibbs sampling with JAGS language, ML = maximum likelihood estimates with GDINA package, and error bars representing SEs.