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2 **On the Mathematical Foundations of Colorimetry:**

3 **Color-matching, Object Colors and Optimal Colors**

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ABSTRACT

The psychophysical description of one of the pillars of color science, the color-matching paradigm, forms the basis of the representation theory of colors. This description entails the weighted integration of a spectral distribution of radiant energy (i.e. a light ray) with three color-matching functions. Here these color-matching functions are conceptualized as a vector-valued measure ϕ , which grounds the representation theory of colors in the mathematical frameworks of measure theory and functional analysis. Properties like the convexity and compactness of the tristimulus space follow readily from this *color-matching measure* model. Notably, the model quite naturally encapsulates metamerism as ϕ -equivalence classes of spectral distributions. A subsequent extension of the model to object colors results in similar properties for the object color solid. Also following from this extension is a novel expression for optimal colors. This entails both a comprehensive mathematical formulation of these colors as well as a practical calculation tool for their reflection coefficients.

ON THE MATHEMATICAL FOUNDATIONS OF COLORIMETRY:

COLOR-MATCHING, OBJECT COLORS AND OPTIMAL COLORS

Colorimetry studies color in isolation, also known as aperture color, stripped of the contextual influences which are typically present in a naturalistic scene. The canonical measurement instrument of colorimetry is the color-matching experiment (Wyszecki & Stiles, 2000; see also Koenderink & van Doorn, 2000). In such an experiment, two singleton rays of light (the colors to be matched) excite only a small part of the retina (effectively only the fovea). In particular this is the case in the 2° split visual field color-matching experiment. It is assumed that the input to the visual system in a single trial of a color-matching experiment can be characterized completely by the light rays' spectral distributions of some radiometric unit directly related to the radiant energy, such as luminance or lightness. Psychophysical color can then be expressed in terms of *tristimulus values*, connected to every light ray, which are geometrically interpreted as coordinates in a three-dimensional real space (Wyszecki & Stiles, 2000).

This approach forms the basis of the representation theory of color perception, and finding suitable mathematical descriptions for it is a scientific endeavor that continues to date (e.g., Logvinenko, 2009; Romney & Chiao, 2009; Rizzi, 2021). The particular description presented in this paper seeks to ground it in measure theory and functional analysis. Several well-known properties of the representation theory for aperture colors follow in a straightforward way from basic theorems of these two mathematical frameworks. Analogous properties of the representation theory for object colors are derived, based on an extension of the basic model. As a corollary, this also leads to an algorithm for calculating the functional form of reflection-coefficients of optimal colors. For purposes of manuscript readability and compactness,

preliminaries, additional results and proofs have been placed in Appendices B to D, respectively, instead of in the main text. In addition, some more general mathematical background is presented in Appendix A.

COLOR-MATCHING

As is explained in Appendix B, all spectral distribution functions that correspond to realistic colors form a convex cone $\mathbf{K}$ in the function space $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$. It was also shown there that the peripheral visual system can mathematically be described by a column vector-valued system consisting of three bounded linear functionals $\mathfrak{G}(\cdot) = (\mathfrak{G}_L(\cdot), \mathfrak{G}_M(\cdot), \mathfrak{G}_S(\cdot))^T$ acting on $\mathbf{K}$. More relevant for colorimetry however, are the related tristimulus functionals $\{\Phi_X, \Phi_Y, \Phi_Z\}$ that generate tristimulus vectors Φ during color-matching with primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. The associated color-matching functions $x(\lambda)$, $y(\lambda)$ and $z(\lambda)$, which are linear combinations of the cone fundamentals, are bounded (equivalence-class) functions in the space $L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$.

COLOR-MATCHING MEASURE

The core of the model presented here is an alternative interpretation of the color-matching functions $x(\lambda)$, $y(\lambda)$ and $z(\lambda)$, which naturally follows from the formulation above (see Appendix B). For this, the following vector-valued set function will be introduced first:

$$\begin{aligned} \phi: \quad \mathbf{B}(\Lambda) &\rightarrow \mathbb{R}^3; \\ d\lambda &\mapsto \phi(d\lambda) \equiv (x(\lambda), y(\lambda), z(\lambda))^T. \end{aligned} \tag{1}$$

This set-function obeys all the properties of a *non-atomic vector-valued measure*. It connects with every Borel set $d\lambda$ of the visual spectrum a three-dimensional column vector $\phi(d\lambda)$. This can be understood in the sense of the Radon-Nikodým theorem (see, e.g., Bennett & Lehman, 2001, Section 4) applied to each of the three tristimulus functionals, and by constructing from the resulting measures this vector-valued measure. Because of its relevance for the description of color-matching, this will be called the *color-matching measure*. The integral of a spectral distribution function $e_P(\lambda)$, with respect to this color-matching measure, equals the tristimulus vector $\Phi(\mathbf{P})$ that is appointed to the corresponding color $\mathbf{P}$:

$$\Phi(\mathbf{P}) = \int_{\Lambda} e_P(\lambda) \phi(d\lambda). \quad (2)$$

As a result of the measure theoretical interpretation of the color-matching functions, a connection has arisen between the convex cone $\mathbf{K}$ in the space $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$ of all spectral distribution functions and the visual system (i.e. its sensorial and transducer properties) in a particular color-matching setting. In other words, a connection between the colors that are described by their physical properties and the colorimetric properties of the visual system.

The color-matching measure ϕ defines a novel structure on the space of all spectral distribution functions. This structure consists of equivalence classes of ϕ -equivalent, integrable functions on the visual spectrum Λ . The space with this structure, generated by the color-matching measure ϕ , will be denoted by $L^1(\phi)$. The equivalence classes each consist of all the spectral distribution functions of colors which are mapped onto the same tristimulus vector. These are precisely all

the metamer colors belonging to that particular tristimulus vector. As a result, the space $L^1(\phi)$ has a mathematical structure that is defined by the colorimetric (color-matching) properties of the human visual system, whereas the space $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$ has a structure that is purely defined by the physics of light. Note that belonging to each equivalence class are also all the non-realistic ‘colors’. Because of this, the model will be restricted to the convex cone $\mathbf{K}$. As a result of the color-matching measure ϕ it has the same structure and will be denoted as $\mathbf{K}_\phi$. It is important to mention that the ϕ does not change the elements themselves. It merely defines another structure on the same space, related to the metamer equivalence classes. The space therefore has the same elements, but is now divided into groups that are formed according to the properties of the visual system. These groups, therefore, are directly related to the ‘color-appointing’ properties of color vision.

COLOR-MATCHING STIMULI

In order to further develop the model introduced above, a set of relevant stimuli for color-matching is necessary, which is experimentally fully accessible and at the same time is large enough to be theoretically interesting. This set is defined by:

$$\mathbf{K}_E = \{e \in L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda) \mid e(\lambda) \geq 0, \forall \lambda \in \Lambda ; \|e\| \leq E\}. \quad (3)$$

This means that $\mathbf{K}_E \subset \mathbf{K}$ is the convex subset of all non-negative spectral distribution functions with a norm less than or equal to some constant E . The parameter E is, according to the definition of the norm, directly related to the energy content of the colors. The set $\mathbf{K}_E$ arises from a plausible demand, namely that all the colors that can be used in a (realistic) experimental

setting have an energy content that is bounded by some fixed maximum. The actual value of this maximum energy is of course something the experimenter controls. But other than that it holds as a constant within each specific experimental setting. It is thus by no means a restriction on the experimental possibilities, but rather an obvious condition that will turn out to be mathematically advantageous. Note that because it is a physical condition, the structure on the space $L^1(\Phi)$ is not altered by it, nor on all its subsets for that matter.

REPRESENTATIONAL COLOR-SPACE

The set of all tristimulus vectors $\Phi(\cdot)$, corresponding to realistic colors that are represented by spectral distribution functions $e(\lambda)$ from the convex set $\mathbf{K}_E$ (i.e., with maximum energy content directly related to E) is defined as the tristimulus space $\mathbf{T}$. Symbolically:

$$\mathbf{T} \equiv \{\Phi \mid e \in \mathbf{K}_E\} = \Phi(\mathbf{K}_E). \quad (4)$$

In other words, the tristimulus space $\mathbf{T}$ is defined as the three-dimensional space representing all realistic colors with a bounded energy-content. It has an interesting structure that follows directly from the model's foundation in measure theory and functional analysis. This structure can be presented most conveniently by ways of a number of mathematical propositions. Each of the propositions correspond to a specific property of the representation theory of aperture colors. In that sense they therefore also represent properties of the chromatic visual system.

Property T_1 *The tristimulus space $\mathbf{T}$ is a convex and compact subset of the three-dimensional real space $\mathbb{R}^3$.[†]*

This property is based on Liapunov's theorem on the integration-mapping and on the definition of the Liapunov measure (see Appendix A; see also e.g., Hermes & LaSalle, 1969; Kluvánek & Knowles, 1975). It is a direct consequence of the fact that the color-matching functions as a triple constitute the non-atomic vector-valued color-matching measure $\phi : \mathbf{B}(\Lambda) \rightarrow \mathbb{R}^3$.

Property T_2 *For every tristimulus vector Φ in the tristimulus space $\mathbf{T}$ a spectral distribution function $e(\lambda)$ can be found with the functional form of a characteristic function on the visual spectrum Λ .[‡]*

The set of all extreme elements of the tristimulus space $\mathbf{T}$ is denoted by $\partial_E(\mathbf{T})$. For the extreme elements of the tristimulus space the **Property T_2** can be further specified:

Property T_3 *The extreme elements $\partial_E(\mathbf{T})$ of the tristimulus space $\mathbf{T}$ each correspond to a unique spectral distribution function $e(\lambda)$ with the functional form of a characteristic function on the visual spectrum Λ .*

This property is equivalent with the theorem of extreme points of the range of a Liapunov integration-mapping (see Appendix A; see also e.g., Kluvánek & Knowles, 1975). As will be

[†] An equivalent property will be derived for the chromaticity diagram $\mathbf{C}$ in Appendix C.

[‡] This property is analogous to the famous Bang-Bang principle from control theory (Hermes & LaSalle, 1969; Kluvánek & Knowles, 1975).

shown in the next section, an analogous property for the representation theory for object colors can be derived.

OBJECT COLORS

In this section the color-matching measure model for aperture colors, as described above will be extended to capture object colors. It will become clear that this offers an attractive formalization of the representation theory for object colors as well. The properties that follow from the extension of the model are analogous to the corresponding properties for aperture colors, but obviously have a different color-theoretical meaning. Fundamentally however, in both versions of the model, the results follow directly from the measure theoretical grounding of colorimetry. In order to construct the extended model it is necessary to first introduce a suitable interpretation of reflection coefficients.

REFLECTION COEFFICIENTS

The radiant field arriving at the retina after it has been reflected at the surface of an object, is altered by the electromagnetic properties of the surface. This physical process is formally described by applying Maxwell's law of electromagnetism on the combined system of light and matter (see, e.g., Jackson, 1975). Briefly, the reflective and refractive properties of the surface depend in the general case on the incident angle and spectral specification of the incident light ray, as described by Fresnel's law.

For perfectly diffuse reflecting (i.e. Lambertian), non-emitting surfaces, the reflection is completely described by a single coefficient, which is wavelength dependent: the reflection

coefficient $r(\lambda)$. Because reflection coefficients are bounded functions that only take values in the interval $[0,1]$ on the visual spectrum Λ , we can denote the set of all reflection coefficients as follows:

$$\mathbf{R} = \{r \in L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda) \mid 0 \leq r(\lambda) \leq 1, \forall \lambda \in \Lambda\}. \quad (5)$$

The set $\mathbf{R}$ is a convex subset of $L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$, and is the mathematical representation of all Lambertian surfaces.

OBJECT COLORS

The straightforward colorimetric setup for investigating object colors can be described mathematically by combining the reflection coefficient $r(\lambda) \in \mathbf{R}$ associated with a Lambertian surface with the spectral distribution function $I(\lambda)$ of an illuminant $\mathbf{I}$. As before, $I(\lambda)$ is an element of the convex set $\mathbf{K}_E$ of all spectral distribution functions with a total energy content less than or equal to a (experiment dependent) maximum E . Note that $\mathbf{I}$ is assumed to be fixed in a specific experiment and to illuminate the scene uniformly. The ‘surface-adjusted’ light ray arriving at the retina is thus given by the resulting spectral distribution function $r(\lambda)I(\lambda)$, representing the reflected illuminant $\mathbf{rI}$. The tristimulus vector $\Phi(\mathbf{rI})$ associated with this is given by the operation of the system of color-matching functionals:

$$\Phi(\mathbf{rI}) = \int_{\Lambda} r(\lambda)I(\lambda) \phi(d\lambda). \quad (6)$$

Since the spectral distribution function $I(\lambda)$ of the illuminant $\mathbf{I}$ is taken to be fixed in a given experiment, it can be said that the reflection coefficient $r(\lambda)$ determines the object color. In other words, the color a participant assigns to an observed object in the experiment is ultimately determined by how the spectral energy distribution of the illuminant is reflected by the object's surface. In that sense, object-color perception for non-emitting Lambertian scattering under uniform illumination, can be seen as a process of 'gauging' surface reflection coefficients. From a mathematical point of view this interpretation is particularly attractive for colorimetry, as will become clear. Note that alternative views which put different interpretive emphasis on the illuminant or the surface are formally equivalent and do not provide other tristimulus specifications.

The system of color-matching functionals can now be extended by incorporating the critical (yet fixed) contribution of the illuminant $\mathbf{I}$. For each spectral distribution function $I(\lambda)$ associated with an illuminant $\mathbf{I}$ and each system of color-matching functionals $\{\Phi_X, \Phi_Y, \Phi_Z\}$ associated with a set of primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$, a system of so-called *object-color functionals* $\{\Psi_{\mathbf{I},\mathbf{X}}, \Psi_{\mathbf{I},\mathbf{Y}}, \Psi_{\mathbf{I},\mathbf{Z}}\}$ is defined, as follows:

$$\Psi_{\mathbf{I},\mathbf{X}}(r) = \int_{\Lambda} r(\lambda) \psi_{\mathbf{I},\mathbf{X}}(d\lambda), \quad (7)$$

and likewise for $\Psi_{\mathbf{I},\mathbf{Y}}(r)$ and $\Psi_{\mathbf{I},\mathbf{Z}}(r)$. Here the functions $\psi_{\mathbf{I},\mathbf{X}}(d\lambda)$, $\psi_{\mathbf{I},\mathbf{Y}}(d\lambda)$ and $\psi_{\mathbf{I},\mathbf{Z}}(d\lambda)$ are equal to $I(\lambda)x(d\lambda)$, $I(\lambda)y(d\lambda)$, $I(\lambda)z(d\lambda)$, respectively. In other words, these are the color-matching functions 'adjusted' by the illuminant $\mathbf{I}$. These Lebesgue functions are integrable and therefore

also belong to $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$. This system of object-color functionals assigns to every reflection coefficient $r(\lambda)$ in the set $\mathbf{R}$ a column vector in three-dimensional real space:

$$\begin{aligned} \Psi_{\mathbf{I}}: \mathbf{R} &\rightarrow \mathbb{R}^3; \\ r(\lambda) &\mapsto \Psi_{\mathbf{I}}(r) = \int_{\Lambda} r(\lambda) \psi_{\mathbf{I}}(d\lambda), \end{aligned} \tag{8}$$

with the (column) vector-valued measure $\psi_{\mathbf{I}}(d\lambda) = (I(\lambda)x(d\lambda), I(\lambda)y(d\lambda), I(\lambda)z(d\lambda))^T$. That is to say, the three-dimensional column vector $\Psi_{\mathbf{I}}(r)$ is the object color which has been assigned to a surface with reflection coefficient $r(\lambda)$.

It is important to note that each illuminant $\mathbf{I}$ corresponds to a different vector-valued measure $\psi_{\mathbf{I}}(d\lambda)$ (provided that $I(\lambda)$ is realistic), and therefore results in a different set of object-color functionals. In order to keep notation light, however, the index $\mathbf{I}$ denoting the illuminant will be omitted as from here.

OBJECT COLOR SOLID

The set of all object colors $\Psi(\cdot)$ of all reflection coefficients in $\mathbf{R}$ is called the object color solid $\mathbf{O}$, or symbolically:

$$\mathbf{O} \equiv \{\Psi \mid r \in \mathbf{R}\} = \Psi(\mathbf{R}). \tag{9}$$

In words: The object color solid $\mathbf{O}$ (formally: $\mathbf{O_I}$) is the representation space of all object colors (under illumination of illuminant $\mathbf{I}$). Next, two properties of the representation theory of object colors will be given, in the form of mathematical propositions of the object color solid. (Some of the proofs can be found in Appendix D.) The first property is analogous to **Property T_1** of the tristimulus space. The second property gives the relationship between the object color solid and the tristimulus space.

Property O_1 *The object color solid $\mathbf{O}$ for every illuminant is a convex and compact subset of the three-dimensional real space $\mathbb{R}^3$.*

Property O_2 *The object color solid $\mathbf{O}$ for every illuminant is a subset of the tristimulus space $\mathbf{T}$. That is, $\mathbf{O} \subset \mathbf{T}$.*

With respect to **Property O_2** , let $\mathbf{I}(\lambda) \in \mathbf{K_E}$ be the spectral distribution function associated with a chosen illuminant $\mathbf{I}$. The spectral distribution functions $r(\lambda)\mathbf{I}(\lambda)$ are therefore elements of $\mathbf{K_E}$, for all $r(\lambda) \in \mathbf{R}$. This follows from the fact that $r(\lambda)\mathbf{I}(\lambda) \leq \mathbf{I}(\lambda)$, for all $\lambda \in \Lambda$. The tristimulus vectors $\Phi(\mathbf{rI})$ of $r(\lambda)\mathbf{I}(\lambda)$ therefore all will be contained within the tristimulus space $\mathbf{T} = \Phi(\mathbf{K_E})$.

OPTIMAL COLORS AND SCHRÖDINGER COLORS

Optimal colors are object colors with maximum luminance for a given chromaticity specification. They are located on the surface of the object color solid. Schrödinger colors have ‘bivalent’ reflectance functions with only the values 0 and 1 on the visual spectrum and with no more than two transitions between these values. A famous theorem states that optimal colors are

identical to Schrödinger colors. This has been proven in several different ways over the years. The first was of course Schrödinger's own proof in 1920 (Schrödinger, 1920; see also Koenderink & Kappers, 1996). MacAdam constructed an alternative proof of the theorem in 1981 (see also West & Brill, 1983). MacAdam's is a typical color-theoretical proof, whereas Schrödinger's proof is constructed ad absurdum.

This section will show how the theorem follows directly from the mathematical setup of the color measure model applied to object colors. Moreover, within the model a simple and effective expression can be derived for the exact functional form of the reflection coefficient corresponding to an optimal color. To begin with, the bang-bang principle of control theory which also applies to color-matching is formulated in a slightly different way. Subsequently, said expression is derived and discussed, with the aid of a few auxiliary theorems. It will become clear that this expression implies Schrödinger's theorem.

First, a third property of the object color solid will be given:

Property O_3 *For every object color Ψ in the object color solid O there exists a reflection coefficient $r(\lambda)$ that has the functional form of a characteristic function on the visual spectrum Λ .*

Property O_3 thus says that every object color of a surface can be matched (in the sense of metamerism) with a surface that is color-typified by a simple reflection coefficient that is only zero or one on the entire visual spectrum. Note that this is not yet proof of the statement that the

optimal colors and the Schrödinger colors are equal, but that it does not contradict with that statement either. In a way, this statement is somewhat stronger. For each set of metameric object colors, at least one surface can be found with a reflection coefficient that meets Schrödinger's first requirement, not just for the optimal colors. It will now be investigated how to further specify **Property O_3** for those object colors that are on the surface $\partial(\mathbf{O})$ of the object color solid $\mathbf{O}$. That is, for the optimal colors, which will be labeled by Ψ_o .

Let Ψ_o be an optimal color. Optimal colors are object colors that are on the surface of the object color solid, so for which $\Psi_o \in \partial(\mathbf{O})$. Based in the theory of convex sets, and with the regular inner product on $\mathbb{R}^3$, it follows that:

$$(\mathbf{n}, \Psi_o) \geq (\mathbf{n}, \Psi), \quad (10)$$

for all $\Psi \in \mathbf{O}$ and with $\mathbf{n} = (n_x, n_y, n_z)^T (\neq 0)$ the normal-vector of the support surface at $\mathbf{O}$ in Ψ_o in the tristimulus coordinates defined by the primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. Filling in the definition of an object color as the Lebesgue integral from the reflection coefficient $r(\lambda)$ to the 'adjusted' color-matching measure $\psi_I(d\lambda)$ gives:

$$\left(\mathbf{n}, \int_{\Lambda} r_o(\lambda) \psi_I(d\lambda) \right) \geq \left(\mathbf{n}, \int_{\Lambda} r(\lambda) \psi_I(d\lambda) \right), \quad (11)$$

for all $r(\lambda) \in \mathbf{R}$. In this expression $r_o(\lambda) \in \mathbf{R}$ is the reflection coefficient belonging to the optimal color Ψ_o . Expanded a little further this reads as:

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}, \int_{\Lambda} r_o(\lambda) \begin{pmatrix} \psi_{I,x}(\lambda) \\ \psi_{I,y}(\lambda) \\ \psi_{I,z}(\lambda) \end{pmatrix} d\lambda \geq \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}, \int_{\Lambda} r(\lambda) \begin{pmatrix} \psi_{I,x}(\lambda) \\ \psi_{I,y}(\lambda) \\ \psi_{I,z}(\lambda) \end{pmatrix} d\lambda, \quad (12)$$

By applying the linearity property of the inner product this leads to the following expression:

$$\int_{\Lambda} (\mathbf{n}, \boldsymbol{\psi}_I(\lambda)) \cdot r_o(\lambda) d\lambda \geq \int_{\Lambda} (\mathbf{n}, \boldsymbol{\psi}_I(\lambda)) \cdot r(\lambda) d\lambda, \quad (13)$$

for all $r(\lambda) \in \mathbf{R}$, in which:

$$(\mathbf{n}, \boldsymbol{\psi}_I(d\lambda)) = n_x \cdot I(\lambda)x(d\lambda) + n_y \cdot I(\lambda)y(d\lambda) + n_z \cdot I(\lambda)z(d\lambda). \quad (14)$$

It is easy to see that (13) is only satisfied if the reflection coefficient $r_o(\lambda)$ has the functional form of a sign function with the argument being the inner product $(\mathbf{n}, \boldsymbol{\psi}_I(\lambda))$, which is a function of λ .

This means that $r_o(\lambda) \sim \text{sgn}((\mathbf{n}, \boldsymbol{\psi}_I(\lambda)))$. However, the sign function itself is not an element of the set $\mathbf{R}$, because of the value -1 in its range. Therefore, it needs to be rescaled so that instead of values in $\{-1, +1\}$ only values in $\{0,1\}$ can be attained. This results in:

$$r_o(\lambda) = \frac{1}{2}[1 + \text{sgn}((\mathbf{n}, \boldsymbol{\psi}_I(\lambda)))] , \quad (15)$$

for all $\lambda \in \Lambda$. All the previous is summed up by the following auxiliary statement:

Lemma O_1 Any optimal color $\boldsymbol{\Psi}_o \in \partial(\mathbf{O})$ is represented by a reflection coefficient of the form

$$r_o(\lambda) = \frac{1}{2}[1 + \text{sgn}((\mathbf{n}, \boldsymbol{\psi}_I(\lambda)))] , \text{ with } \boldsymbol{\psi}_I(\lambda) = (I(\lambda)x(\lambda), I(\lambda)y(\lambda), I(\lambda)z(\lambda))^T \text{ and } \mathbf{n} =$$

340 $(n_x, n_y, n_z)^T$, the outward pointing normal-vector of the support surface (in the
341 tristimulus coordinates defined by the primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$) located at the optimal
342 color Ψ_o .

343

344 This lemma does not indicate whether this expression uniquely determines the reflection
345 coefficient of an optimal color. However, it can be demonstrated on geometric grounds that the
346 following is also true:

347

348 **Lemma O_2** The expression $r_o(\lambda) = \frac{1}{2}[1 + \text{sgn}((\mathbf{n}, \boldsymbol{\psi}(\lambda)))]$ determines the reflection
349 coefficient of the optimal color Ψ_o in a unique way, if and only if Ψ_o is an
350 exposed point (*exposed object color*) of $\mathbf{O}$.

351

352 The defining expression for an exposed object color Ψ_o must therefore strictly contain a ‘greater
353 than’ sign, so: $(\mathbf{n}, \Psi_o) > (\mathbf{n}, \Psi)$, for all $\Psi \in \mathbf{O}$. This comment, and thus the general definition of
354 an exposed point of a collection, obviously has to do with the fact that the optimal colors are
355 located on the surface of the object color solid.

356

357 Note that for the extreme elements $\partial_E(\mathbf{O})$ of the object color solid $\mathbf{O}$, analogous to **Property T_3**
358 above, it can be proven that they are achieved by a unique reflection coefficient having the
359 functional form of a characteristic function. All exposed points of a convex set are extreme
360 points. In this case, however, the exposed points also give rise to an explicit formula to uniquely
361 determine this characteristic function. In particular, from **Lemma O_2** it follows that all points on

the edge of the object color solid are uniquely determined by the expression (15) if the object color solid is a strictly convex set.

Since the sign function is not defined if the argument takes the value zero, the reflection coefficient associated with an optimal color is only uniquely defined if the set $G(\mathbf{n}) = \{\lambda \mid (\mathbf{n}, \boldsymbol{\psi}_I(\lambda)) = 0, \forall \lambda \in \Lambda\}$ has measure zero. For a color-matching experiment with well-chosen primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$, the color-matching functions $\bar{x}(\lambda)$, $\bar{y}(\lambda)$ en $\bar{z}(\lambda)$ are linearly independent on the interval Λ . With a realistic illuminant $\mathbf{I}$ (in practice this means that the spectral distribution function $I(\lambda)$ is zero at no more than a finite number of wavelengths) the linear combination $(\mathbf{n}, \boldsymbol{\psi}_I(\lambda)) = n_x \cdot I(\lambda)x(\lambda) + n_y \cdot I(\lambda)y(\lambda) + n_z \cdot I(\lambda)z(\lambda)$ is identical to zero on the interval Λ , only if $n_x = n_y = n_z = 0$, which means that $\mathbf{n} = \mathbf{0}$. Moreover, this means that in only a finite number of points the value zero is assumed on the interval Λ . The latter follows from the fact that the function $(\mathbf{n}, \boldsymbol{\psi}_I(\lambda))$, for realistic color-matching functions and a realistic illuminant, is an analytic function on the interval Λ .

A necessary condition for this is that the definition area of the (column) vector-valued function $\boldsymbol{\psi}_I(\lambda)$ (i.e. the visual spectrum Λ) is chosen for the interval at which always at least one of the 'adjusted' color-matching functions is unequal to zero. This is a straightforward requirement that is certainly satisfied for the human visual system on the interval $\Lambda = [380, 800]$ nm.

Finally, this leads to the insight that the set $G(\mathbf{n})$ in that case indeed has measure zero (for all $\mathbf{n} \neq \mathbf{0}$). So it follows that the reflection coefficient $r_o(\lambda)$ associated with each object color Ψ_o on the surface of the object color solid (i.e. an optimal color) can uniquely be calculated using (15).

The wavelengths λ_s at which there is a switch between the values zero and one are given by the zeros of the function $(\mathbf{n}, \boldsymbol{\psi}_1(\lambda))$, for $\mathbf{n} \neq \mathbf{0}$. The number of switches taking place on the visual spectrum is therefore equal to the number of zeros of this function. For the human visual system this is a maximum of two on the visual spectrum Λ , as is easy to verify. For a similar conclusion see Logvinenko (2009), who also presents a comparable albeit less general and comprehensive expression for optimal colors.

The derivations presented here are important enough to be formulated explicitly again, as a property of the representation theory of object colors:

Property O₄ Every optimal color Ψ_o is uniquely determined by a reflection coefficient $r_o(\lambda) = \frac{1}{2}[1 + \text{sgn}((\mathbf{n}, \boldsymbol{\psi}_1(\lambda)))]$, with $\boldsymbol{\psi}_1(\lambda) = (I(\lambda)x(\lambda), I(\lambda)y(\lambda), I(\lambda)z(\lambda))^T$ the for illuminant $\mathbf{I}$ adjusted color-matching functions and $\mathbf{n} = (n_x, n_y, n_z)^T$ an outwards pointing normal-vector of the support surface at the location Ψ_o , in the tristimulus coordinates defined by the primaries $\{X, Y, Z\}$.

Note that this goes beyond the statement that the optimal colors are equal to the Schrödinger colors. **Property O₄** provides an explicit algorithm for determining the functional form of the reflection coefficient associated with an optimal color, based on a given tristimulus specification.

IDEAL TRICHROMATIC SYSTEM

In this section *Property O₄* will be demonstrated for a fictional trichromatic visual system with ideal mathematical properties (Koenderink & Kappers, 1996). Color-matching functions are chosen which enable straightforward and insightful calculations, in order to showcase the properties of the representation theory of object colors. In the following, the color-matching functions, tristimulus space and object color solid will be given. The main goal, however, is to explicitly derive the reflection coefficients of a number of example optimal colors for this fictional visual system, by applying expression (15).

The color-matching functions that will be used here are the first three Hermite functions $\{H_n(x) \mid n = 0, 1, 2\}$, which are orthogonal functions related to the Hermite polynomials (see e.g. Kreyszic, 1993):

$$\begin{aligned} H_0(x) &\sim \exp(-x^2), \\ H_1(x) &\sim -x \cdot \exp(-x^2), \\ H_2(x) &\sim (x^2 - \frac{1}{2}) \cdot \exp(-x^2). \end{aligned} \tag{16}$$

The color-matching functions are shown in Figure 1.

*** FIGURE 1 ***

The boundary of the tristimulus space can be seen in Figure 2:

*** FIGURE 2 ***

Finally, Figure 3 shows the object color solid (with a uniform spectral distribution function as illuminant).

*** FIGURE 3 ***

For the Hermite trichromatic system, the algorithm (15) has the following functional form:

$$\begin{aligned} r_o(\lambda) &= \frac{1}{2}[1 + \text{sgn}(n_x \cdot H_0(x) + n_y \cdot H_1(x) + n_z \cdot H_2(x))] \\ &= \frac{1}{2}[1 + \text{sgn}((n_x \cdot 2^{-1/2} - n_y \cdot x + n_z \cdot (x^2 - 1/2)) \cdot \pi^{-3/4} \cdot \exp(-x^2))] \\ &= \frac{1}{2}[1 + \text{sgn}(n_x \cdot 2^{-1/2} - n_y \cdot x + n_z \cdot (x^2 - 1/2))], \end{aligned} \quad (16)$$

Or, written differently: $r_o(\lambda) = \frac{1}{2}[1 + \text{sgn}(a \cdot x^0 + b \cdot x^1 + c \cdot x^2)]$, with $a = 2^{-1/2}n_x - \frac{1}{2}n_z$, $b = -n_y$ and $c = n_z$. Figure 4 shows how different numerical values of the coefficients n_x , n_y en n_z , that is, different normal-vectors $\mathbf{n}$, give rise to specific reflection coefficients of optimal colors for the Hermite trichromatic system.

*** FIGURE 4 ***

DISCUSSION

In this paper, reasonable assumptions regarding the physics of light and the nature of the visual system are combined with principles from the mathematical frameworks of measure theory and functional analysis. The resulting model is relevant for color science since a number of the fundamental properties of the representation theory of colors follow in a natural way. The mathematical model outlined above takes as its most prominent axiom that the sensorial and transducer properties of the chromatic visual system can be captured by a set of three bounded linear functionals. The three functionals, in turn, are each completely characterized by a bounded function, as a consequence of a fundamental theorem of functional analysis. These functions can be identified as the spectral sensitivity functions of each type of cone. The description of the representation theory of aperture colors based on the color-matching paradigm is a direct result of this. Because all linear combinations of these functions (i.e., both realistic or fictional color-matching functions) can be interpreted as non-atomic vector-valued measures, a couple of fundamental theorems for the general representation theory of aperture colors can be derived. Moreover, the extension of the model to object colors leads to similar results. Notably, the convexity and compactness of the tristimulus space, the chromaticity diagram and the object color solid, which are generally accepted empirical properties, now become mathematical consequences from the measure theoretical grounding of colorimetry. In addition, the extended framework leads to a more comprehensive description of optimal colors, which has practical significance in the form of *Property O₄*.

The introduction of functional analysis and measure theory into color science opens up novel opportunities for mathematical exploration. The introduction of these well-developed and rich

mathematical frameworks can shed a new light on theoretical concepts of the representation theory of colors. Within functional analysis there exists an extensive body of theorems on operators working on or between many different spaces. Operators on the space of all spectral distribution functions $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$ and between this space and the representation spaces $\mathbf{T}$ and $\mathbf{O}$ of all the aperture colors and object colors, respectively, can provide the mathematical description of numerous experimental manipulations and empirical findings in color science. This can go well beyond the color-matching experiments described here. It is not impossible that the analysis of adaptation effects, like Von Kries laws (see Wyszecki & Stiles, 2000) or color-constancy, can benefit from this approach. In these cases, let A be an operator that depends on a number of aspects in an experimental setting, such as the reflection coefficients of surrounding surfaces $r_i(\lambda)$, spectral distribution functions of multiple light sources $I_j(\lambda)$, and possibly on other spatial (x) and temporal (t) aspects of the scene. Symbolically:

$$A = A(r_i(\lambda), I_j(\lambda), x, t) . \quad (17)$$

This operator maps the spectral distribution function of the target stimulus onto a, for these circumstances, metameric spectral distribution function. Exploring the structure of the range and the domain of such an operator can produce valuable insights. For these sophisticated properties of the human visual system, different processes on different levels in the nervous system are involved, ranging from the retina to the visual cortex. The color-matching measure model is primarily a mathematical description of the representation theory based on color-matching experiments. This representation is in terms of a point in the three-dimensional tristimulus space that is accomplished by the color-matching measure. This vector-valued measure is in its essence

a linear combination of the three spectral sensitivity functions, representing the output of the cones in the retina as a reaction on the absorption of the energy-content in a color distributed according to a spectral distribution function. In other words, this model does not ‘leave’ the retina and therefore can never be a full account of all adaptation effects. But the color-matching measure model maybe powerful enough to give some qualitative statements and provide limiting constraints.

The color-matching measure model is quite different and, arguably, more parsimonious than Krantz’ influential approach of color theory based on n-chromatic Grassmann structures (Krantz, 1975; Grassmann, 1853). To be able to derive a representation space that is in correspondence with experimental findings, Krantz postulated an equivalence relation to represent color-matches. In the color-matching measure model, however, this equivalence relation presents itself as a consequence of the measure theoretical approach. Because of this, it is in a sense more fundamental. Moreover, the color-matching measure model makes the set of all stimuli that can be used in a specific color-matching setting more explicit. It is the convex set $\mathbf{K}_E$ in the infinite-dimensional vector space $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$ of all Lebesgue integrable functions. $\mathbf{K}_E$ consists of all non-negative spectral distribution functions (of some radiometric unit related to the energy) that are bounded from above by a maximum total energy content. With the identification of three functionals that describe the three receptors, the corresponding spectral sensitivity functions follow directly from mathematical principles. The critical step is the interpretation of these functions and any linear combination of these functions (known as the color-matching functions) as a non-atomic vector-valued measure. This viewpoint automatically provides the model with an equivalence relation on the set of all colors in the set $\mathbf{K}_E$. The strength of the color-matching

518 measure model, which makes it more powerful than the Grassmann structures introduced by
519 Krantz, presents itself directly from the fact that a number of fundamental properties of the
520 representation theory of aperture colors and also of object colors can readily be derived. These
521 properties are a direct consequence of the measure-theoretical formulation of the model.
522 Moreover, the axioms about the equivalence relation are redundant, because they are also
523 inherited from measure theory. The only assumption replacing them is that the sensorial and
524 transducer qualities of the receptors in the periphery of the human visual system can be described
525 by three bounded linear functionals.
526

527 **REFERENCES**

- 528 Arend, L. E. (1993). How much does illuminant color affect unattributed colors? *Journal of the*
529 *Optical Society of America A*, **10**, 2134-2147.
- 530 Baylor, D. A., Nunn, B. J., & Schnapf, J. L. (1987). Spectral Sensitivity of Cones of the Monkey
531 Macaca Fascicularis. *Journal of Physiology*, **390**, 145-160.
- 532 Bennett, B. M., & Lehman, R. C. (2001). Directed Convergence in Stable Percept Acquisition.
533 *Journal of Mathematical Psychology*, **45**, 732-779.
- 534 Boynton, R. M. (1979). *Human Color Vision*. Holt, Rinehart, Winston.
- 535 Cone, R. A. (1965). The Early Receptor Potential of the Vertebrate Eye. *Cold-Spring-Harbor-*
536 *symposia-on-quantitative-biology*.
- 537 Dunford, N. S., & Schwartz, J. T. (1958). *Linear Operators. Part I: General Theory*. New York:
538 Interscience Publishers.
- 539 Grassmann, H. (1853). Zur theorie der farbenmischung. *Annals of Physics*, *165*(5), 69-84.
- 540 Halmos, P. R. (1974). *Measure Theory*. (2nd Edition) New York, Berlin Heidelberg: Springer-
541 Verlag.
- 542 Hermes, H., & LaSalle, J. P. (1969). *Functional Analysis and Time Optimal Control*. New York:
543 Academic Press.
- 544 Jackson, J. D. (1999). *Classical Electrodynamics*. (3rd Edition) John Wiley & Sons, Inc.
- 545 Kluvánek, I., & Knowles, G. (1975). *Vector Measures and Control Systems*. Amsterdam: North-
546 Holland Publishing Company.
- 547 Koenderink, J., & Kappers, A. (1996). *Color Space*. Bielefeld: Universität Bielefeld Report Nr.
548 16/96.

549 Koenderink, J. J., & van Doorn, A. J. (1996). Local Image Operators and Iconic Structure. In G.
 550 Sommer, & J. J. Koenderink (Eds.), *Algebraic Frames for the Perception-Action Cycles*,
 551 (pp. 66-93). Berlin: Springer-Verlag.

552 Koenderink, J. J., & van Doorn, A. J. (2000). The Structure of Colorimetry. In: Sommer G.,
 553 Zeevi Y.Y. (eds) *Algebraic Frames for the Perception-Action Cycle*. AFPAC 2000.
 554 Lecture Notes in Computer Science, vol 1888. Springer, Berlin, Heidelberg.

555 Krantz, D. H. (1975). Color Measurement and Color Theory: I. Representation Theorem for
 556 Grassmann Structures. *Journal of Mathematical Psychology*, **12**, 283-303.

557 Kreyszig, E. (1993). *Advanced Engineering Mathematics*. (7th Edition). John Wiley & Sons, Inc.

558 Logvinenko, A. D. (2009). An object-color space. *Journal of Vision*, 9(11): 5.

559 MacAdam, D. L. (1981). *Color Measurement: Theme and Variations*. New York, Berlin,
 560 Heidelberg: Springer-Verlag.

561 Riesz, F., & Sz.-Nagy, B. (1955). *Functional Analysis*. New York: F. Ungar Publishing
 562 Company.

563 Rizzi, A. (2021). Colour after colorimetry. *Coloration Technology*, 137, 22–28.

564 Romney, A., & Chiao, C. (2009). Functional computational model for optimal color coding.
 565 *Proceedings of the National Academy of Sciences*, **106**, 10376-10381.

566 Schrödinger, E. (1920). Theorie der Pigmente von größter Leuchtkraft. *Annalen der Physik*, **62**,
 567 603-622.

568 Trigt, C. van (1990). Smoothest Reflectance Functions. I. Definition and Main Results. *Journal*
 569 *of the Optical Society of America A*, **7**, 1891-1904.

570 Trigt, C. van (1990). Smoothest Reflectance Functions. II. Complete Results. *Journal of the*
 571 *Optical Society of America A*, **7**, 2208-2222.

- 572 Wandell, B. A. (1995). *Foundations of Vision*. Stanford University Press.
- 573 West, G., & Brill, M. H. (1983). Conditions under which Schrödinger object colors are optimal.
- 574 *Journal of the Optical Society of America*, **73(9)**, 1223-1225.
- 575 Wyszecki, G., & Stiles, W. S. (2000). *Color Science. Concepts and Methods, Quantitative Data*
- 576 *and Formulae*. (2nd Edition). New York: Wiley.
- 577

578 APPENDIX A

579 MATHEMATICAL PRELIMINARIES

580 In this appendix several concepts and theorems related to convexity in finite-dimensional real
581 spaces and measure theory will be provided. Theorems will not be proven here, but proofs can be
582 found in many standard texts on the subjects. A detailed treatise of measure theory can be found
583 in the classical book by Halmos (1974). Also, Bennett and Lehman (2001) provide an excellent
584 summary of definitions and fundamental results in the first part of Section 4 (including the
585 Radon-Nikodým theorem). For a number of more advanced measure theoretical concepts, see
586 Kluvánec and Knowles (1975, esp. Chapter 2) and Dunford and Schwartz (1958, esp. Chapter
587 IV.10). Chapter V.1 of Dunford and Schwartz also holds as a reference on convexity.

588

589 CONVEXITY

590 Let $\mathbf{C}$ be a subset and $\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_m\}$ a finite set of vectors of a finite-dimensional real linear space
591 $\mathbf{V}$. A *convex combination* is a linear combination $\lambda_1 \cdot \mathbf{s}_1 + \lambda_2 \cdot \mathbf{s}_2 + \dots + \lambda_m \cdot \mathbf{s}_m$, with $0 \leq \lambda_i \leq 1$ for all
592 $i = 1, \dots, m$ and $\sum_i \lambda_i = 1$. The set $\mathbf{C}$ is called *convex* if it coincides with the collection of all convex
593 combinations of finite subsets of $\mathbf{C}$; the *convex hull* $H(\mathbf{C})$. This means that if $\mathbf{s}_1, \dots, \mathbf{s}_m$ are
594 elements of a convex set $\mathbf{C}$, their convex combination is also in $\mathbf{C}$. Moreover, every line-element
595 $[\mathbf{s}, \mathbf{t}] = \{\alpha \cdot \mathbf{s} + (1 - \alpha) \cdot \mathbf{t} \mid \alpha \in [0, 1]\}$ falls completely within the convex set $\mathbf{C}$ if the endpoints $\mathbf{s}$
596 and $\mathbf{t}$ are in $\mathbf{C}$.

597

598 A point $\mathbf{e}$ of a set $\mathbf{K}$ is an *extreme element* of $\mathbf{K}$ if $\mathbf{e} = \alpha \cdot \mathbf{x} + (1 - \alpha) \cdot \mathbf{y}$, for $\mathbf{x}, \mathbf{y} \in \mathbf{K}$ and $\alpha \in [0, 1]$
599 is not possible for $\mathbf{x} \neq \mathbf{y}$, but instead it must hold that $\mathbf{e} = \mathbf{x} = \mathbf{y}$. The set of extreme elements of a

convex set C is denoted by $\partial_E(C)$. (It can be proven that every non-void compact convex subset of a linear topological space has extreme elements.)

Theorem (Minkowski). *Let C be a compact (i.e. bounded and closed) convex set of a finite-dimensional linear space X and let $\partial_E(C)$ be the set of all extreme elements of C . Then $H(\partial_E(C)) = C$.*

In general, this convex representation in terms of extreme elements will not be unique. However, the number of extreme elements in Minkowski's theorem is at most one larger than the dimension of the space X :

Theorem (Carathéodory). *Let n be the dimension of a linear space X . An element s of the convex hull $H(K)$ of a set $K \subset X$ can then be written as the convex combination of at most $n + 1$ elements from K .*

MEASURE THEORY

A set $A \in \Sigma$ is called an *atom* of the measure m if $m(A) \neq 0$ and $B \in \Sigma$, $B \subset A$ implies that $m(B) = m(A)$ or $m(B) = 0$. If m has no atoms, it will be called *non-atomic*.

The concept of a measure can be expanded so as to include set-functions that map into more general spaces. In this paper only a finite-dimensional normed space will be used as the image-space, but more generally, locally convex topological vector spaces are possible in this respect.

Let X be a normed space and A a set from the σ -algebra of subsets Σ of some underlying set S .

623 The vector-valued set-function $\mathbf{m}: \Sigma \rightarrow \mathbf{X}; A \mapsto \mathbf{m}(A)$ will be a *vector-valued measure* if each
624 (vector-)component of $\mathbf{m}$ is a measure. The notation S_A is used for the restriction in S to the
625 subset A of S , so $S_A = \{B \mid B \in S, B \subset A\}$. Then, the following theorem can be proven:

626

627 **Theorem (Liapunov).** *Let X be a finite-dimensional normed space and Σ a σ -algebra of subsets*
628 *of a set S . If the mapping $\mathbf{m}: \Sigma \rightarrow X$ is a non-atomic vector-valued measure, than the set of all*
629 *images $\mathbf{m}(S_A)$ of the measure $\mathbf{m}$, for every $A \in S$ is compact and convex.*

630

631 This theorem can now be used as a definition for the so-called *Liapunov measures* as being
632 measures with a compact and convex image-set.

633

634 If the space $\mathbf{X}$ is complete, the integral of a measurable function f with respect to a vector-valued
635 measure $\mathbf{m}$ can be defined as follows: A real valued Σ -measurable function f on a set S is $\mathbf{m}$ -
636 integrable if there is a row of simple functions $\{f_n\}$, with $\{f_n\}$ converging to f and the series of
637 integrals $\{\int f_n d\mathbf{m}\}$ converging in the norm of $\mathbf{X}$. The limit of the latter series of integrals is
638 defined as the integral of f with respect to $\mathbf{m}$. With this, the Liapunov theorem for the
639 integration–mapping can be formulated:

640

641 **Theorem (Liapunov integration-mapping).** *Let $\mathbf{m}: \Sigma \rightarrow X$ be a Liapunov measure and f an $\mathbf{m}$ -*
642 *integrable function. The integration-mapping $\mathbf{n}$ with respect to the measure $\mathbf{m}$, defined by:*

643

644

645 $\mathbf{n} : L^1(S, \Sigma, \mathbf{m}) \rightarrow X;$

646 $f \mapsto \mathbf{n}(f) = \int_S f d\mathbf{m},$

647

648 *is also a Liapunov measure.*

649

650 Finally, a theorem about the image-set of Liapunov measures:

651

652 **Theorem (Liapunov measure, extreme points).** *Let $\mathbf{e}$ be an extreme element of the image-set of*
653 *an integration-mapping $\mathbf{n}(L^1(S, \Sigma, \mathbf{m}))$, with $f(x) \in [0, 1]$ for all x , and $\mathbf{m}$ a Liapunov measure.*

654 *There exists a unique element f of $L^1(S, \Sigma, \mathbf{m})$, such that $\mathbf{e} = \mathbf{n}(f) = \int f d\mathbf{m}$. This element has the*
655 *form of a characteristic function.*

656

657 The reverse is also true: Let $\mathbf{e}$ be the image (i.e., $\mathbf{e}$ is in the image-set of the integration-mapping
658 $\mathbf{n}(L^1(S, \Sigma, \mathbf{m}))$) of an element of the characteristic functions of $L^1(S, \Sigma, \mathbf{m})$, than $\mathbf{e}$ is a unique
659 extreme element of $\mathbf{n}(L^1(S, \Sigma, \mathbf{m}))$.

660

APPENDIX B

MATHEMATICAL PROPERTIES OF COLORS

The spectral distribution of radiant energy of a single color can be described by a non-negative function $e(\lambda)$ that represents the energy per unit wavelength of the light ray. The integral of this spectral distribution function $e(\lambda)$ with respect to the wavelength λ , taken over the interval $\Lambda (\subset \mathbb{R}^+)$ of the entire electromagnetic spectrum on which the function is defined, is therefore directly related to the total energy-content of the color. This integral has to be a non-negative and finite value, because light rays with a negative or infinite energy-content, do not support a sensible physical interpretation. According to this demand, the following constraint on the spectral distribution functions will be placed:

$$\int_{\Lambda} |e(\lambda)| d\lambda < \infty. \quad (\text{B1})$$

Note that taking the absolute-value $e(\lambda)$ is not strictly necessary because of the non-negative definiteness of the spectral distribution functions for realistic color stimuli. But for mathematical correctness it is nevertheless added.

It is physically possible, of course, to mix two light rays into a single color. This means that every spectral distribution function can be written as the superposition of two other distribution functions. Also, if the total energy content of a color is scaled with a non-negative factor α , the corresponding spectral distribution of energy of the new color is related to the former by a multiplication of the energy-value at each wavelength by the factor α :

$$e_1(\lambda) + e_2(\lambda) = (e_1 + e_2)(\lambda), \quad (B2)$$

$$\alpha \cdot e(\lambda) = (\alpha \cdot e)(\lambda),$$

686

687 for all α non-negative and real. The spectral distribution functions that correspond to realistic
688 colors, and by that allow an actual physical realization, are members of a subset of the more
689 general space of all functions $e(\lambda)$ obeying (B2). The latter space also contains all functions with
690 negative ‘energy’ values at some wavelengths, which do not allow a physical interpretation. All
691 the functions that obey this less strict demands make up a vector space and together with a norm
692 defined in terms of the integral (B1) this vector space forms a so-called Banach space, label as
693 $L^1(\Lambda, \mathbf{B}(\Lambda), \mu)$ (see Riesz & Nagy, 1955; Dunford & Schwartz, 1958). Although the norm does
694 not play an essential role in the model of this manuscript, for spectral distribution functions that
695 correspond to realistic colors it has a physical interpretation related to total energy content. The
696 measure space on which these functions are defined consists of the visual part of the
697 electromagnetic spectrum $\Lambda = [\lambda_{\min}, \lambda_{\max}]$, with $\lambda_{\min} = 380$ nm and $\lambda_{\max} = 800$ nm, equipped with
698 the Borel structure $\mathbf{B}(\Lambda)$. The measure μ will be written in the usual notation $d\lambda$ for the Lebesgue
699 measure of an infinitesimal Borel set. Finally, resulting from the physical ‘restriction’ of non-
700 negativity, all spectral distribution functions that correspond to realistic colors form a convex
701 cone $\mathbf{K}$ in the space $L^1(\Lambda, \mathbf{B}(\Lambda), d\lambda)$.

702

703 **MATHEMATICAL PROPERTIES OF COLOR RECEPTORS**

704 Essential sensorial elements of the visual system, as far as color vision is concerned, are the three
705 cones (Boynton, 1979; Wandell, 1995). These are three independent light-sensitive cells that
706 constitute the visual system’s initial spectral analysis of incoming light. Roughly this process

707 goes as follows: A light ray passes through the ocular media (including lens and macular
708 pigment) and hits the light-sensitive photopigment on the outer layers of the cone. As a
709 consequence of the radiant energy contained in the light ray, a photochemical reaction takes
710 place, resulting in differences in the electric potential on the efferent side of the cell. These
711 potential differences can in turn lead to a series of action potentials that obey the ‘all-or-none’
712 law of the electrochemical signal theory of neurons. So effectively this means that the light ray is
713 transformed into an electrochemical binary signal.

714
715 At this stage, it is important to mention the close correspondence between the binary
716 representation $\{0,1\}^{\mathbb{N}}$ of the real numbers and the more familiar decimal one. Every element of
717 $\{0,1\}^{\mathbb{N}}$ is a (possibly infinite) sequence of zeros and ones, that can represent any possible real
718 number. Consequently, the sensorial and transducer processes taking place inside a cone connect
719 to every color the binary representation of a real number. Mathematically this process constitutes
720 a mapping from the convex cone $\mathbf{K}$ in the Banach space of all spectral distribution functions to
721 the real numbers, that is, a *functional*. Clearly this functional will be a bounded one, since the
722 chemical and physical processes inside a cone cannot generate an infinitely large value.
723 Grassmann’s laws, which are founded on a firm experimental basis, state that color-matching is a
724 linear process. It is reasonable to suppose that the cones behave linearly in such a matching
725 setting as well (see e.g. Wandell, 1995). Moreover, there is experimental evidence for this claim
726 according to Baylor, Nunn, and Schnapf (1987) and Cone (1965): In electrophysiological
727 experiments, the latter measured the early receptor potentials (ERPs) of the eye, which turned out
728 to be linearly dependent on the radiant energy in the stimulation. Taken together, the sensorial

and transducer process that takes place in a single cone can mathematically correct be described as a *bounded linear functional*.

An elegant theorem from functional analysis (Riesz & Sz.-Nagy, 1955) states that every bounded linear functional on $L^1(\Lambda, \mathbf{B}(\Lambda), \mu)$ can uniquely be written as an inner product. This inner product, represented as $(\cdot, \cdot)$, has the form of a Lebesgue integral of two functions, one from $L^1(\Lambda, \mathbf{B}(\Lambda), \mu)$ and the other from $L^\infty(\Lambda, \mathbf{B}(\Lambda), \mu)$. Here this means that there is a unique mathematical description of the receptor output signal ϑ_i of the i^{th} receptor in terms of the following functional:

$$\begin{aligned} \vartheta_i : \quad \mathbf{K} &\rightarrow \quad \mathbb{R}; \\ e(\lambda) &\mapsto \quad \vartheta_i(e) = (e, \varphi_i) = \int_{\Lambda} e(\lambda) \varphi_i(\lambda) d\lambda. \end{aligned} \tag{B3}$$

The unique generating functions $\varphi_i(\lambda) \in L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$ are bounded (equivalence-class) functions defining the functionals ϑ_i . The functions $\varphi_i(\lambda)$ can be recognized as the *cone fundamentals*, which describe the combined effect of the ocular media's transmittance spectrum and that cone's spectral absorption. The exact functional form of the functions $\varphi_i(\lambda)$ obviously remains an empirical matter. The only thing this theorem states is that the output of each cone to an incoming light ray is uniquely characterized in this way.

Since the three different types of cones are generally be labeled by the wavelength interval at which they are most sensitive, as ‘large’ (L), ‘medium’ (M) and ‘small’ (S) wavelength type (Boynton, 1979; Wandell, 1995), we now have the following system:

$$\{\vartheta_L, \vartheta_M, \vartheta_S\} = \{\vartheta_i(.) = (. , \varphi_i) \mid \varphi_i(\lambda) \in L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda); i = L, M, S\} . \quad (B4)$$

Finally, the receptor output signals resulting from the interaction with light can be represented as the three-dimensional column vector:

$$\mathfrak{V}(.) = (\vartheta_L(.), \vartheta_M(.), \vartheta_S(.))^T . \quad (B5)$$

Summarized, the sensorial and transducer processes of the peripheral visual system can mathematically be described by a column vector-valued system (B5) consisting of three bounded linear functionals (B3), each acting on the spectral distribution functions in the convex cone $\mathbf{K}$ of all the realistic colors.

FROM COLOR RECEPTORS TO TRISTIMULUS VALUES

For purposes of color matching, the three receptor output functionals $\{\vartheta_L, \vartheta_M, \vartheta_S\}$, generating the receptor output signals $\mathfrak{V}$, need to be transformed into a system of three tristimulus functionals $\{\Phi_X, \Phi_Y, \Phi_Z\}$, generating the tristimulus vectors Φ . These tristimulus functionals are defined by the particular set of primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$ that is chosen in some experimental or imaginary color-matching setting, and which always form some linear combination of the receptor output

functionals[§]. The coefficients in this linear combination are ultimately dependent on the receptor output signals $\mathfrak{G}(\mathbf{ex})$, $\mathfrak{G}(\mathbf{ey})$ and $\mathfrak{G}(\mathbf{ez})$ of the spectral distribution functions $\mathbf{ex}(\lambda)$, $\mathbf{ey}(\lambda)$ and $\mathbf{ez}(\lambda)$, corresponding to the primaries $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, respectively.

Let $x(\lambda)$, $y(\lambda)$ and $z(\lambda)$ be the color-matching functions describing the color-matching properties of an average (fictional) subject for the primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. Because of the linearity of the functionals $\{\mathfrak{G}_L, \mathfrak{G}_M, \mathfrak{G}_S\}$, these are linear combinations of the cone fundamentals $\phi_L(\lambda)$, $\phi_M(\lambda)$ and $\phi_S(\lambda)$. Therefore $x(\lambda)$, $y(\lambda)$ and $z(\lambda)$ are also elements of $L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$.

The corresponding tristimulus functionals are in agreement with the color equations that are constructed by (fictional) average subjects. Take a color $\mathbf{P}$, with a spectral distribution function $\mathbf{e_P}(\lambda)$ and a color-matching setting defined by the primaries $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$, with spectral distribution functions $\mathbf{ex}(\lambda)$, $\mathbf{ey}(\lambda)$ and $\mathbf{ez}(\lambda)$, respectively. Suppose a subject constructs the following color equation:

$$\mathbf{P} = X \cdot \mathbf{X} + Y \cdot \mathbf{Y} + Z \cdot \mathbf{Z} . \quad (\text{B6})$$

The coefficients X , Y and Z in this equation are given by the tristimulus functionals $\Phi_{\mathbf{X}}$, $\Phi_{\mathbf{Y}}$, respectively $\Phi_{\mathbf{Z}}$:

$$X = \Phi_{\mathbf{X}}(\mathbf{P}) = \int_{\Lambda} \mathbf{e_P}(\lambda) x(\lambda) d\lambda , \quad (\text{B7})$$

[§] Nonlinear transformations are also possible of course, but will not be considered here.

792

793 and analogously for Y and Z. In vector-notation:

794

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \Phi(\mathbf{P}) = \begin{pmatrix} \Phi_X(\mathbf{P}) \\ \Phi_Y(\mathbf{P}) \\ \Phi_Z(\mathbf{P}) \end{pmatrix} = \int_{\Lambda} \mathbf{e}_P(\lambda) \begin{pmatrix} x(\lambda) \\ y(\lambda) \\ z(\lambda) \end{pmatrix} d\lambda . \quad (\text{B8})$$

795

796 The vector $\Phi(\mathbf{P})$ is called the tristimulus vector belonging to the color $\mathbf{P}$. Every color
797 corresponds to a single tristimulus vector. As is well-known, this is not true the other way
798 around, however. That this correspondence is not one-to-one is the quintessence of *metamerism*,
799 on which the color-matching paradigm is based. As will be clarified in the main text, the
800 mathematical framework of measure theory leads to a natural and coherent account of
801 metamerism.

802

803

804 APPENDIX C

805 CHROMATICITY DIAGRAM

806 *Property T_I* of the tristimulus space is immediately inherited by the chromaticity diagram **C**.

807 This a consequence of the fact that the mapping connecting the tristimulus space and the
808 chromaticity diagram is projective. For general tristimulus coordinates defined by the set of
809 primaries **{X,Y,Z}**, this mapping is given by:

810

$$x = \frac{X}{X+Y+Z}, \quad y = \frac{Y}{X+Y+Z}, \quad z = \frac{Z}{X+Y+Z}, \quad (C1)$$

811

812 The coordinates (x,y,z) are the chromaticity coordinates that obey the equation:

813

$$x + y + z = 1. \quad (C2)$$

815

816 This means that just two coordinates are needed for a graphical representation in the chromaticity
817 diagram. The chromaticity diagram is the projective space spanned by the projective coordinates
818 $(x_0, x_1, x_2) = (L, x, y)$. The coordinates x and y form the chromaticity diagram **C** and $L = X + Y + Z$
819 is related to the luminance. From this, the following property can be derived:

820

821 *Property C₁* The chromaticity diagram **C** is a convex and compact subset of the three-
822 dimensional real space $\mathbb{R}^3$.

823

824 **PROOF**

825 According to *Property T₁*, the tristimulus space **T** is a convex and compact subset of the three-
826 dimensional real space $\mathbb{R}^3$. The chromaticity diagram **C** is constructed by a projective mapping
827 from the tristimulus space **T**. Projective mappings like this one, where the tristimulus values
828 become the chromaticity coordinates are continuous mappings.

829

830 *Lemma 1:* The projective image of a convex set **T** in tristimulus coordinates (X,Y,Z):

831

$$x = \frac{X}{X+Y+Z}, \quad y = \frac{Y}{X+Y+Z}, \quad z = \frac{Z}{X+Y+Z},$$

832

833 is a convex set **C** in chromaticity coordinates (x,y,z).

834

835 *Proof:* Let $P_1 = (X_1, Y_1, Z_1)^T$ and $P_2 = (X_2, Y_2, Z_2)^T$ be elements of **T**. Let $Q = \alpha \cdot P_1 + (1-\alpha) \cdot P_2 \in \mathbf{T}$
836 be a convex combination. The projective mapping of P_1 and P_2 results in $p_i = (x_i, y_i, z_i)^T =$
837 $(X_i, Y_i, Z_i)^T / (X_i + Y_i + Z_i)$, for $i = 1, 2$. The projective mapping of the convex combination Q
838 looks like:

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$$\mathbf{q} = \frac{(\alpha X_1 + (1-\alpha)X_2, \alpha Y_1 + (1-\alpha)Y_2, \alpha Z_1 + (1-\alpha)Z_2)^T}{\alpha L_1 + (1-\alpha)L_2},$$

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841 with $L_i = X_i + Y_i + Z_i$ (Luminance), for $i = 1, 2$. This can be written as:

842

$$\mathbf{q} = \frac{\alpha L_1}{\alpha L_1 + (1-\alpha)L_2} \cdot \mathbf{p}_1 + \frac{(1-\alpha)L_2}{\alpha L_1 + (1-\alpha)L_2} \cdot \mathbf{p}_2 .$$

843

844 For realistic colors $L_i > 0$ ($i = 1,2$) and this expression becomes a convex combination. This

845 means that the projective mapping of any convex combination of tristimulus points (coming

846 from realistic colors from $\mathbf{K}_E$) is a convex combination in the chromaticity diagram.

847

848 From ***Lemma 1*** and the fact that the continuous mapping of a compact set is compact, it follows

849 that the chromaticity diagram $\mathbf{C}$ is convex and compact.

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851

852 APPENDIX D

853 PROOF OF PROPERTY O_1

854 The set of all reflection coefficients $\mathbf{R}$ is weak * compact. This follows from the fact that $\mathbf{R}$ is the
855 translation over the constant function $\frac{1}{2}$ of the closed sphere of radius $\frac{1}{2}$ in $L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$,
856 which is compact in the weak * topology. The mapping Ψ from $\mathbf{R}$ to $\mathbb{R}^3$ is continuous in the
857 topology generated by the norm on $L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$. Since $(L^1)' = L^\infty$, it follows from the theorem
858 about the continuity of such a 'vector mapping ' in the weak * topology, that Ψ is continuous
859 relative to the weak * topology of $L^\infty(\Lambda, \mathbf{B}(\Lambda), d\lambda)$.

860 The mapping Ψ is continuous and linear, and the set $\mathbf{R}$ is weak * compact and convex. This
861 means that the image $\mathbf{O} = \Psi(\mathbf{R})$ is compact and convex in $\mathbb{R}^3$.

862

863 PROOF OF LEMMA O_2

864 The expression determines an exposed object color since every exposed point of a convex set is
865 located at the surface of that set. Therefore **Lemma O_1** applies. That there is unicity follows from
866 this reasoning: The normal vector $\mathbf{n} (\neq \mathbf{0})$ is perpendicular to the hyper plane $\pi(\mathbf{n})$ which is a
867 support plane of $\mathbf{O}$ for Ψ_o . Suppose that $\pi(\mathbf{n})$, besides for Ψ_o , is also a support plane for another
868 point Ψ'_o of $\mathbf{O}$. ($\Psi_o, \Psi'_o \in \partial(\mathbf{O})$) Than, by definition, also that point $(\mathbf{n}, \Psi'_o) \geq (\mathbf{n}, \Psi)$, for all Ψ
869 $\in \mathbf{O}$. As a result, in a similar way as before, we get $r'_o(\lambda) = \frac{1}{2}[1 + \text{sgn}((\mathbf{n}, \bar{\Psi}_1(\lambda)))]$. In addition,
870 $r'_o(\lambda) \neq r_o(\lambda)$, because otherwise $\Psi_o \equiv \Psi'_o$. So every optimal color in the set $\pi(\mathbf{n}) \cap \mathbf{O}$ satisfies
871 the expression $\frac{1}{2}[1 + \text{sgn}((\mathbf{n}, \bar{\Psi}_1(\lambda)))]$. And this with the same normal vector $\mathbf{n}$. In case $\pi(\mathbf{n}) \cap$
872 $\mathbf{O}$ is a 'plateau', there is an infinite number of points that satisfy the expression with the same
873 normal vector. If this set consists of only a single point, that is, $\pi(\mathbf{n}) \cap \mathbf{O} = \{\Psi_o\}$, than this

specification is complete and unique. This is of course the case when Ψ_o is an exposed point, because for these point this property is true by definition.

PROOF OF PROPERTY O_4

The property is proven to be correct when it has been shown that:

$$\mathbf{O} = \Psi(\mathbf{R}^0),$$

In which $\mathbf{R}^0 = \{1_{\Delta}(\lambda) \mid \Delta \in \mathbf{B}(\Lambda)\}$, the set of all reflection coefficients with the functional form of a characteristic function. This means that both $\Psi(\mathbf{R}^0) \subset \Psi(\mathbf{R})$ and $\Psi(\mathbf{R}^0) \supset \Psi(\mathbf{R})$ need to be true. The first is clear since $\mathbf{R}^0$ is a subset of $\mathbf{R}$, and because Ψ is a continuous mapping.

The second follows from this: Let $\mathbf{T} \in \mathbf{O}$ be a tristimulus vector. This means that $\mathbf{T}$ is the colorimetric representation of all those object surfaces of which the corresponding reflection coefficients $r(\lambda)$ belong to the equivalence class $[\mathbf{r}] = \Psi^{-1}(\mathbf{T}) = \{\mathbf{r} \in \mathbf{R} \mid \Psi(\mathbf{r}) \in \mathbf{T}\}$. This equivalence class contains at least one characteristic function. The non-empty, closed, convex set $\Psi^{-1}(\mathbf{T})$ has extreme elements. These extreme elements appear to have the functional form of a characteristic function, which can be proven ad absurdum in combination with an induction argument. Note that the latter can also be derived directly from the definition of the Lebesgue integral.