A Gibbs Sampler for a Hierarchical Dirichlet Process Mixture Model

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THE *hierarchical Dirichlet process mixture model* (HDPMM), when its mixture components are categorical distributions, is a probabilistic model of multinomial data. It was first described as part of a more general description of Hierarchical Dirichlet Process (HDP) models by Teh, Jordan, Beal, and Blei (2004, 2006), and is the Bayesian nonparametric generalization of the Latent Dirichlet Allocation (LDA) model of Blei, Ng, and Jordan (2003).

The aim of this note is to describe in detail a Gibbs sampler for the HDPMM when used with multinomial data. This Gibbs sampler is based on what was described in Teh et al. (2004, 2006) for general HDP models. However, as they did not deal in detail with HDP mixture models for multinomial data, important details of the sampler required for this particular case were not described. On the other hand, Newman, Asuncion, Smyth, and Welling (2009) do deal explicitly with the case of the HDPMM for multinomial data, and the Gibbs sampler that we describe here is almost identical to theirs. However, for some hyper-parameters, Newman et al. (2009) either make simplifying assumptions or assume that their values are known, which are assumptions that we do not make here. As such, the sampler we describe here is an extension, though minor, of that described in Newman et al. (2009).

The probabilistic model

One of the most straightforward applications of the multinomial data HDPMM is as a *bag-of-words* probabilistic language model, and in what follows we'll describe it with this application in mind. However, modulo some possible changes in notation, this will in fact constitute a general description of the HDPMM for multinomial data.

In general, according to a bag-of-words language model, a corpus of natural language is a set of J documents or texts

$$w_1, w_2 \dots w_j \dots w_J,$$

where text j , i.e., w_j , is a set of n_j words from a finite vocabulary of V word types. For simplicity, this vocabulary can be represented as the V integers $\{1, 2 \dots V\}$. From this, we have each w_j defined as

$$w_j = w_{j1}, w_{j2} \dots w_{ji} \dots w_{jn_j},$$

with each $w_{ji} \in \{1 \dots V\}$. The *bag-of-words* assumption is that, for each text, $w_{j1}, w_{j2} \dots w_{ji} \dots w_{jn_j}$ are exchangeable random variables, i.e. their joint probability distribution is invariant to any permutation of the indices. By this assumption therefore, as the name implies, each text is modelled as an unordered set, or *bag*, of words.

As a generative model of this language corpus, the HDPMM treats each observed word w_{ji} as a sample from one of an underlying set of text or discourse topics:

$$\phi = \phi_1, \phi_2 \dots \phi_k \dots,$$

where each ϕ_k is a probability distribution over $\{1 \dots V\}$. The identity of the particular topic distribution from which w_{ji} is drawn is determined by the value of a discrete latent variable $x_{ji} \in \{1, 2 \dots k \dots\}$ that corresponds to w_{ji} . As such, for each w_{ji} , we model it as

$$w_{ji}|x_{ji}, \phi \sim \text{dcat}(\phi_{x_{ji}}).$$

To be clear, the HDPM assumes that there are an unlimited number of topic distributions from which the observed data are drawn, and so each x_{ji} can take on infinitely many discrete values. The probability distribution over the infinitely possible values of each x_{ji} is given by an infinite length categorical distribution π_j , i.e., $\pi_j = \pi_{j1}, \pi_{j2} \dots \pi_{jk} \dots$, where $0 \leq \pi_{jk} \leq 1$ and $\sum_{k=1}^{\infty} \pi_{jk} = 1$, that is specific to text j . In other words,

$$x_{ji}|\pi_j \sim \text{dcat}(\pi_j).$$

Each π_j is assumed to be drawn from a Dirichlet process prior whose base distribution, m , is a categorical distribution over the positive integers and whose scalar concentration parameter is α :

$$\pi_j|\alpha, m \sim \text{ddp}(\alpha, m).$$

The m base distribution is assumed to be drawn from a stick breaking distribution with a parameter γ :

$$m|\gamma \sim \text{dstick}(\gamma).$$

The prior distributions of the Dirichlet process concentration parameter α and the stick-breaking parameter γ are Gamma distributions, both with shape and scale parameters equal to 1. For the topic distributions, $\phi_1, \phi_2 \dots \phi_k \dots$, we can assume they are independently and identically drawn from a Dirichlet distribution with a length V location parameter ψ and concentration parameter b . In turn, ψ is drawn from a symmetric Dirichlet distribution with concentration parameter c . Finally, both b and c , like α and γ , can be given Gamma priors, again with shape and scale distributions equal to 1.

Sampling each latent variable x_{ji}

The posterior probability that x_{ji} takes the value k , for any $k \in 1, 2 \dots$, is

$$\begin{aligned} P(x_{ji} = k|w_{ji} = v, x_{-ji}, w_{-ji}, b, \psi, \alpha, m), \\ \propto P(w_{ji} = v|x_{ji} = k, w_{-ji}, x_{-ji}, b, \psi)P(x_{ji} = k|x_{-ji}, \alpha, m), \end{aligned}$$

where x_{-ji} denotes all latent variables excluding x_{ji} , with an analogous meaning for w_{-ji} . Here, the likelihood term is

$$P(w_{ji} = v|x_{ji} = k, x_{-ji}, w_{-ji}, b, \psi) = \int P(w_{ji} = v|\phi_k)P(\phi_k|x_{-ji}, w_{-ji}, b, \psi)d\phi_k.$$

This is the expected value of ϕ_{kv} according the Dirichlet posterior

$$P(\phi_k|x_{-ji}, w_{-ji}, b, \psi) = \frac{\Gamma(S_{k\cdot}^{-ji} + b)}{\prod_{v=1}^V \Gamma(S_{kv}^{-ji} + b\psi_v)} \prod_{v=1}^V \phi_{kv}^{S_{kv}^{-ji} + b\psi_v - 1},$$

where $S_{kv}^{-ji} \triangleq \sum_{j' \neq ji} \mathbb{I}(w_{j'i'} = v, x_{j'i'} = k)$ and $S_{k\cdot}^{-ji} = \sum_{v=1}^V S_{kv}^{-ji}$. As such,

$$P(w_{ji} = v|x_{ji} = k, x_{-ji}, w_{-ji}, b, \psi) = \frac{S_{kv}^{-ji} + b\psi_v}{S_{k\cdot}^{-ji} + b}.$$

The prior term, on the other hand, is

$$P(x_{ji} = k | x_{-ji}, a, m) = \int P(x_{ji} = k | \pi_j) P(\pi_j | x_{-ji}, a, m) d\pi_j,$$

and this the expected value of π_{jk} according to

$$P(\pi_j | x_{-ji}, a, m) \propto P(x_{-ji} | \pi_j) P(\pi_j | a, m).$$

As $P(\pi_j | a, m)$ is a Dirichlet process, by the definition of a Dirichlet process, we have

$$P(\pi_{jk} | a, m) = \text{Beta}(a m_k, a \sum_{\{k' \neq k\}} m_{k'}),$$

and therefore,

$$P(\pi_{jk} | x_{-ji}, a, m) = \text{Beta}(R_{jk}^{-ji} + a m_k, a \sum_{\{k' \neq k\}} R_{jk'}^{-ji} + m_{k'}),$$

where $R_{jk}^{-ji} \triangleq \sum_{i' \neq i}^{n_j} \mathbb{I}(x_{ji'} = k)$. As such, the expected value of π_{jk} is

$$P(x_{ji} = k | x_{-ji}, a, m) = \frac{R_{jk}^{-ji} + a m_k}{R_{j\cdot}^{-ji} + a},$$

where $R_{j\cdot}^{-ji} = \sum_{k=1}^{\infty} R_{jk}^{-ji}$.

Given these likelihood and prior terms, the posterior is simply

$$\begin{aligned} P(x_{ji} = k | w_{ji} = v, x_{-ji}, w_{-ji}, b, \psi, a, m) &\propto \frac{S_{kv}^{-ji} + b \psi_v}{S_{k\cdot}^{-ji} + b} \frac{R_{jk}^{-ji} + a m_k}{R_{j\cdot}^{-ji} + a}, \\ &\propto \frac{S_{kv}^{-ji} + b \psi_v}{S_{k\cdot}^{-ji} + b} \times (R_{jk}^{-ji} + a m_k). \end{aligned}$$

Note that from this, we also have

$$P(x_{ji} > K | x_{-ji}, w, b, \psi, a, m) \propto \sum_{\{k > K\}} \frac{S_{kv}^{-ji} + b \psi_v}{S_{k\cdot}^{-ji} + b} \times (R_{jk}^{-ji} + a m_k).$$

Given that for all $k > K$, where K is the maximum value of the set $\{x_{ji} : j \in 1 \dots J, i \in 1 \dots n_j\}$, we have $R_{jk}^{-ji} = 0$ and $S_{kv}^{-ji} = 0$, then

$$\begin{aligned} P(x_{ji} > K | x_{-ji}, w, b, \psi, a, m) &\propto \sum_{\{k > K\}} \frac{b \psi_v}{b} \times a m_k, \\ &= \psi_v \times a m_u \end{aligned}$$

where $m_u = \sum_{\{k > K\}} m_{jk}$.

As a practical matter of sampling, for each latent variable x_{ji} , we calculate

$$f_{jik} \propto \frac{S_{kv}^{-ji} + b \psi_v}{S_{k\cdot}^{-ji} + b} \times (R_{jk}^{-ji} + a m_k),$$

for $k \in 1, 2 \dots K$, and then

$$f_{jiu} = \psi_v \times a m_u$$

where K, m_u are defined as above and $v = w_{ji}$. Now,

$$P(x_{ji} \leq K | x_{-ji}, w, b, \psi, a, m) = \frac{\sum_{k=1}^K f_{jik}}{\sum_{k=1}^K f_{jik} + f_{jiu}}$$

and

$$P(x_{ji} > K | x_{-ji}, w, b, \psi, a, m) = \frac{f_{jiu}}{\sum_{k=1}^K f_{jik} + f_{jiu}},$$

and so a single random sample will be sufficient to decide if $x_{ji} \leq K$ or $x_{ji} > K$.

If $x_{ji} \leq K$, then

$$P(x_{ji} = k | x_{-ji}, w, b, \psi, a, m, x_{ji} \leq K) = \frac{f_{jik}}{\sum_{k=1}^K f_{jik}}.$$

On the other hand, if $x_{ji} > K$, the probability that $x_{ji} = k^{\text{new}}$ for $k^{\text{new}} > K$ is

$$P(x_{ji} = k^{\text{new}} | x_{-ji}, w, b, \psi, a, m, x_{ji} > K) = \frac{\psi_v \times a m_{k^{\text{new}}}}{f_{jiu}} = \frac{m_{k^{\text{new}}}}{m_u}.$$

Sampling m and a

The posterior distribution over the infinite length array m is

$$P(m | x_{1:J}, a, \gamma) \propto P(x_{1:J} | a, m) P(m | \gamma),$$

while the posterior over the scalar parameter a is

$$P(a | x_{1:J}) \propto P(x_{1:J} | a, m) P(a),$$

with the priors as stated above. The likelihood term in both cases is

$$P(x_{1:J} | a, m) = \prod_{j=1}^J \int \prod_{i=1}^{n_j} P(x_{ji} | \pi_j) P(\pi_j | a, m) d\pi_j,$$

where $\prod_{i=1}^{n_j} P(x_{ji} | \pi_j) = \prod_{k=1}^K \pi_{jk}^{R_{jk}}$, with $R_{jk} = \sum_{i=1}^{n_j} \mathbb{I}(x_{ji} = k)$ and K is, as stated above, the maximum value attained by any latent variable. The prior $P(\pi_j | a, m)$ is a Dirichlet process prior and so, by definition of the Dirichlet process,

$$P(\pi_{j1}, \pi_{j2} \dots \pi_{jk}, \pi_{ju} | a, m) = \frac{\Gamma(a)}{\Gamma(am_u) \prod_{k=1}^K \Gamma(am_k)} \pi_{ju}^{am_u-1} \prod_{k=1}^K \pi_{jk}^{am_k-1},$$

where $m_u = \sum_{\{k > K\}} m_{jk}$, as stated above, and $\pi_u = \sum_{\{k > K\}} \pi_{jk}$. Therefore,

$$\begin{aligned} P(x_{1:J} | a, m) &= \prod_{j=1}^J \int \prod_{i=1}^{n_j} P(x_{ji} | \pi_j) P(\pi_j | a, m) d\pi_j, \\ &= \prod_{j=1}^J \frac{\Gamma(a)}{\Gamma(am_u) \prod_{k=1}^K \Gamma(am_k)} \pi_{ju}^{am_u-1} \int \prod_{k=1}^K \pi_{jk}^{R_{jk}+m_k-1} d\pi_j, \\ &= \prod_{j=1}^J \frac{\Gamma(a)}{\Gamma(am_u) \prod_{k=1}^K \Gamma(am_k)} \frac{\Gamma(am_u) \prod_{k=1}^K \Gamma(R_{jk} + am_k)}{\Gamma(R_{j\cdot} + a)}, \\ &= \prod_{j=1}^J \frac{\Gamma(a)}{\Gamma(R_{j\cdot} + a)} \prod_{k=1}^K \frac{\Gamma(R_{jk} + am_k)}{\Gamma(am_k)}, \end{aligned}$$

and this can be re-written as

$$= \prod_{j=1}^J \frac{1}{\Gamma(R_{j\cdot})} \int_0^1 (\tau_j^r)^{a-1} (1 - \tau_j^r)^{R_{j\cdot}-1} d\tau_j^r \prod_{k=1}^K \sum_{\sigma_{jk}^r=0}^{R_{jk}} \mathbb{S}(R_{jk}, \sigma_{jk}^r) (am_k)^{\sigma_{jk}^r},$$

given that

$$\int_0^1 (\tau_j^r)^{a-1} (1 - \tau_j^r)^{R_{j\cdot}-1} d\tau_j^r \doteq \frac{\Gamma(a)\Gamma(R_{j\cdot})}{\Gamma(R_{j\cdot} + a)},$$

and

$$\sum_{\sigma_{jk}^r=0}^{R_{jk}} \mathbb{S}(R_{jk}, \sigma_{jk}^r) (am_k)^{\sigma_{jk}^r} \doteq \frac{\Gamma(R_{jk} + am_k)}{\Gamma(am_k)},$$

where $\mathbb{S}$ is an unsigned Stirling number of the first kind. By treating τ^r and σ^r as unobserved auxiliary variable, this leads to the augmented likelihood term

$$P(x_{1:J}|a, m, \tau^r, \sigma^r) = \prod_{j=1}^J \frac{1}{\Gamma(R_{j\cdot})} (\tau_j^r)^{a-1} (1 - \tau_j^r)^{R_{j\cdot}-1} \prod_{k=1}^K \mathbb{S}(R_{jk}, \sigma_{jk}^r) (am_k)^{\sigma_{jk}^r}.$$

With the augmented likelihood treated as a function of σ_{jk}^r , we have

$$P(x_{1:J}|a, m, \sigma^r) \propto \mathbb{S}(R_{jk}, \sigma_{jk}^r) (am_k)^{\sigma_{jk}^r}.$$

and with a uniform prior over the values of σ_{jk}^r , the posterior probability of σ_{jk}^r given all other variables is therefore

$$P(\sigma_{jk}^r|x_{1:J}, a, m) = \frac{\mathbb{S}(R_{jk}, \sigma_{jk}^r) (am_k)^{\sigma_{jk}^r}}{\sum_{\sigma_{jk}^r=0}^{R_{jk}} \mathbb{S}(R_{jk}, \sigma_{jk}^r) (am_k)^{\sigma_{jk}^r}}.$$

On the other hand, when the augmented likelihood is treated as a function of τ_j^r , we have

$$P(x_{1:J}|a, \tau_j^r) \propto (\tau_j^r)^{a-1} (1 - \tau_j^r)^{R_{j\cdot}-1},$$

and so, with a uniform prior on τ_j^r , the posterior is

$$P(\tau_j^r|x_{1:J}, a) \propto (\tau_j^r)^{a-1} (1 - \tau_j^r)^{R_{j\cdot}-1} = \text{Beta}(a, R_{j\cdot}).$$

Similarly, with the augmented likelihood treated as function of m , we have

$$P(x_{1:J}|m, \sigma^r) \propto \prod_{k=1}^K m_k^{\sum_{j=1}^J \sigma_{jk}^r} = \prod_{k=1}^K m_k^{\sigma_{\cdot k}^r}.$$

The prior on m is a stick-breaking prior, and so the probability distribution over $m_1 \dots m_K, m_u$ is $m_1 \sim \omega_1^0$, and then $m_k \sim \omega_k^0 (1 - \sum_{k'=1}^{k-1} m_{k'})$ for $k \in 2 \dots K$, and finally $m_u = 1 - \sum_{k=1}^K m_k$, where $\omega_1^0 \dots \omega_K^0$ are independently and identically distributed as $\text{Beta}(1, \gamma)$. This finite stick-breaking distribution is a $K + 1$ dimensional Generalized Dirichlet distribution, see Connor and Mosimann (1969); Wong (1998), whose parameter vectors are a vector of K 1's and a vector of K γ 's. As described in Wong (1998), the Generalized Dirichlet distribution is a conjugate prior for a multinomial likelihood. As such, the posterior over $m_1, m_2 \dots m_K, m_u$ is a Generalized Dirichlet distribution with length K parameter vectors $\alpha_1, \alpha_2 \dots \alpha_K$ and $\beta_1, \beta_2 \dots \beta_K$, where

$$\alpha_k = 1 + \sigma_{\cdot k}^r, \quad \beta_k = \gamma + \sum_{k'=k+1}^K \sigma_{\cdot k'}^r, \quad \text{for } k \in 1, 2 \dots K.$$

We can sample from this Generalized Dirichlet distribution by using a finite stick-breaking construction, as was used for the prior, i.e. for $m_1 \sim \omega_1$, $m_k \sim \omega_k(1 -$

$\sum_{k'=1}^{k-1} m_{k'}$ for $k \in 1 \dots K$, and $m_u = 1 - \sum_{k=1}^K m_k$, where $\omega_k \sim \text{Beta}(\alpha_k, \beta_k)$. Finally, when treated as a function of α , the augmented likelihood is

$$P(x_{1:J}|m, \sigma^r) = \prod_{j=1}^J (\tau_j^r)^\alpha \prod_{k=1}^K \alpha^{\sigma_{jk}^r} = e^{\alpha \sum_{j=1}^J \log \tau_j^r} \alpha^{\sum_{jk} \sigma_{jk}^r}.$$

As stated above, the prior on α is a Gamma distribution whose shape and scale parameters are both equal to 1.0. Therefore, the posterior is a Gamma distribution with shape and scale

$$\sum_{jk} \sigma_{jk}^r + 1, \quad \frac{1}{1 - \sum_{j=1}^J \log \tau_j^r},$$

respectively.

Sampling ψ and b

The posterior distribution over ψ is

$$P(\psi|w_{1:J}, x_{1:J}) \propto P(w_{1:J}|x_{1:J}, \psi, b)P(\psi|b),$$

while the posterior over the concentration parameter b is

$$P(b|w_{1:J}, x_{1:J}) \propto P(w_{1:J}|x_{1:J}, \psi, b)P(b).$$

In both cases, the likelihood term is

$$\begin{aligned} P(w_{1:J}|x_{1:J}, \psi, b) &= \int \prod_{j=1}^J \prod_{i=1}^{n_j} P(w_{ji}|x_{ji}, \phi_1, \phi_2, \dots) P(\phi_1, \phi_2, \dots | \psi, b) d\phi_1 d\phi_2 \dots, \\ &= \int \prod_{j=1}^J \prod_{i=1}^{n_j} P(w_{ji}|\phi_{x_{ji}}) \prod_{k=1}^\infty P(\phi_k | \psi, b) d\phi_1 d\phi_2 \dots, \\ &= \underbrace{\prod_{k=K+1}^\infty \int P(\phi_k | \psi, b) d\phi_k}_{=1} \int \prod_{j=1}^J \prod_{i=1}^{n_j} P(w_{ji}|\phi_{x_{ji}}) \prod_k^K P(\phi_k | \psi, b) d\phi_1 d\phi_2 \dots, \\ &= \frac{\Gamma(b)}{\prod_{v=1}^V \Gamma(b\psi_v)} \prod_{k=1}^K \int \prod_{v=1}^V \phi_{kv}^{S_{kv} - b\psi_v - 1} d\phi_k, \\ &= \prod_{k=1}^K \frac{\Gamma(b)}{\Gamma(S_{k\cdot} + b)} \prod_{v=1}^V \frac{\Gamma(S_{kv} + b\psi_v)}{\Gamma(b\psi_v)}, \end{aligned}$$

and, as was the case for $P(x_{1:J}|\alpha, m)$, this likelihood can be rewritten as

$$P(w_{1:J}|x_{1:J}, \psi, b) = \prod_{k=1}^K \frac{1}{\Gamma(S_{k\cdot})} \int_0^1 (\tau_k^s)^{b-1} (1 - \tau_k^s)^{S_{k\cdot}-1} d\tau_k^s \prod_{v=1}^V \sum_{\sigma_{kv}^s=0}^{S_{kv}} \mathbb{S}(S_{kv}, \sigma_{kv}^s) (b\psi_v)^{\sigma_{kv}^s}.$$

and by treating σ^s and τ^s as auxiliary variables, we obtain the augmented likelihood

$$\begin{aligned} P(w_{1:J}|x_{1:J}, \sigma^s, \tau^s, \psi, b) \\ = \prod_{k=1}^K \frac{1}{\Gamma(S_{k\cdot})} (\tau_k^s)^{b-1} (1 - \tau_k^s)^{S_{k\cdot}-1} \prod_{v=1}^V \mathbb{S}(S_{kv}, \sigma_{kv}^s) (b\psi_v)^{\sigma_{kv}^s}. \end{aligned}$$

Here, we assume the following parameterization of the Gamma distribution:

$$P(x|\alpha, s) = \frac{1}{s^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{s}},$$

where α and s are the shape and scale parameters, respectively.

As the prior for ψ is a symmetric Dirichlet with concentration parameter c , its posterior is the Dirichlet distribution

$$P(\psi|\sigma^s, c, V) \propto \prod_{v=1}^V \psi_v^{\sum_k \sigma_{kv}^s + c/V - 1}.$$

On the other hand, just like with the case of σ_{jk}^s , the posterior for each σ_{kv}^s is

$$P(\sigma_{kv}^s | w, x, \psi, b) = \frac{\mathbb{S}(S_{kv}, \sigma_{kv}^s) b \psi_v^{\sigma_{kv}^s}}{\sum_{\sigma_{kv}^s=0}^{S_{kv}} \mathbb{S}(S_{kv}, \sigma_{kv}^s) b \psi_v^{\sigma_{kv}^s}}$$

For the case of b , its prior is a Gamma distribution with shape and scale parameters equal to 1. The augmented likelihood treated as a function of b is

$$P(w_{1:j}|x_{1:j}, b) \propto \prod_k \tau_k^b \prod_v b^{\sigma_{kv}^s} = e^{b \sum_k \log \tau_k^s} b^{\sum_{kv} \sigma_{kv}^s},$$

and so the posterior is a Gamma distribution with shape and scale

$$\sum_{kv} \sigma_{kv}^s + 1, \quad \frac{1}{1 - \sum_k \log \tau_k^s},$$

respectively. Finally, with a uniform prior on each τ_k^s , and given that the augmented likelihood treated as a function of τ_k^s is

$$P(w_{1:j}|x_{1:j}, b, \tau_j^s) = (\tau_k^s)^{b-1} (1 - \tau_k^s)^{S_{k\cdot} - 1}$$

the posterior for each τ_k^s is $\text{Beta}(b, S_{k\cdot})$.

Sampling c

Rather than inferring c , the hyperprior of ψ , directly from the sampled value of ψ , we can infer c on the basis of the remaining observed variables, integrating over ϕ and ψ . The probability of the observed variables, integrating over ϕ and ψ , is

$$P(w_{1:j}|x_{1:j}, b, c) = \underbrace{\int \int P(w_{1:j}|x_{1:j}, \phi) P(\phi|b, \psi) d\phi P(\psi|c) d\psi}_{P(w_{1:j}|x_{1:j}, \psi, b)}.$$

Using the augmented version of $P(w_{1:j}|x_{1:j}, \psi, b)$ from the previous section, i.e., $P(w_{1:j}|x_{1:j}, \sigma^s, \tau^s, \psi, b)$, and treating this augmented likelihood as a function of ψ , we then have

$$\begin{aligned} \int P(w_{1:j}|\sigma^s, \psi) P(\psi|c) d\psi &\propto \int \prod_{v=1}^V \psi_v^{\sum_{k=1}^K \sigma_{kv}^s} \prod_{v=1}^V \psi_v^{c/V-1} d\psi, \\ &= \frac{\Gamma(c)}{\Gamma(c + \sigma_{\cdot}^s)} \prod_{v=1}^V \frac{\Gamma(\sigma_{\cdot v}^s + c/V)}{\Gamma(c/V)}, \end{aligned}$$

which is the likelihood function for c . As before, we can re-write this likelihood as

$$\frac{1}{\Gamma(\sigma_{\cdot}^s)} \int (\tau^q)^{c-1} (1 - \tau^q)^{\sigma_{\cdot}^s - 1} d\tau^q \prod_{v=1}^V \sum_{\sigma_v^q=0}^{\sigma_v^s} \mathbb{S}(\sigma_{\cdot v}^s, \sigma_v^q) \left(\frac{c}{V}\right)^{\sigma_v^q}$$

and treating τ^q and σ^q as auxiliary variables, we have the augmented likelihood

$$\frac{1}{\Gamma(\sigma^s)} (\tau^q)^{c-1} (1 - \tau^q)^{\sigma^s-1} \prod_{v=1}^V \mathbb{S}(\sigma^s_v, \sigma^q_v) \left(\frac{c}{V}\right)^{\sigma^q_v}.$$

Following the same procedure here as we used for the case of a and m , and b and ψ , the posterior for each σ^q_v is

$$\frac{\mathbb{S}(\sigma^s_v, \sigma^q_v) \frac{c}{V}^{\sigma^q_v}}{\sum_{\sigma^q_{v'}=0}^{\sigma^s_v} \mathbb{S}(\sigma^s_v, \sigma^q_{v'}) \frac{c}{V}^{\sigma^q_{v'}}}.$$

The posterior for c is the Gamma distribution with shape and scale

$$\sum_{v=1}^V \sigma^q_v + 1, \quad \frac{1}{1 - \log \tau^q},$$

respectively, and the posterior for τ^q is $\text{Beta}(c, \sigma^s)$.

Sampling γ

The posterior distribution for the stick-breaking parameter γ is

$$P(\gamma|m) \propto P(m|\gamma)P(\gamma).$$

As $m_1, m_2 \dots m_K, m_u$ is a deterministic function of $\omega_1, \omega_2 \dots \omega_K$, the likelihood of m given γ is equivalent to the likelihood of ω given γ , which is a product of Beta distributions

$$\begin{aligned} P(\omega|\gamma) &= \prod_{k=1}^K \text{Beta}(\omega_k|1, \gamma) \\ &\propto \frac{\Gamma(1+\gamma)}{\Gamma(\gamma)} \prod_{k=1}^K (1 - \omega_k)^{\gamma-1}, \\ &= \gamma^K e^{\gamma \sum_k \log(1 - \omega_k)}, \end{aligned} \quad \text{For any } a, \frac{\Gamma(1+a)}{\Gamma(a)} = a.$$

With a Gamma prior on γ , whose shape and scale equal 1, the posterior is also a Gamma distribution with shape and scale

$$K + 1, \quad \frac{1}{1 - \sum_k \log(1 - \omega_k)}.$$

respectively.

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