Designing bivariate timeseries with controlled Granger causality

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Abstract

In this manuscript, we analyzed vector auto-regressive (VAR) model in order to draw a design principle for bivariate timeseries with a controlled statistical inter-relationship. We show how to generate bivariate timeseries with given covariance and Granger causality, and showed the trade-off relationship between these two types of statistical interaction. In principle, covariance and Granger causality are controllable independently, but the ranges of their values, which let VAR be proper and have stationary distribution, are constrained by each other. Thus, there is an essential tri-lemma among the stability and properness of VAR, the controllability of covariance, and that of Granger causality.

1 Background and motivation

In a field of cognitive psychology, human perception of life-likeness, called *animacy perception*, of one or multiple moving geometric patterns has been studied for decades [5, 2, 3, 8, 6]. There are multiple findings on the effect of “synchrony” or “temporal contingency” between the multiple moving points on animacy perception. Findings from one line of research [2] have suggested that higher degree of “temporal contingency” of the moving objects is related to higher likelihood of animacy perception. Findings from the other line of research [7] have suggested that the highest “temporal contingency”, presented in a form of perfect synchronization, would decrease the likelihood of animacy perception.

These two lines of research together have suggested the existence of multiple types of “temporal contingency”, which may not be yet clarified in these past research, and two lines of apparent conflicting effects of “temporal contingency” may be caused by confusion of these two distinct types of effects.

With this potential conflict in the literature on animacy perception in our mind, we explore a theoretical framework, which can generate time-series of multiple random variables with multiple distinct types of statistical dependency. One of such systems, which is simple enough and readily manipulable, is vector auto-regression (VAR). Vector auto-regression is a random process generating multivariate timeseries for a given set of parameters. In this manuscript, we specifically consider only bivariate VAR, which is a minimal system with interaction between two moving points.

Importantly, bivariate VAR, a series of paired random variables (x_t, y_t) for $t = 0, 1, \dots$, has two types of statistical dependency – correlation and Granger causality of timeseries [4]. Correlation between univariate series x and y is statistical dependency between x_t and y_t in the limit $t \rightarrow \infty$ (if exists), while (one side of) Granger causality from y to x is that between x_t and y_{t-1} given x_{t-1} in the limit $t \rightarrow \infty$ (if exists). Conceptually, correlation captures a type of similarity between two timeseries, meanwhile, Granger causality captures “reactiveness” of one timeseries to the other.

Thus, given these difference in theory, we hypothesize that these two theoretically distinct statistical dependency may clarify the potential multiple sources of temporal contingency being involved with human animacy perception of multiple moving points. In order to test this hypothesis, we need to design a bivariate timeseries generated by VAR with controlled their variance, correlation, and two ways of Granger causality. In this manuscript, thus, we explore a theoretical structure of bivariate VAR from the designer’s perspective, and analyzed mathematical limit to which we can control a bivariate timeseries with a certain statistical dependency.

This manuscript is written as the following structure. In Section 2, VAR model is defined, and derive a set of basic statistical properties of VAR model such as Granger causality (Section 3) with a set of parameters. In Section 4, the existence of stationary distribution of VAR is analyzed. This is a foundation which gives the limit of controllable set of parameters. In Section 5, we give a method to derive the parameters of VAR for a given set of statistics in bivariate timeseries. In Section 6, the mathematical analysis provided in this manuscript is summarized and add a remark on a design principle of bivariate timeseries generated by VAR.

2 Vector Auto-Regression (VAR)

In theory, Granger causality (GC) is transfer entropy of the random variables in the bivariate vector auto-regression (VAR) model up to the constant

factor of 2, if the VAR model has a stationary distribution [1]. Thus, it is straight-forward to start with the bivariate VAR and derive its transfer entropy. In this way, we can derive a rich mathematical relationship between GC and the properties of VAR, not just as a statistics of bivariate timeseries.

Definition 1. For some real vector $\mu \in \mathbb{R}^2$ and some positive definite matrix $\Sigma \in \mathbb{R}^{2 \times 2}$, suppose the random variable ϵ_t for every integer $t = 0, 1, \dots$ is drawn from the bivariate normal distribution $N(\epsilon_t | \mu, \Sigma)$ with mean $\mu \in \mathbb{R}^2$ and variance Σ . Define the initial vector by $v_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, and for $t \geq 0$ and a given matrix with real entries $A = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$, define the random variable v_t by

$$v_{t+1} := Av_t + \epsilon_t. \quad (1)$$

Then, bivariate vector auto-regression is defined by the semi-infinite series of random variables $V = (v_0, v_1, v_2, \dots)$.

2.1 Marginal distribution of the VAR at each step

Lemma 1 (Marginal distribution of the VAR random variables at each step). The VAR model with the initial vector $v_0 \in \mathbb{R}^2$ and the coefficient matrix $A \in \mathbb{R}^{2 \times 2}$ has the bivariate normal distribution

$$N(v_t | \mu_t, \Sigma_t)$$

as its marginal distribution of the random variable v_t at each step $t = 0, 1, \dots$, where

$$\mu_t := A^t v_0, \quad \Sigma_t = \sum_{s=0}^t A^s \Sigma A^{\top s}.$$

Proof. By Definition 1, Lemma 1 holds for $t = 0$. For $t + 1 > 0$, we prove Lemma 1 by assuming that it holds up to $t \geq 0$. By this assumption held for t , we have the distribution of $v_t \in \mathbb{R}^2$ as the bivariate normal distribution

$$N(v_t | \mu_t, \Sigma_t) = (2\pi)^{-1} |\Sigma_t|^{-\frac{1}{2}} e^{-\frac{1}{2}(v_t - \mu_t)^\top \Sigma_t^{-1} (v_t - \mu_t)}$$

with its mean μ_t , and its covariance matrix Σ_t . Then the random variable Av_t is distributed by the normal distribution

$$\begin{aligned} N(Av_t | A\mu_t, A\Sigma_t A^\top) &= (2\pi)^{-1} |A\Sigma_t A^\top|^{-\frac{1}{2}} \\ &\times e^{-\frac{1}{2}(A(v_t - \mu_t))^\top (A\Sigma_t A^\top)^{-1} (A(v_t - \mu_t))} \end{aligned} \quad (2)$$

with its mean $A\mu_t$, and the covariance matrix $A\Sigma_t A^\top$. The random variable ϵ_t is distributed by the following normal distribution

$$N(\epsilon_t|0, \Sigma) = (2\pi)^{-1} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2} \epsilon_t^\top \Sigma^{-1} \epsilon_t}.$$

Thus, by VAR Eq (1), we have the random variable

$$v_{t+1} := Av_t + \epsilon_t$$

has the distribution calculated by the following integral:

$$P(v_{t+1}) = \int_{\epsilon_t \in \mathbb{R}^2} N(v_{t+1} - \epsilon_t | A\mu_t, A\Sigma_t A^\top) N(\epsilon_t | 0, \Sigma) d\epsilon_t.$$

Calculating this, we have

$$P(v_{t+1}) = N\left(v_{t+1} | A\mu_t, A\Sigma_t A^\top + \Sigma\right). \quad (3)$$

Thus defining by $\mu_{t+1} := A\mu_t$ and $\Sigma_{t+1} = A\Sigma_t A^\top + \Sigma$, Lemma 1 holds for $t+1$. By expanding this, we have the Lemma 1 for any integer $t \geq 0$. $\square$

2.2 Lyapunov equation

By Lemma 1

$$\mu_t = A^t \text{ and } \Sigma_t = \sum_{s=0}^t A^s \Sigma (A^s)^\top.$$

From this, we have stationary distribution

$$\lim_{t \rightarrow \infty} N(v_t | \mu_t, \Sigma_t),$$

if and only if the absolute values of all the eigenvalues λ_0, λ_1 of the coefficient matrix A are less than 1. If there is such stationary distribution, we call VAR is *stable*, and its stationary distribution is the bivariate normal distribution

$$N(\hat{v} | \mathbf{0}_2, \hat{\Sigma}).$$

Thus, if the VAR is stable, we have the following Lyapunov equation of the stationary covariance matrix $\hat{\Sigma} \in \mathbb{R}^{2 \times 2}$:

$$\hat{\Sigma} = \Sigma + A\hat{\Sigma}A^\top. \quad (4)$$

The Lyapunov equation is solved analytically by

$$\text{vec}(\hat{\Sigma}) = (I_4 - A \otimes A)^{-1} \text{vec}(\Sigma), \quad (5)$$

where $\otimes$ denotes the Kronecker product, and $\text{vec}(X)$ for any matrix $X = (x_{i,j})_{i=1,\dots,n,j=1,\dots,m}$ is the vectorization operator $\text{vec}() : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{nm \times 1}$ defined by

$$\text{vec}(X) := (x_{1,1}, x_{2,1}, \dots, x_{n,1}, \dots, x_{1,m}, x_{2,m}, \dots, x_{n,m})^\top.$$

The Lyapunov equation 4 has the solution for $\hat{\Sigma}$, if the VAR is stable. This is shown by Lemma 4.

3 Transfer entropy and Granger causality

In [1], transfer entropy of an appropriate triplet of variables in VAR model is shown to be equivalent to Granger causality up to the constant factor 2. Following this guid, we defined this quantity as Granger causality of VAR model as follows.

Although this relationship has been known in a more general form [1], we re-derive it for bivariate VAR in order to analyze the structure of VAR and GCs in depth, such as its upper and lower bound (Lemma 3), stability (Section 4), and design principle (Section 5), later on.

Definition 2. *If VAR with its random variables $v_t = (x_t, y_t)^\top \in \mathbb{R}^2$ is stable, transfer entropy from y to x is defined by*

$$T_{y \rightarrow x} := \lim_{t \rightarrow \infty} (H(x_{t+1}|x_t) - H(x_{t+1}|x_t, y_t)),$$

and the transfer entropy from x to y is defined by

$$T_{x \rightarrow y} := \lim_{t \rightarrow \infty} (H(y_{t+1}|y_t) - H(y_{t+1}|y_t, x_t)),$$

where the differential entropy of random variable x with its probability density function P is

$$H(x) := - \int_{x \in \Omega} P(x) \log P(x) dx,$$

and the conditional entropy is

$$H(x|y) := H(x, y) - H(y).$$

In particular, we call two times of transfer entropy Granger causality denoted by

$$G_0 = 2T_{y \rightarrow x} \quad G_1 = 2T_{x \rightarrow y}. \quad (6)$$

Specifically, GCs are given by the following lemma.

Lemma 2 (Granger causality). *If a stable VAR has its covariance matrix, coefficient matrix, and stationary matrix*

$$\Sigma = \begin{pmatrix} \sigma_{0,0} & \sigma_{0,1} \\ \sigma_{1,0} & \sigma_{1,1} \end{pmatrix}, \quad A = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{0,0} & \hat{\sigma}_{0,1} \\ \hat{\sigma}_{1,0} & \hat{\sigma}_{1,1} \end{pmatrix},$$

each Granger causality of this VAR for $i = 0, 1$ is

$$G_i = \log \left(1 + \frac{a_{i,1-i}^2 \det(\hat{\Sigma})}{\hat{\sigma}_{i,i} \sigma_{i,i}} \right). \quad (7)$$

Proof. In general, the differential entropy of multivariate normal distribution $N(v|\mu, \Sigma)$ is

$$H(v) = \frac{1}{2} \log |2\pi e \Sigma|,$$

where $e \approx 2.71$ is Napier's constant. For the joint probability distribution of $v_t = (x_t, y_t)^\top$

$$P(v_{t+1}|v_t) = N(v_{t+1}|Av_t, \Sigma),$$

the two marginal probability distributions of x_t, y_t are

$$P(x_{t+1}|v_t) = N(x_{t+1}|(1, 0)Av_t, \sigma_{0,0}) \text{ and } P(y_{t+1}|v_t) = N(y_{t+1}|(0, 1)Av_t, \sigma_{1,1}).$$

Thus, the conditional entropy of x_{t+1} and y_{t+1} given $v_t = (x_t, y_t)^\top$ are

$$H(x_{t+1}|x_t, y_t) = \frac{1}{2} \log |2\pi e \sigma_{0,0}| \text{ and } H(y_{t+1}|x_t, y_t) = \frac{1}{2} \log |2\pi e \sigma_{1,1}|.$$

With the conditional probability distribution

$$P(v_{t+1}|v_t) = N(v_{t+1}|Av_t, \Sigma)$$

and the marginal probability distribution

$$P(v_t) = N(v_t|A^t v_0, \Sigma_t),$$

the joint probability distribution of v_t and v_{t+1} is their product

$$P(v_{t+1}, v_t) = N(v_{t+1}|Av_t, \Sigma) N(v_t|A^t v_0, \Sigma_t).$$

Specifically, this quad-variate normal distribution is

$$P(v_{t+1}, v_t) = e^{-\frac{1}{2}(v_{t+1}-Av_t)^\top \Sigma^{-1}(v_{t+1}-Av_t) - \frac{1}{2}(v_t-A^t v_0)^\top \Sigma_t^{-1}(v_t-A^t v_0)} (2\pi)^{-2} |\Sigma|^{-\frac{1}{2}} |\Sigma_t|^{-\frac{1}{2}}.$$

Applying the identities

$$v_{t+1} - Av_t = v_{t+1} - A^{t+1}v_0 - A(v_t - A^t v_0),$$

and

$$\Sigma'_t := \begin{pmatrix} \Sigma + A\Sigma_t A^\top & A\Sigma_t \\ \Sigma_t A^\top & \Sigma_t \end{pmatrix} = \begin{pmatrix} \Sigma^{-1} & -\Sigma^{-1}A \\ -A^\top \Sigma^{-1} & \Sigma_t^{-1} + A\Sigma^{-1}A^\top \end{pmatrix}^{-1}$$

and $|\Sigma||\Sigma_t| = |\Sigma'_t|$ to $P(v_{t+1}, v_t)$, we have

$$P(v_{t+1}, v_t) = N(v'_t | \mu'_t, \Sigma'_t),$$

where

$$v'_t := \begin{pmatrix} v_{t+1} \\ v_t \end{pmatrix}, \quad \mu'_t := \begin{pmatrix} A^{t+1}v_0 \\ A^t v_0 \end{pmatrix}, \quad \Sigma'_t := \begin{pmatrix} \Sigma + A\Sigma_t A^\top & A\Sigma_t \\ \Sigma_t A^\top & \Sigma_t \end{pmatrix}.$$

From this joint probability distribution $P(v_{t+1}, v_t)$, we drive the marginal distributions

$$P(x_{t+1}, x_t) = N(x_{t+1}, x_t | \mu_{t,0}, \Sigma_{t,0}), \quad P(y_{t+1}, y_t) = N(y_{t+1}, y_t | \mu_{t,1}, \Sigma_{t,1}).$$

where for $i = 0, 1$ the mean vectors and covariance matrices are defined as follows:

$$\begin{aligned} \mu_{t,i} &:= \begin{pmatrix} e_i^\top A^{t+1}v_0 \\ e_i^\top A^t v_0 \end{pmatrix} = (I_2 \otimes e_i) \mu'_t, \\ \Sigma_{t,i} &:= \begin{pmatrix} e_i^\top (\Sigma + A\Sigma_t A^\top) e_i & e_i^\top A\Sigma_t e_i \\ e_i^\top \Sigma_t A^\top e_i & e_i^\top \Sigma_t e_i \end{pmatrix}, = (I_2 \otimes e_i)^\top \Sigma'_t (I_2 \otimes e_i), \end{aligned}$$

with the unit vectors $e_0 := (1, 0)^\top$, $e_1 := (0, 1)^\top$.

Thus, we have the joint entropy of x_t and x_{t+1}

$$H(x_{t+1}, x_t) = \frac{1}{2} \log |2\pi e \Sigma_{t,0}| \quad (8)$$

$$= \frac{1}{2} \log(2\pi e)^2 |e_0^\top (\Sigma + A\Sigma_t A^\top) e_0 e_0^\top \Sigma_t e_0 - (e_0^\top A\Sigma_t e_0)^2|, \quad (9)$$

and the marginal distribution of x_t

$$H(x_t) = \frac{1}{2} \log 2\pi e |e_0^\top \Sigma_t e_0|. \quad (10)$$

Using these, we have the conditional entropy

$$H(x_{t+1}|x_t) = \frac{1}{2} \log 2\pi \frac{e_0^\top (\Sigma + A\Sigma_t A^\top) e_0 e_0^\top \Sigma_t e_0 - (e_0^\top A\Sigma_t e_0)^2}{|e_0^\top \Sigma_t e_0|}. \quad (11)$$

By stability of VAR, the Lyapunov equation (4) holds, and this conditional entropy in the limit $t \rightarrow \infty$ is

$$\lim_{t \rightarrow \infty} H(x_{t+1}|x_t) = \frac{1}{2} \log 2\pi \frac{(e_0^\top \hat{\Sigma} e_0)^2 - (e_0^\top A \hat{\Sigma} e_0)^2}{|e_0^\top \hat{\Sigma} e_0|} \quad (12)$$

Applying Definition 2 and denoting by entries in the stationary covariance matrix $\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{0,0} & \hat{\sigma}_{0,1} \\ \hat{\sigma}_{1,0} & \hat{\sigma}_{1,1} \end{pmatrix}$, we have

$$G_0 = 2T_{y \rightarrow x} = \log \frac{(\hat{\sigma}_{0,0})^2 - (a_{0,0}\hat{\sigma}_{0,0} + a_{0,1}\hat{\sigma}_{1,0})^2}{\hat{\sigma}_{0,0}\sigma_{0,0}} \quad (13)$$

Similarly, we have

$$G_1 = 2T_{x \rightarrow y} = \log \frac{(\hat{\sigma}_{1,1})^2 - (a_{1,0}\hat{\sigma}_{0,1} + a_{1,1}\hat{\sigma}_{1,1})^2}{\hat{\sigma}_{1,1}\sigma_{1,1}}. \quad (14)$$

Let us define for $i = 0, 1$

$$\delta_i := (\hat{\sigma}_{i,i})^2 - (a_{i,i}\hat{\sigma}_{i,i} + a_{i,1-i}\hat{\sigma}_{1-i,i})^2 \text{ and } \delta'_i := \hat{\sigma}_{i,i}\sigma_{i,i}. \quad (15)$$

By Lyapunov equation, $\sigma_{i,i} = \hat{\sigma}_{i,i} - e_i^\top A \hat{\Sigma} A^\top e_i$. Applying this to δ'_i , we have

$$\delta_i = \hat{\sigma}_{i,i}^2 - e_i^\top A \begin{pmatrix} \hat{\sigma}_{i,i}^2 & \hat{\sigma}_{i,i}\hat{\sigma}_{1-i,i} \\ \hat{\sigma}_{i,i}\hat{\sigma}_{1-i,i} & \hat{\sigma}_{1-i,i}^2 \end{pmatrix} A^\top e_i \quad (16)$$

$$\delta'_i = \hat{\sigma}_{i,i}^2 - e_i^\top A \begin{pmatrix} \hat{\sigma}_{i,i}\hat{\sigma}_{0,0} & \hat{\sigma}_{i,i}\hat{\sigma}_{0,1} \\ \hat{\sigma}_{i,i}\hat{\sigma}_{1,0} & \hat{\sigma}_{i,i}\hat{\sigma}_{1,1} \end{pmatrix} A^\top e_i \quad (17)$$

As $\delta_i - \delta'_i = a_{i,1-i}^2 \det(\hat{\Sigma})$ and $G_0 = \log \left(1 + \frac{\delta_i - \delta'_i}{\delta'_i} \right)$, we

$$G_i = \log \left(1 + \frac{a_{i,1-i}^2 \det(\hat{\Sigma})}{\hat{\sigma}_{i,i}\sigma_{i,i}} \right).$$

□

Lemma 3 (The upper and lower bound for Granger causality). *For each $i = 0, 1$, Granger causality G_i has the following bounds:*

$$0 \leq G_i \leq \log \gamma_i, \quad (18)$$

where $\gamma_i := \frac{\hat{\sigma}_{i,i}}{\sigma_{i,i}} \geq 1$ due to the Lyapunov equation (4). The lower bound $G_i = 0$ is given only if

$$a_{i,1-i}^2 \det(\hat{\Sigma}) = 0. \quad (19)$$

The upper bound $G_i = \log \gamma_i$ is given only if

$$a_{i,i} \hat{\sigma}_{i,i} + a_{i,1-i} \hat{\sigma}_{1-i,i} = 0. \quad (20)$$

Proof. As the stationary covariance matrix is (semi-)positive definite, $\det(\hat{\Sigma}) \geq 0$. Thus the lower bound of Granger causality is $G_i \geq \log(1) = 0$ and this bound is only reachable when $a_{i,1-i}^2 \det(\hat{\Sigma}) = 0$.

Modifying (13) and (14), for $i = 0, 1$ we have

$$(a_{i,i} \hat{\sigma}_{i,i} + a_{i,1-i} \hat{\sigma}_{1-i,i})^2 = \hat{\sigma}_{i,i} (\hat{\sigma}_{i,i} - \sigma_{i,i} e^{G_i}). \quad (21)$$

As $(a_{i,i} \hat{\sigma}_{i,i} + a_{i,1-i} \hat{\sigma}_{1-i,i})^2 \geq 0$ and $\hat{\sigma}_{i,i} > 0$,

$$G_i \leq \log \gamma_i.$$

This lower bound holds, only if $a_{i,i} \hat{\sigma}_{i,i} + a_{i,1-i} \hat{\sigma}_{1-i,i} = 0$. $\square$

4 Stability and constraints of VAR

In this study, we primarily consider the class of stable VAR model with a proper set of the parameters. In this class, statistical nature of any VAR is characterized with the base covariance matrix $\Sigma \in \mathbb{R}^{2 \times 2}$, coefficient matrix $A \in \mathbb{R}^{2 \times 2}$, and stationary covariance matrix $\hat{\Sigma} \in \mathbb{R}^{2 \times 2}$. Let us denote the set of (strictly) positive definite matrices by

$$\mathbb{R}_+^{2 \times 2} := \{M \in \mathbb{R}^{2 \times 2} | \det(M) > 0 \text{ and } \text{tr}(M) > 0\},$$

and the set of coefficient matrices of stable VAR models

$$\mathbb{R}_*^{2 \times 2} := \{M \in \mathbb{R}^{2 \times 2} | -1 + |\text{tr}(M)| < \det(M) < 1\}.$$

We will shortly show the stable set $\mathbb{R}_*^{2 \times 2}$ include all and only coefficient matrices of stable VAR models.

With this notation of the set of matrices, the two conditions any proper VAR model needs to satisfy are as follows.

Stability Any stable VAR model has both of the eigenvalues λ_0, λ_1 of its coefficient matrix A be $|\lambda_0|, |\lambda_1| < 1$.

Properness To have a proper (non-degenerated) bivariate normal distribution in a VAR model, its base covariance matrix Σ and stationary covariance matrix $\hat{\Sigma}$ needs to satisfy $\Sigma, \hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$. The set of positive definite matrices is equivalently written with the entries of matrix

$$\mathbb{R}_+^{2 \times 2} = \{C \in \mathbb{R}^{2 \times 2} \mid C_{0,0} > 0, C_{1,1} > 0, \text{ and } C_{0,0}C_{1,1} - C_{0,1}C_{1,0} > 0\}. \quad (22)$$

4.1 Stability of VAR

As stated previously in Section 2.2, the stability of VAR is primarily characterized by the eigenvalues of the coefficient matrix A . But this condition is equivalent with $A \in \mathbb{R}_*^{2 \times 2}$, as shown by the following lemma.

Lemma 4. *A given bivariate VAR model with its coefficient matrix $A \in \mathbb{R}^{2 \times 2}$ is stable, if and only if*

$$|\text{tr}(A)| - 1 < \det(A) < 1. \quad (23)$$

Proof. Let λ be an eigenvalue of the coefficient matrix A . Then such an eigenvalue satisfies

$$f(\lambda) = |A - \lambda I_2| = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0. \quad (24)$$

If a VAR is stable, this eigenvalue needs to satisfy $|\lambda| < 1$. As (24) is rewritten by

$$f(\lambda) = \left(\lambda - \frac{1}{2}\text{tr}(A)\right)^2 - \frac{1}{4}\left(\text{tr}(A)^2 - 4\det(A)\right), \quad (25)$$

we analyze this condition on (24) for the following two cases with λ being real or non-real:

1. If λ is real, this stability condition is equivalent with

$$\text{tr}(A)^2 \leq 4\det(A), f(1) > 0, f(-1) > 0, |\text{tr}(A)| < 2. \quad (26)$$

2. If λ is not real, this stability condition is equivalent with

$$\text{tr}(A)^2 < 4\det(A), |\lambda|^2 < 1. \quad (27)$$

If λ of (24) is non-real (Case 2.), λ (and its conjugate) is

$$\lambda = \frac{1}{2}\text{tr}(A) \pm \frac{j}{2}\sqrt{|\text{tr}(A)^2 - 4\det(A)|}, \quad (28)$$

with imaginary unit denoted by j .

With the inequality (27), the stability condition in this case is

$$\left(\frac{\text{tr}(A)}{2}\right)^2 < |\lambda|^2 = \det(A) < 1. \quad (29)$$

If λ of (24) is real, $\text{tr}(A)^2 - 4\det(A) \geq 0$ and

$$f(1) = 1 - \text{tr}(A) + \det(A) > 0 \quad (30)$$

$$f(-1) = 1 + \text{tr}(A) + \det(A) > 0 \quad (31)$$

$$|\text{tr}(A)| < 2. \quad (32)$$

Combining (30) and (31), we have $|\text{tr}(A)| - 1 < \det(A)$. This inequality with (26),

$$C_0 < \det(A) \leq \left(\frac{\text{tr}(A)}{2}\right)^2 < C_1, \quad (33)$$

where

$$\begin{aligned} C_0 &:= |\text{tr}(A)| - 1, \\ C_1 &:= \min\left(1, \left(\frac{1 + \det(A)}{2}\right)^2\right). \end{aligned}$$

Find for an arbitrary $A \in \mathbb{R}^{2 \times 2}$ we have the following two inequalities

$$|\text{tr}(A)| - 1 \leq \left(\frac{1}{2}\text{tr}(A)\right)^2, \quad (34)$$

and

$$\det(A) \leq \left(\frac{1 + \det(A)}{2}\right)^2. \quad (35)$$

The inequality (34) holds equality for and only for $\text{tr}(A) = 2$, and the inequality (35) holds equality for and only for $\det(A) = 1$. As both of these equality conditions do not hold under (29), (29) is equivalent with

$$C_0 < \left(\frac{\text{tr}(A)}{2}\right)^2 < \det(A) < C_1. \quad (36)$$

Integrating the two inequalities (33) for real λ and (36) for non-real λ , the VAR with the coefficient matrix A is stable, if

$$C_0 < \det(A) < C_1 \quad (37)$$

and

$$C_0 < \left(\frac{\operatorname{tr}(A)}{2} \right)^2 < C_1. \quad (38)$$

As the inequality (37) implies $0 < \frac{1+\det(A)}{2} < 1$ and $\operatorname{tr}(A) < 2$, (38) is equivalent with

$$\left(\frac{\operatorname{tr}(A)}{2} \right)^2 < \left(\frac{1+\det(A)}{2} \right)^2. \quad (39)$$

As the upper bound for $\det(A)$ in (37) can be implied by $\det(A) < 1$, it is equivalent with

$$|\operatorname{tr}(A)| - 1 < \det(A) < 1. \quad (40)$$

Thus, the pair of the inequalities (37) and (38) for A is equivalent with the single inequality (40) for A . $\square$

4.2 Stability and existence of the solution for the Lyapunov equation

Intuitively, it would be reasonable that there is a stationary covariance matrix $\hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$ satisfying the Lyapunov equation (4), if the coefficient matrix is $A \in \mathbb{R}_*^{2 \times 2}$. However, this is not too trivial, as the opposite may not be always true: The existence of $\hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$ does not imply $A \in \mathbb{R}_*^{2 \times 2}$. This relationship between A and $\hat{\Sigma}$ is stated by the following Theorem 1.

Theorem 1. *There is a stationary covariance matrix $\hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$ satisfying the Lyapunov equation (4), if the coefficient matrix is $A \in \mathbb{R}_*^{2 \times 2}$. However, the existence of $\hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$ does not imply $A \in \mathbb{R}_*^{2 \times 2}$.*

Proof. By Lemma 4 and 5, $\det((I_4 - A \otimes A)) > 0$. Thus Lyapunov equation 5 has the solution for $\hat{\Sigma}$, as the matrix $(I_4 - A \otimes A)$ is invertible. The converse of this theorem does not hold, as we construct a counter-example of the coefficient matrix A such that $\det((I_4 - A \otimes A)) < 0$, with which there is a $\hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$, but such a VAR is not stable. $\square$

In the proof of Theorem 1, we use the following lemma.

Lemma 5. *For an arbitrary VAR model with a base covariance matrix $\Sigma \in \mathbb{R}_+^{2 \times 2}$, a coefficient matrix $A \in \mathbb{R}^{2 \times 2}$, and stationary covariance matrix $\hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$, there is the identity:*

$$\det((I_4 - A \otimes A)) = (1 - \det(A))^2 \left\{ (1 + \det(A))^2 - \text{tr}(A)^2 \right\}. \quad (41)$$

Proof. Expand the determinant as follows:

$$\det((I_4 - A \otimes A)) = \det(\{(I_2 - a_{0,0}A)(I_2 - a_{1,1}A) - a_{0,1}a_{1,0}A^2\}) \quad (42)$$

$$= \det((I_2 - A \text{tr}(A) + A^2 \det(A))). \quad (43)$$

$$= (1 - \det(A))^2 ((1 - a_{0,1} - a_{0,0}^2)(1 - a_{0,1} - a_{1,1}^2) - a_{0,1}a_{1,0} \text{tr}(A)). \quad (44)$$

$$= (1 - \det(A))^2 \left\{ (1 + \det(A))^2 - \text{tr}(A)^2 \right\}. \quad (45)$$

□

5 Design of bivariate timeseries given GCs

The goal of this manuscript is to derive a design principle of bivariate timeseries generated by a VAR model. The set of parameters in VAR include

- coefficient matrix A ,
- the base covariance matrix Σ ,
- stationary covariance matrix $\hat{\Sigma}$,
- and two ways of Granger causality G_0, G_1 .

There are equality constraints on these variables:

- The variables $A, \Sigma, \hat{\Sigma}$ needs to satisfy the Lyapunov equation (4).
- Granger causality G_i is a function of $a_{i,1-i}$, $\sigma_{i,i}$, and $\hat{\Sigma}$ (Lemma 2).

Besides, it is important to know the feasibility of a set of parameters in VAR, which constrains the range of these variables:

- the stability: $A \in \mathbb{R}_*^{2 \times 2}$ (Section 4),
- properness: $\Sigma, \hat{\Sigma} \in \mathbb{R}_+^{2 \times 2}$ (Section 4) and $\sigma_{i,i} \leq \hat{\sigma}_{i,i}$ due to the existence of a solution for the Lyapunov equation,
- and the bound for each Granger causality: $G_i \in [0, \log \gamma_i]$ (Lemma 3).

The Lyapunov equation (4) on the matrices can be decomposed into the three equations on the scalar variables as follows. For a coefficient matrix $A = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix}$, let us define two vectors by

$$a_0 := \begin{pmatrix} a_{0,0} \\ a_{0,1} \end{pmatrix}, \quad a_1 := \begin{pmatrix} a_{1,0} \\ a_{1,1} \end{pmatrix}.$$

Then Lyapunov equation is equivalently written with these vectors a_0, a_1 by

$$\hat{\sigma}_{0,0} - \sigma_{0,0} = a_0^\top \hat{\Sigma} a_0 \quad (46)$$

$$\hat{\sigma}_{1,1} - \sigma_{1,1} = a_1^\top \hat{\Sigma} a_1 \quad (47)$$

$$\hat{\sigma}_{0,1} - \sigma_{0,1} = a_0^\top \hat{\Sigma} a_1. \quad (48)$$

Equation (46) and (47) in the above implies that each of the vectors a_0 and a_1 is on an ellipsis on their plane. This gives the lower bound for $\hat{\sigma}_{i,i} \geq \sigma_{i,i}$ (i.e., one condition of the properness above), as $x^\top \hat{\Sigma} x \geq 0$ for any $x \in \mathbb{R}^2$ with a positive definite matrix $\hat{\Sigma}$.

By fixing G_0 and G_1 , it imposes each of the two vectors a_0 and a_1 on the two parallel lines by

$$(a_i^\top \hat{\sigma}_i)^2 = \tau_i^2, \quad (49)$$

where

$$\hat{\sigma}_i := (\hat{\sigma}_{i,0}, \hat{\sigma}_{i,1})^\top, \quad \tau_i^2 := \hat{\sigma}_{i,i}^2 (1 - \gamma_i^{-1} e^{G_i}).$$

Thus, the solution of a_i to satisfies the Lyapunov equation and the fixed Granger causality is the four intersections of the ellipsis and the two parallel lines (Figure 1). This ellipsis is obtained by scaling and shearing transformation to the standard circle $a_{0,0}^2 + a_{0,1}^2 = 1$. This observation gives the angular parametrization of the solution vector $(a_{0,0}, a_{0,1})^\top$, which is stated by Lemma 6 in the next section.

5.1 Solution

In what follows, we start with derivation of the coefficient matrix A as a root of the equality constraint by the Lyapunov equation (4) and the Granger causality, for a fixed proper $\hat{\Sigma}$, $\sigma_{i,i}$ and G_i for each $i = 0, 1$. The following Lemma 6 gives a necessary condition for the coefficient matrix $A \in \mathbb{R}^{2 \times 2}$ satisfying the equality conditions above. Note that, however, a solution A of this equation does not guarantee $A \in \mathbb{R}_*^{2 \times 2}$.

Figure 1: The ellipsis (46) and two parallel lines (49) on the plane $(a_{0,0}, a_{0,1}) \in \mathbb{R}^2$. The solution (a_0, a_1) is four intersections of these two. Granger causality takes its maximum with the largest $|a_{0,1}|$ and its minimum with $|a_{0,1}| = 0$.

Lemma 6. For a given set of parameters, a positive definite matrix $\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{0,0} & \hat{\sigma}_{0,1} \\ \hat{\sigma}_{1,0} & \hat{\sigma}_{1,1} \end{pmatrix} \in \mathbb{R}_+^{2 \times 2}$, $\sigma_{i,i} \in (0, \hat{\sigma}_{i,i})$, $G_i \in [0, \log \gamma_i]$ for each $i = 0, 1$, suppose that a coefficient matrix $A = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$, satisfies the set of the equations

$$\begin{cases} a_0^\top \hat{\Sigma} a_0 = \hat{\sigma}_{0,0} - \sigma_{0,0} \\ a_1^\top \hat{\Sigma} a_1 = \hat{\sigma}_{1,1} - \sigma_{1,1} \\ (\hat{\sigma}_0^\top a_0)^2 = \tau_0^2 \\ (\hat{\sigma}_1^\top a_1)^2 = \tau_1^2 \end{cases}, \quad (50)$$

where for $i = 0, 1$

$$\hat{\sigma}_i := (\hat{\sigma}_{i,0}, \hat{\sigma}_{i,1})^\top, \quad \tau_i^2 := \hat{\sigma}_{i,i}^2 (1 - \gamma_i^{-1} e^{G_i}).$$

Any coefficient matrix A of a root of this equation (50) is in the form of

$$A = \begin{pmatrix} S_0 \begin{pmatrix} \cos \theta_0 \\ \sin \theta_0 \end{pmatrix} & P_2 S_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \end{pmatrix}^\top \quad (51)$$

where each pair of the angles $\theta_0 \in [0, 2\pi)$ and $\theta_1 \in [0, 2\pi)$ takes one of the two or four pairs satisfying for each $i = 0, 1$

$$\sin^2 \theta_i = \frac{e^{G_i} - 1}{\gamma_i - 1} \quad (52)$$

and

$$P_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_i := \sqrt{1 - \gamma_i^{-1}} \begin{pmatrix} 1 & -\frac{\hat{\sigma}_{i,1-i}}{\hat{\sigma}_{i,i}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\hat{\sigma}_{i,i}}{\sqrt{\det(\hat{\Sigma})}} \end{pmatrix}$$

Proof. Find that the following pair of equations in the (50) is symmetric under exchange of $i = 0, 1$:

$$\begin{cases} a_i^\top \hat{\Sigma} a_i = \hat{\sigma}_{i,i} - \sigma_{i,i} \\ (\hat{\sigma}_i^\top a_i)^2 = \tau_i^2 \end{cases}. \quad (53)$$

Thus, we solve this for $i = 0$ below, and it holds for $i = 1$.

Solving the second equation of (53) for $a_{0,0}$, we have

$$a_{0,0} = \frac{\pm \tau_0 - a_{0,1} \hat{\sigma}_{0,1}}{\hat{\sigma}_{0,0}}. \quad (54)$$

Inserting this to the first equation of (53), we have

$$a_{0,1}^2 = \frac{\hat{\sigma}_{0,0}^2(1 - \gamma_0^{-1}) - \tau_0^2}{\det(\hat{\Sigma})}. \quad (55)$$

Inserting $\tau_0^2 = \hat{\sigma}_{0,0}^2(1 - \gamma_0^{-1}e^{G_0})$,

$$a_{0,1} = \pm \sqrt{\frac{\hat{\sigma}_{0,0}\sigma_{0,0}(e^{G_0} - 1)}{\det(\hat{\Sigma})}} \quad (56)$$

Inserting this to (54), we have at most four vectors $a_0 = (a_{0,0}, a_{0,1})^\top$ as the solution of (53) for $i = 1$:

$$a_0 = \begin{pmatrix} c_0 - \frac{\hat{\sigma}_{0,1}}{\hat{\sigma}_{0,0}}d_0 \\ d_0 \end{pmatrix}, \begin{pmatrix} c_0 + \frac{\hat{\sigma}_{0,1}}{\hat{\sigma}_{0,0}}d_0 \\ -d_0 \end{pmatrix}, \begin{pmatrix} -c_0 - \frac{\hat{\sigma}_{0,1}}{\hat{\sigma}_{0,0}}d_0 \\ d_0 \end{pmatrix}, \begin{pmatrix} -c_0 + \frac{\hat{\sigma}_{0,1}}{\hat{\sigma}_{0,0}}d_0 \\ -d_0 \end{pmatrix}, \quad (57)$$

where for $i = 0, 1$

$$c_i := \sqrt{1 - \gamma_i^{-1}e^{G_i}}, \quad d_i := \sqrt{\frac{\hat{\sigma}_{i,i}\sigma_{i,i}(e^{G_i} - 1)}{\det(\hat{\Sigma})}}.$$

By symmetry to $i = 0, 1$, there are at most four vectors as the solution of (53) for $i = 1$:

$$P_2 a_1 = \begin{pmatrix} a_{1,1} \\ a_{1,0} \end{pmatrix} = \begin{pmatrix} c_1 - \frac{\hat{\sigma}_{1,0}}{\hat{\sigma}_{1,1}}d_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} c_1 + \frac{\hat{\sigma}_{1,0}}{\hat{\sigma}_{1,1}}d_1 \\ -d_1 \end{pmatrix}, \begin{pmatrix} -c_1 - \frac{\hat{\sigma}_{1,0}}{\hat{\sigma}_{1,1}}d_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} -c_1 + \frac{\hat{\sigma}_{1,0}}{\hat{\sigma}_{1,1}}d_1 \\ -d_1 \end{pmatrix} \quad (58)$$

Find that these four solution vectors parameterized by

$$a_0 = S_0 \begin{pmatrix} \cos \theta_0 \\ \sin \theta_0 \end{pmatrix}, \quad a_1 = P_2 S_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix},$$

satisfy (53), if (θ_0, θ_1) holds (52), with the trigonometric identity $\cos^2 \theta_i + \sin^2 \theta_i = 1$ and

$$S_0^\top \hat{\Sigma} S_0 = (\hat{\sigma}_{0,0} - \sigma_{0,0}) I_2, \quad S_1^\top P_2^\top \hat{\Sigma} P_2 S_1 = (\hat{\sigma}_{1,1} - \sigma_{1,1}) I_2. \quad (59)$$

□

5.2 Sufficiency of the solution

For a given set of the parameters, Lemma 6 in the previous section gives a set of solutions of the coefficient matrix A for the Lyapunov equation. Note that not all of these solutions A is feasible, in the sense that it satisfies all constraints such as stability of A and properness of Σ . The following lemmas provide the sufficient condition for a solution A by checking properness of Σ and stability of A .

Lemma 7. *Suppose A is a solution of the equation (50) in Lemma 6, represented by a pair of (θ_0, θ_1) . In this case, $\Sigma \in \mathbb{R}_+^{2 \times 2}$, if and only if*

$$\cos \hat{\eta} - (\gamma_0 \gamma_1)^{-\frac{1}{2}} \leq \hat{\gamma}_0 \hat{\gamma}_1 \cos(\hat{\eta} - \theta_0 - \theta_1) \leq \cos \hat{\eta} + (\gamma_0 \gamma_1)^{-\frac{1}{2}} \quad (60)$$

where $\hat{\eta} \in [0, 2\pi]$ is the angler parameterization of the correlation coefficient defined by $\cos \hat{\eta} := \frac{\hat{\sigma}_{0,1}}{\hat{\sigma}_{0,0} \hat{\sigma}_{1,1}}$ and $\hat{\gamma}_i = \sqrt{1 - \gamma_i^{-1}}$.

Proof. Applying the polar representation of the a_0, a_1 in (51) in Lemma 6 to the third equation (48) of the Lyapunov equation, we have

$$\hat{\sigma}_{0,1} - \sigma_{0,1} = \sqrt{(\hat{\sigma}_{0,0} - \sigma_{0,0})(\hat{\sigma}_{1,1} - \sigma_{1,1})} \cos(\hat{\eta} - \theta_0 - \theta_1). \quad (61)$$

By the positive definiteness of Σ , $\sigma_{0,1}^2 \leq \sigma_{0,0} \sigma_{1,1}$. This inequality applied to (61) gives the lemma. $\square$

If we have

$$\hat{\gamma}_0 \hat{\gamma}_1 \leq \cos \hat{\eta} + (\gamma_0 \gamma_1)^{-\frac{1}{2}} \wedge \cos \hat{\eta} - (\gamma_0 \gamma_1)^{-\frac{1}{2}} \leq -\hat{\gamma}_0 \hat{\gamma}_1,$$

(60) holds for any pair of angles (θ_0, θ_1) . This condition is

$$|\cos \hat{\eta}| \leq (\gamma_0 \gamma_1)^{-\frac{1}{2}} - \hat{\gamma}_0 \hat{\gamma}_1,$$

or

$$|\hat{\sigma}_{0,1}| \leq \sqrt{\sigma_{0,0} \sigma_{1,1}} - \sqrt{(\hat{\sigma}_{0,0} - \sigma_{0,0})(\hat{\sigma}_{1,1} - \sigma_{1,1})}. \quad (62)$$

By Lemma 4, a stable VAR has A satisfying

$$-1 + |\text{tr}(A)| < \det(A) < 1 \quad (63)$$

Using the angular notation of the solution A with θ_0, θ_1 , we have

$$\det(A) = \hat{\gamma}_0 \hat{\gamma}_1 \frac{\sin(\hat{\eta} - \theta_0 - \theta_1)}{\sin \hat{\eta}} \quad (64)$$

and

$$\text{tr}(A) = \frac{\hat{\gamma}_0 \sin(\hat{\eta} - \theta_0) + \hat{\gamma}_1 \sin(\hat{\eta} - \theta_1)}{\sin \hat{\eta}}. \quad (65)$$

Thus, inserting these to the stability condition (23), we have

$$-\sin \hat{\eta} + |\hat{\gamma}_0 \sin(\hat{\eta} - \theta_0) + \hat{\gamma}_1 \sin(\hat{\eta} - \theta_1)| < \hat{\gamma}_0 \hat{\gamma}_1 \sin(\hat{\eta} - \theta_0 - \theta_1) < \sin \hat{\eta}. \quad (66)$$

6 Summary and remarks

In this manuscript, we analyzed bivariate VAR model, and explore the relationship between the parameters in VAR and its pair of Granger causality. Granger causality G_0, G_1 (in the form of angular θ_0, θ_1) is independently from the stationary covariance $\hat{\sigma}_{0,1}$ of the bivariate timeseries up to the constraint Lemma 7. Or the stationary covariance and variance of $\hat{\Sigma}$ and Σ needs to be constrained by (62), in order to let Granger causality be any value $G_i \in [0, \log \gamma_i]$. Thus, our analysis revealed the trade-off like relationship of the stationary covariance and Granger causality.

This half-constraint relationship – Granger causality is almost free from covariance of the bivariate timeseries, but needs to satisfy the range constraint imposed by covariance matrix. Perhaps, this loose logical connection between Granger causality and covariance would cause a conceptual confusion of temporal inter-relationship between two moving objects in literature. In literature, a sort of terms such as “synchronization”, “contingency”, “interaction”, and “lead-follow relationship” has been used to express some temporal inter-relationship between motions. Some of these terms was intended to indicate covariance, and the other was intended to indicate Granger causality, or a mixture of them.

Even if animacy perception is related to Granger causality but not to covariance, a carefully designed experiment needs to be conducted to clarify this conceptual confusion. Although covariance and Granger causality have the mutually constraining relationship as analyzed in this manuscript, Granger causality can be manipulable to a large extent with a fixed covariance value. Thus, we can discriminate the effect of Granger causality from the effect of covariance on animacy perception by carefully designing the psychological experiment.

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