Approximate Objective Bayes Factors From P-Values and Sample Size: The $3p\sqrt{n}$ Rule

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Abstract

In 1936, Sir Harold Jeffreys proposed an approximate objective Bayes factor that quantifies the degree to which the point-null hypothesis $\mathcal{H}_0$ outpredicts the alternative hypothesis $\mathcal{H}_1$. This approximate Bayes factor (henceforth JAB_{01}) depends only on sample size and on how many standard errors the maximum likelihood estimate is away from the point under test. We revisit JAB_{01} and introduce a piecewise transformation that clarifies the connection to the frequentist two-sided p-value. Specifically, if $p \leq .10$ then $\text{JAB}_{01} \approx 3p\sqrt{n}$; if $.10 < p \leq .50$ then $\text{JAB}_{01} \approx \sqrt{pn}$; and if $p > .50$ then $\text{JAB}_{01} \approx p^{1/4}\sqrt{n}$. These transformation rules present p-value practitioners with a straightforward opportunity to obtain Bayesian benefits such as the ability to monitor evidence as data accumulate without reaching a foregone conclusion. Using the JAB_{01} framework we derive simple and accurate approximate Bayes factors for the t-test, the binomial test, the comparison of two proportions, and the correlation test.

Keywords: Evidence; standard error; Wald test; optional stopping

This research was supported by a Netherlands Organisation for Scientific Research (NWO) grant (016.Vici.170.083) and an Advanced ERC grant (743086 UNIFY). Correspondence should be sent to Eric-Jan Wagenmakers, University of Amsterdam, Nieuwe Achtergracht 129 B, 1018 WT Amsterdam, The Netherlands. E-mail may be sent to EJ.Wagenmakers@gmail.com. I am especially grateful to Alexander Ly for extensive suggestions for improvement. I also thank František Bartoš, Maarten Marsman, and Balazs Aczel for their comments on an earlier draft.

Afterwards when I was first exposed to Jeffreys's book, *Theory of Probability*, I had the feeling that if one carefully scrutinized the work one could very likely find Bayesian solutions to most statistical problems – but exasperatingly, the search often seemed to require nearly as much effort as the research.

Geisser, 1980

Introduction

For better or for worse, the null-hypothesis significance test remains firmly entrenched as the dominant statistical procedure across the empirical sciences. In the prototypical scenario, a null-hypothesis $\mathcal{H}_0$ assigns a parameter of interest θ the fixed value θ_0. This fixed value usually denotes the absence of an effect (e.g., $\mathcal{H}_0 : \theta_0 = {}^1\!/_2$ when θ is the chance parameter in a binomial model; $\mathcal{H}_0 : \theta_0 = 0$ when θ is a correlation coefficient). The alternative hypothesis $\mathcal{H}_1$ relaxes the restriction imposed by $\mathcal{H}_0$ and allows θ to be estimated from the data. For convenience and conceptual clarity we suppress notation concerning nuisance parameters; in addition, we consider only the case where a single parameter is subject to test.[1]

In frequentist statistics, the test of $\mathcal{H}_0$ centers on the much-maligned p-value (e.g., Wasserstein & Lazar, 2016; Wasserstein, Schirm, & Lazar, 2019):

$$p := p(y^n) = \Pr(|T| \geq |t(y^n)| \,;\, \mathcal{H}_0 : \theta = \theta_0)$$

where $p(y^n)$ denotes the two-sided p-value for observed data y of sample size n, T is a test statistic that quantifies the discrepancy from $\mathcal{H}_0$, and $t(y^n)$ is the observed test statistic. In words, the p-value is the probability under $\mathcal{H}_0$ of encountering a test statistic at least as extreme as the one that is observed. When the p-value is lower than a significance threshold α (usually $\alpha = .05$) $\mathcal{H}_0$ is said to be "rejected". Note that the p-value depends on more extreme outcomes of the test statistic that were not observed, violating the likelihood principle (e.g., Berger & Wolpert, 1988; Jeffreys, 1961, pp. 385–387). Also note that the p-value does not take into account the values of the test statistic that can be expected under $\mathcal{H}_1$.

In Bayesian statistics, the test of $\mathcal{H}_0$ is conducted using the Bayes factor (which is also much-maligned; e.g., Bernardo, 1980; Robert, 2016). Developed by Sir Harold Jeffreys, the Bayes factor contrasts the predictive performance of $\mathcal{H}_0$ against that of $\mathcal{H}_1$, in which θ has been assigned a prior distribution $g(\theta)$ (e.g., Etz & Wagenmakers, 2017; Jeffreys, 1935, 1939, 1961; Kass & Raftery, 1995; Ly, Verhagen, & Wagenmakers, 2016):

$$\mathrm{BF}_{01} = \frac{p(y^n \mid \mathcal{H}_0)}{p(y^n \mid \mathcal{H}_1)} = \frac{p(y^n \mid \theta_0)}{\int_\Theta p(y^n \mid \theta)\, g(\theta)\, \mathrm{d}\theta}. \tag{1}$$

[1]The recurring use of 'I' and 'mine' grates, and the remainder of this paper therefore uses 'we' and 'our' instead (for an alternative solution see Hetherington & Willard, 1975).

In contrast to the p-value, the Bayes factor is based on the (average) likelihood for the observed data; hence, the Bayes factor does obey the likelihood principle. Also, the Bayes factor is comparative, as it contrasts $\mathcal{H}_0$ against a well-defined $\mathcal{H}_1$. Thus, as a model evaluation method, the Bayes factor is relative whereas the p-value is absolute.

A long line of statistical research has explored the relation between p-values and Bayes factors. The purpose of such research is generally either to attempt some sort of reconciliation (i.e., a Bayes-frequentist compromise), to demonstrate that one method is superior to the other, or to achieve a deeper understanding on when and why the methods may lead to different conclusions. The most important insights from this line of research are the following:

1. The one-sided p-value is an approximate Bayes factor test of direction, that is, a Bayes factor that contrasts $\mathcal{H}_- : \theta < \theta_0$ against $\mathcal{H}_+ : \theta > \theta_0$ (Jeffreys, 1939, pp. 317-320, echoed in Jeffreys, 1961, pp. 387-390; see also Berger & Mortera, 1999; Casella & Berger, 1987; Greenland & Poole, 2013; P. M. Lee, 2012; Lindley, 1965; Morey & Wagenmakers, 2014; Pratt, 1965; Pratt, Raiffa, & Schlaifer, 1995; Rouanet, 1996). This is relevant insofar as it counters the popular objection to p-values that the point null hypothesis is never true exactly (e.g., Cohen, 1990; Tukey, 1991). From a Bayesian point of view, the p-value does not test $\mathcal{H}_0$ at all; rather, the p-value has a Bayesian interpretation only when $\mathcal{H}_0$ is known to be false from the outset – exactly the situation in which p-value detractors have argued that the p-value should *not* be used (see also Marsman & Wagenmakers, 2017). However, in many applications the key question concerns not the sign of the effect, but its presence. For example, in clinical trials the question of interest is usually not whether a new drug helps or hurts; the question is whether it helps or achieves nothing. From a Bayesian point of view, these questions are fundamentally different (Berger & Delampady, 1987; Berger & Sellke, 1987; Jefferys, 1990; Marsman & Wagenmakers, 2017), as a point null hypothesis makes predictions that are less extreme than those from both $\mathcal{H}_-$ and $\mathcal{H}_+$.

2. The p-value is based on a constant multiple of the standard error, whereas the Bayes factor is not (e.g., Jeffreys, 1935; Wagenmakers & Ly, 2021 and references therein). For instance, the test statistic W in the Wald test is given by

$$W = \left[\frac{\hat{\theta} - \theta_0}{\text{se}(\hat{\theta})}\right]^2,$$

where $\hat{\theta}$ is the maximum likelihood estimate and $\text{se}(\hat{\theta})$ its standard error. Under $\mathcal{H}_0$, W has a χ^2 distribution with one degree of freedom, and the corresponding quantile function yields the p-value. As will be clarified below, Jeffreys proposed an approximate Bayes factor which is proportional to a function of W, but multiplied by $\sqrt{n}$. The inclusion of the $\sqrt{n}$ terms means that conflicts between p-values and Bayes factors are certain to arise when the p-value remains fixed and n increases. For example, $p = .001$ usually signals a highly confident rejection of $\mathcal{H}_0$; yet when n is sufficiently high (say $n = 1,000,000$), the data may actually be less likely under $\mathcal{H}_1$ than under $\mathcal{H}_0$, that is, $\text{BF}_{01} > 1$. This phenomenon is known as the Jeffreys-Lindley paradox (e.g., Bartlett, 1957; Cousins, 2017; Jeffreys, 1935; Lindley, 1957), although it is worth emphasizing that Jeffreys already discovered, studied, explained, illustrated, and emphasized the paradox across a multitude of papers and books published in the second half of the 1930s (Wagenmakers & Ly, 2021).

3. If the prior $g(\theta)$ is calibrated (i.e., proportional to real-world frequency of occurrence), then the total number of Type I and Type II errors (i.e., $\alpha + \beta$) is minimized by using the criterion $\text{BF}_{01} = 1$ (Jeffreys, 1939, p. 327, echoed in Jeffreys, 1961, pp. 396-397). In general, minimizing the weighted sum of the two errors (i.e., $\lambda\alpha + \beta$) brings Bayesian and frequentist testing procedures in closer alignment (e.g., Cornfield, 1966; DeGroot, 1975; Leamer, 1978; Lehmann, 1958; Lindley, 1953; Pérez & Pericchi, 2014; Pericchi & Pereira, 2016). Intuitively, when power $1 - \beta$ is very high (e.g., because of high sample size), some of it can be sacrificed in order to lower α, a procedure that Egon Pearson called a "quite legitimate device" (Pearson, 1953, p. 69). Although rarely applied in practice, the total-error-minimization rule does provide a frequentist justification for decreasing α with n – and in this case the critical p-value threshold would no longer be a constant multiple of the standard error.

4. Motivated by the work of Jeffreys, Jack Good repeatedly advocated a "Bayes/non-Bayes compromise" where the frequentist p-value is adjusted using the Bayes factor $\sqrt{n}$ term (e.g., Good, 1982; Good, 1984b; Good, 1988, p. 391; Good, 1992). Specifically,

   > "Since a P-value for a sample of size N would convert to about the same Bayes factor as one of $\text{P}\sqrt{\text{N}/100}$ for a sample of size 100, when this is less than say 0.5, we may write $\text{Q} = \min(0.5, \sqrt{\text{N}}\,\text{P}/10)$ and call it a *standardized* P-*value*, or *standardized tail-area probability* or a Q-*value* for short. Its interpretation in non-Bayesian terms (though based on some Bayesian thinking) is: *the evidence against the null hypothesis is (about) the same as if a tail-area probability of* Q *had been obtained from a sample of size* 100." (Good, 1982, p. 65; italics in original)

   Good (1992, p. 600) concluded that

   > "we have empirical evidence that sensible *P* values are related to weights of evidence and, therefore, that *P* values are not entirely without merit. The real objection to *P* values is not that they usually are utter nonsense, but rather that they can be highly misleading,

> especially if the value of N is not also taken into account and is large."

The present work may be considered a more precise and easier to interpret version of Good's 1982 proposal.

5. When misinterpreted as $p(\mathcal{H}_0 \,|\, y^n)$, the p-value is "violently biased against the null hypothesis" (Edwards, 1965, p. 400; see also Colquhoun, 2014; Dickey, 1977; Johnson, 2013). After exploring a range of prior distributions $g(\theta)$, Edwards, Lindman, and Savage (1963) conclude that "Even the utmost generosity to the alternative hypothesis cannot make the evidence in favor of it as strong as classical significance levels might suggest." (p. 228). Related work by Berger and Delampady (1987) and Berger and Sellke (1987) showed that when $p = .05$, the upper bound on the Bayes factor BF_{10} is about $2^1/_2$; with equal prior model probabilities, this modest level of evidence translates to a lower bound on the posterior probability for $\mathcal{H}_0$ of about .30. The Bayes factor bounds are obtained by choosing prior distributions $g(\theta)$ that are favorable to $\mathcal{H}_1$. Berger and Delampady (1987, p. 330) leave little doubt about the implications of this relation between p-values and Bayes factors; the section "What should be done?" starts with the following recommendation:

   > "First and foremost, when testing precise hypotheses, formal use of P-values should be abandoned. Almost anything will give a better indication of the evidence provided by the data against H_0."

   To counteract the "violent bias" against $\mathcal{H}_0$, Benjamin et al. (2018) recently proposed to lower the α-level for new discoveries from the usual .05 (a 2-sigma result) to .005 (a 3-sigma result). This is in line with a remark by Jeffreys (1980), who pointed out that for sample sizes from about 10 to 500, the Bayes factor results "are not far from the rough rule long known to astronomers, i.e., that differences up to twice the standard error usually disappear when more or better observations become available, and that those of three or more time usually persist." (p. 453).

6. Sellke, Bayarri, and Berger (2001) proposed a Bayesian calibration of the p-value (see also Vovk, 1993). This calibration was derived by considering the diagnosticity of a p-value, that is, its likelihood of occurrence under $\mathcal{H}_0$ versus that under a most favorable $\mathcal{H}_1$.[2] The key result is that, for $p < {}^1/_e \approx .37$, we obtain the following "Vovk-Sellke bound" on the Bayes factor:

   $$\text{BF}_{01} > -e\, p \log p.$$

   This bound is relatively close to that derived in Berger and Delampady (1987) from a parametric Bayesian perspective. When $p = .05$ ("reject the null"), the

[2]For a Shiny app and a cartoon see `https://www.shinyapps.org/apps/vs-mpr/`.

Vovk-Sellke bound gives $\text{BF}_{01} > .41$ (i.e., $\text{BF}_{10} < 2.46$); when $p = .01$, the bound gives $\text{BF}_{01} > .13$ (i.e., $\text{BF}_{10} < 7.99$); and when $p = .005$, the bound gives $\text{BF}_{01} > .07$ (i.e., $\text{BF}_{10} < 13.89$). A sample-size dependent extension of the Vovk-Sellke bound was presented by Held and Ott (2016); for a review see Held and Ott (2018).

The present work further explores the relation between *p*-values and Bayes factors. Below we first call attention to Jeffreys's 1936 approximate form of the Bayes factor, JAB_{01}, which makes the relation to *p*-values exact. We then show that JAB_{01} stands to the BIC (Schwarz, 1978) as the Wald test stands to the likelihood ratio test. A convenient piecewise approximation to the quantile function of the χ^2_1 distribution is then used to obtain an approximate relation between the *p*-value and JAB_{01} (and, by extension, BIC). Specifically, for $p \leq .10$, we propose that $\text{JAB}_{01} \approx 3p\sqrt{n}$. We illustrate the similarity to existing default Bayes factor tests, and showcase the pragmatic benefits of the proposed transformation for examples featuring the *t*-test, the binomial test, the comparison of two proportions, and the correlation test.

Jeffreys's General Approximation to the Bayes Factor

In the 1930s, Jeffreys's statistical interests turned to the development of a Bayesian hypothesis test. In the 1935 article *Some tests of significance, treated by the theory of probability*, the introductory paragraph states the goal:

> "It often happens that when two sets of data obtained by observation give slightly different estimates of the true value we wish to know whether the difference is significant. The usual procedure is to say that it is significant if it exceeds a certain rather arbitrary multiple of the standard error; but this is not very satisfactory, and it seems worth while to see whether any precise criterion can be obtained by a thorough application of the theory of probability." (Jeffreys, 1935, p. 203)

Jeffreys then proceeds to derive Bayes factor hypothesis tests for contingency tables, for a comparison of two means, for correlation, and for periodicity. In these tests, the Bayesian result is *not* a constant multiple of the standard error, as it is for standard frequentist *p*-value hypothesis tests, but also depends on sample size:

> "It is therefore not correct to say that a systematic difference becomes significant when it reaches any constant multiple of its standard error" (Jeffreys, 1935, p. 207)

One year later, Jeffreys (1936, p. 417) first presents a general approximate form of his Bayesian hypothesis tests, a form he would highlight throughout his career (e.g., Jeffreys, 1937, pp. 250-251; Jeffreys, 1938b, p. 161; Jeffreys, 1938c, p. 310; Jeffreys, 1938a, p. 382; Jeffreys, 1939, pp. 193-194, repeated in Jeffreys, 1948, p. 221 and p. 251 and in Jeffreys, 1961, p. 247 and p. 277; Jeffreys, 1950, p. 316;

Jeffreys, 1955, p. 282; Jeffreys, 1957, p. 349; Jeffreys, 1973, p. 75; Jeffreys, 1977, p. 89; and Jeffreys, 1980, p. 453). For completeness and historical interest, Appendix A provides Jeffreys's original derivation and interpretation; below we provide the result in modern notation.

Assume we wish to obtain the Bayes factor for $\mathcal{H}_0 : \theta = \theta_0$ versus $\mathcal{H}_1 : \theta \sim g(\theta)$. As usual in objective Bayesian statistics (Berger & Delampady, 1987; Consonni, Fouskakis, Liseo, & Ntzoufras, 2018) we take $g(\theta)$ to be continuous, unimodal, symmetric, and centered on θ_0, the value under test. However, we abandon the usual approach of considering predictive performance for the observed data y^n and instead focus solely on the maximum likelihood point estimate (MLE) and its standard error: $\hat{\theta} \pm \mathrm{se}(\hat{\theta}) = \sigma/\sqrt{n}$, where σ indicates the sampling variability of a single observation. Thus, we compare how likely the MLE is to occur under $\mathcal{H}_0$ vis-à-vis $\mathcal{H}_1$. Assume that the sampling distribution of the MLE is normal, and that $\mathrm{se}(\hat{\theta}) \ll \sigma_g$, where σ_g is the scale of the prior distribution $g(\theta)$. Then Jeffreys's approximate Bayes factor in favor of $\mathcal{H}_0$ over $\mathcal{H}_1$ equals

$$
\begin{aligned}
\mathrm{JAB}_{01} &= \frac{1}{\sqrt{2\pi}\ \mathrm{se}(\hat{\theta})\ g(\hat{\theta})}\ \exp\left(-\tfrac{1}{2}\left[\frac{\hat{\theta}-\theta_0}{\mathrm{se}(\hat{\theta})}\right]^2\right) \\
&= \frac{1}{\sqrt{2\pi}\ \sigma\ g(\hat{\theta})}\ \sqrt{n}\ \exp\left(-\tfrac{1}{2}\left[\frac{\hat{\theta}-\theta_0}{\mathrm{se}(\hat{\theta})}\right]^2\right) \\
&= A\ \sqrt{n}\ \exp\left(-\tfrac{1}{2}\left[\frac{\hat{\theta}-\theta_0}{\mathrm{se}(\hat{\theta})}\right]^2\right).
\end{aligned} \tag{2}
$$

The exponential term includes the Wald test statistic that uniquely determines the p-value. As mentioned above, JAB_{01} also includes a $\sqrt{n}$ term which underscores the fact that the Bayesian evidence is not a constant multiple of the standard error: for a fixed p-value, the statistical argument against $\mathcal{H}_0$ decreases in strength with $\sqrt{n}$. Before proceeding it should be stressed that a similar form of JAB_{01} is obtained in the presence of nuisance parameters, except when these are not orthogonal to the test-relevant parameter θ *and* when their prior distribution is relatively peaked (Jeffreys, 1961, pp. 249-251).

Eq. 2 allows several insights about evidence, replications, and sequential planning that are independent of the prior distribution $g(\theta)$, and therefore hold in considerable generality. In order not to interrupt the flow of the main argument, a discussion on these topics has been relegated to an online appendix, which we attach for completeness as Appendix B.

Jeffreys's Simplified Approximation to the Bayes Factor

In the first part of this paper, we take advantage of Jeffreys's insight that for the tests he proposed, the value of A in Eq. 2 is "usually not far from 1" (Jeffreys, 1977, p. 89). This occurs whenever the scale of the prior distribution approximately matches that of the sampling variability for a single observation, which defines the *unit-information prior* (cf. Bové & Held, 2011; Consonni et al., 2018, pp. 643-644; Cousins, 2017; Jeffreys, 1961, p. 268; Kass & Wasserman, 1995; Smith & Spiegelhalter, 1980; Raftery, 1998; Ntzoufras, 2009; Overstall & Forster, 2010; Zellner & Siow, 1980; for an early hint see Jeffreys, 1935, pp. 211-212).

Specifically, Jeffreys derived JAB_{01} by assuming that (1) under $\mathcal{H}_0$, the probability of obtaining $\hat{\theta}$ is subject only to Gaussian sampling variability around θ_0 as quantified by $\text{se}(\hat{\theta})$; under $\mathcal{H}_1$, the probability of obtaining $\hat{\theta}$ is approximately $g(\hat{\theta})$, that is, the height of the prior distribution evaluated at the MLE (see Appendix A for details). Let $g(\theta)$ be a normal prior distribution with mean μ_g and standard deviation σ_g. We then have

$$\begin{aligned}\text{JAB}_{01} &\approx \frac{p(\hat{\theta} \mid \mathcal{H}_0)}{p(\hat{\theta} \mid \mathcal{H}_1)} \\ &\approx \frac{p(\hat{\theta} \mid \hat{\theta} \sim N(\theta_0, \text{se}(\hat{\theta})^2))}{p(\hat{\theta} \mid \hat{\theta} \sim N(\mu_g, \sigma_g^2))} \\ &= \frac{\left[\sqrt{2\pi}\,\sigma\right]^{-1} \sqrt{n}\ \exp\left(-\frac{1}{2}\left[(\hat{\theta}-\theta_0)/\text{se}(\hat{\theta})\right]^2\right)}{\left[\sqrt{2\pi}\,\sigma_g\right]^{-1} \exp\left(-\frac{1}{2}\left[(\hat{\theta}-\mu_g)/\sigma_g\right]^2\right)} \\ &= \frac{\sigma^{-1}\sqrt{n}\ \exp\left(-\frac{1}{2}n\left[(\hat{\theta}-\theta_0)/\sigma\right]^2\right)}{\sigma_g^{-1}\exp\left(-\frac{1}{2}\left[(\hat{\theta}-\mu_g)/\sigma_g\right]^2\right)}\end{aligned} \tag{3}$$

The unit-information prior entails the specification $\sigma_g = \sigma$. When $\mu_g = \theta_0$, as is the customary choice in objective Bayesian testing, Eq. 3 simplifies to

$$\text{JAB}_{01} \approx \sqrt{n}\ \exp\left(-\frac{1}{2}(n-1)\left[\frac{\hat{\theta}-\theta_0}{\sigma}\right]^2\right). \tag{4}$$

Alternatively, one may consider the choice $\mu_g = \hat{\theta}$. Centering the prior on the MLE biases the Bayes factor in favor of $\mathcal{H}_1$ and is antithetical to Jeffreys's approach to testing. Nevertheless, this choice conveniently eliminates the exponential term in the numerator of Eq. 3, resulting in

$$\text{JAB}_{01} \approx \sqrt{n}\ \exp\left(-\frac{1}{2}n\left[\frac{\hat{\theta}-\theta_0}{\sigma}\right]^2\right). \tag{5}$$

Comparing Eq. 4 and Eq. 5 shows that the effect of using the MLE instead of θ_0 as the prior mean is to increase the evidence in favor of $\mathcal{H}_1$ by that contained in a single observation. The bias is usually slight and noticeable only when sample size is low and $(\hat{\theta} - \theta_0)/\sigma$ is large.

As shown above, a normal unit-information prior yields $A = 1$ exactly. Based on his invariance theory, Jeffreys has also suggested the general approximation $A = \pi/\sqrt{2\pi} = \sqrt{\pi/2} \approx 1.253$ (Jeffreys, 1961, p. 277, his Eq. 5), a value that will resurface later. First however we will focus on the simplified version of JAB_{01} with $A = 1$, that is,

$$\begin{aligned} \text{JAB}_{01} &\approx \sqrt{n}\ \exp\left(-\tfrac{1}{2}\left[\frac{\hat{\theta} - \theta_0}{\text{se}(\hat{\theta})}\right]^2\right) \\ &= \sqrt{n}\ \exp\left(-\tfrac{1}{2}W\right), \end{aligned} \tag{6}$$

where $W = t^2$ is the Wald statistic.[3] Remarkably, this simple expression has languished in relative obscurity for 85 years. Over time it has been mentioned by a handful of authors (e.g., Berger & Sellke, 1987; Cousins, 2017; Dawid, 1984, p. 288; Edwards et al., 1963; Good, 1992, p. 600; Kass & Raftery, 1995, p. 778; Lindley, 1957; Raftery, 1995; Zellner, 1971/1996, p. 306) but to the best of our knowledge it has seen no practical application.[4] As detailed in the remarks below, Eq. 6 offers several insights and statistical opportunities.

Remark 1: JAB_{01} as a Precursor to BIC Model Comparison

For model i, the widely-used Bayesian information criterion (BIC; Schwarz, 1978) is defined as

$$\text{BIC}_i = k_i \ln n - 2 \ln p(y^n \mid \hat{\theta}_i),$$

where k_i is the number of adjustable parameters. Consider the scenario under investigation, where $\mathcal{H}_1$ has one free parameter more than $\mathcal{H}_0$. Then the BIC-based approximation to the Bayes factor (Kass & Raftery, 1995; Kass & Wasserman, 1995; Smith & Spiegelhalter, 1980) equals

$$\begin{aligned} \text{BF}_{01}^{\text{BIC}} &= \sqrt{n}\ \frac{p(y^n \mid \theta_0)}{p(y^n \mid \hat{\theta})} \\ &= \sqrt{n}\ \exp\left[-\tfrac{1}{2}\left(-2\ln\frac{p(y^n \mid \theta_0)}{p(y^n \mid \hat{\theta})}\right)\right] \\ &= \sqrt{n}\ \exp\left(-\tfrac{1}{2}\lambda_{LR}\right), \end{aligned} \tag{7}$$

[3] Note the asymptotic equality to the Bayesian z-test of a zero mean under a unit-information prior: $\text{BF}_{01} = \sqrt{n+1}\ \exp\{-\frac{1}{2}\, n/(n+1)\, Z^2\}$ with $Z = \bar{y}/(\sigma/\sqrt{n})$ (Clyde et al., 2021).

[4] Admittedly the expression did make a brief appearance on the popular American television sitcom The Big Bang Theory, S8:E23: "The Maternal Combustion", after 9 minutes and 12 seconds.

where λ_{LR} is the likelihood ratio test statistic. A comparison of Eq. 7 and Eq. 6 shows that JAB_{01} and $\text{BF}_{01}^{\text{BIC}}$ are closely related; tests based on W and λ_{LR} are asymptotically identical, and in the non-asymptotic case they are usually nearly identical (Engle, 1984). Later it will be important that both W and λ_{LR} are asymptotically χ_1^2 distributed under $\mathcal{H}_0$.

In sum, Jeffreys anticipated the BIC by over four decades, in the sense that JAB_{01} is a Wald-style version of the BIC for the comparison of nested models that differ in the presence of a single parameter. Stone (1979) is one of few statisticians to explicitly acknowledge Jeffreys's primacy:

> "He [Schwarz – EWAL] gave a rigorous justification of (1) [Schwarz' criterion – EWAL] for the case of independent observations from an exponential family distribution, without reference to its earlier Bayesian manifestations in Jeffreys (1948: Section 5.0, p. 221; Section 6.2, p. 316)." (Stone, 1979, p. 276)

Jeffreys's Wald-style JAB_{01} formulation offers three advantages over the BIC. First, JAB_{01} is easy to interpret and easy to compute, as it requires only the estimation of parameters under $\mathcal{H}_1$. Second, JAB_{01} can be tuned to priors other than the unit-information prior by adjusting the 'A' factor outside the exponential. Third, JAB_{01} allows a more straightforward assessment of the sample size $\sqrt{n}$ term. The examples that conclude this paper will demonstrate the latter two advantages. Compared to BIC, an obvious limitation of JAB_{01} is that it applies only to nested models.

Remark 2: A Bayesian p-Value

It is tempting to misinterpret the frequentist p-value as $p(\mathcal{H}_0 \mid y^n)$, the posterior probability of the null hypothesis. This posterior probability can be obtained from JAB_{01} as follows:

$$p_{\text{JAB}} = \frac{\text{JAB}_{01} \, \frac{p(\mathcal{H}_0)}{p(\mathcal{H}_1)}}{1 + \left[\text{JAB}_{01} \, \frac{p(\mathcal{H}_0)}{p(\mathcal{H}_1)}\right]}, \tag{8}$$

where the prior odds $\frac{p(\mathcal{H}_0)}{p(\mathcal{H}_1)}$ are usually set to 1, in accordance with Jeffreys's simplicity postulate (e.g., Jeffreys, 1950, p. 316).

Remark 3: Evidence in favor of $\mathcal{H}_0$

It is well-known that large, non-significant p-values do not necessarily indicate *evidence of absence*; they may also result from underpowered studies, in which case these p-values signal *absence of evidence* (e.g., Keysers, Gazzola, & Wagenmakers, 2020). Thus, students are regularly warned against concluding that $p > .05$ constitutes support for $\mathcal{H}_0$ (e.g., Greenland et al., 2016).

This common wisdom was challenged early on by Joseph Berkson:

> "If a test for the difference between means has yielded a large or middle P, it does not merely fail to disprove the null hypothesis that the true means are equal; it furnishes *affirmative evidence* that the means *are* substantially equal. If the numbers on which the test is based are large, the evidence will have convincing weight; otherwise not." (Berkson, 1942, p. 333; italics in original)

Berkson's conjecture is corroborated by JAB_{01}. Eq. 6 shows that the evidence that the data provide for $\mathcal{H}_0$ versus $\mathcal{H}_1$ can be decomposed as the balance between two opposing forces: the $\sqrt{n}$ factor increases the evidence for $\mathcal{H}_0$, whereas the Wald statistic increases the evidence for $\mathcal{H}_1$. The Wald statistic is uniquely associated with a p-value, and hence it follows that the same p-value can be evidence for $\mathcal{H}_1$, no evidence, or evidence for $\mathcal{H}_0$, depending on sample size. When the Wald statistic yields a middling p-value and sample size is high, JAB_{01} will be large and indicate evidence of absence rather than absence of evidence. For instance, when $W = 0$, the evidence for $\mathcal{H}_0$ is maximal and equals $\sqrt{n}$:

> "it remains true that the outside factor in the support for q [$\mathcal{H}_0$ – EWAL] is of order $n^{\frac{1}{2}}$; this factor would be the support provided if the estimates happened to agree exactly with the predictions made by q." (Jeffreys, 1938b, p. 164)

As mentioned above, with W fixed (and hence p fixed), larger sample sizes signal increasingly weak support for $\mathcal{H}_1$ against $\mathcal{H}_0$. Specifically, when the MLE is c standard errors from the point of test, $\text{JAB}_{01} = \sqrt{n} \cdot \exp(-\frac{1}{2}c^2) \propto \sqrt{n}$, such that the increase in evidence for $\mathcal{H}_0$ is proportional to $\sqrt{n}$. This pattern is visualized in Figure 1.

The p-Value Transformation Rules

In order to clarify the relation between p-values and Jeffreys-style approximate Bayes factors we may try to write Eq. 6 in terms of p instead of W, noting that p can be uniquely obtained from W by means of the quantile function of the χ^2_1 distribution. In order to arrive at a simple rule, this quantile function (shown in Figure 2a) needs to be approximated by a function that keeps Eq. 6 simple without being wildly inaccurate. After some attempts we settled on a piecewise approximation with three segments: $p \leq .10$, $.10 < p \leq .50$, and $p > .50$. For each segment, simple expressions for JAB_{01} were obtained by fitting functions of the form $a - b \log p$ and then rounding the results. Figures 2b, 2c, and 2d show the functions used. Note that for the middle segment, we propose two functions: one produces a simple result that is slightly inaccurate, whereas the other produces an accurate result but at the cost of adopting a more complex form.

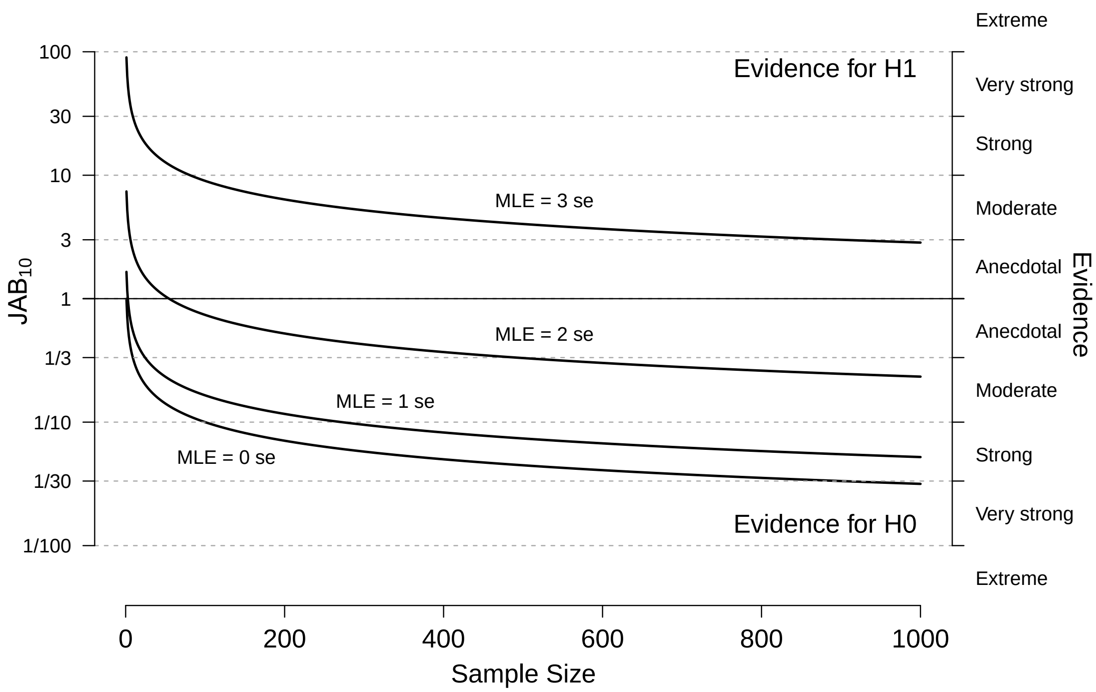
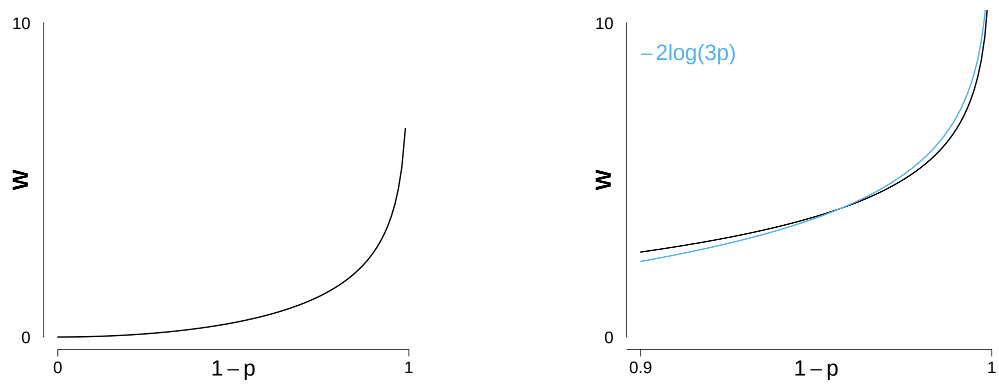


Figure 1. Statistical evidence as quantified by JAB_{10} is a balance between two opposing forces: (1) the number of standard errors that the MLE $\hat{\theta}$ is away from the point under test θ_0, favoring $\mathcal{H}_1$; and (2) sample size n, favoring $\mathcal{H}_0$. Consistent with (1), for a particular fixed sample size the evidence in favor of $\mathcal{H}_1$ increases with $W = \left[(\hat{\theta} - \theta_0)/\mathrm{se}(\hat{\theta})\right]^2$. Consistent with (2), for a particular fixed W the evidence in favor of $\mathcal{H}_1$ decreases with sample size (i.e., the Jeffreys-Lindley paradox).

When the piecewise approximations shown in Figure 2 are inserted in Eq. 6 this yields the following transformation rules for obtaining a Bayes factor from a p-value:

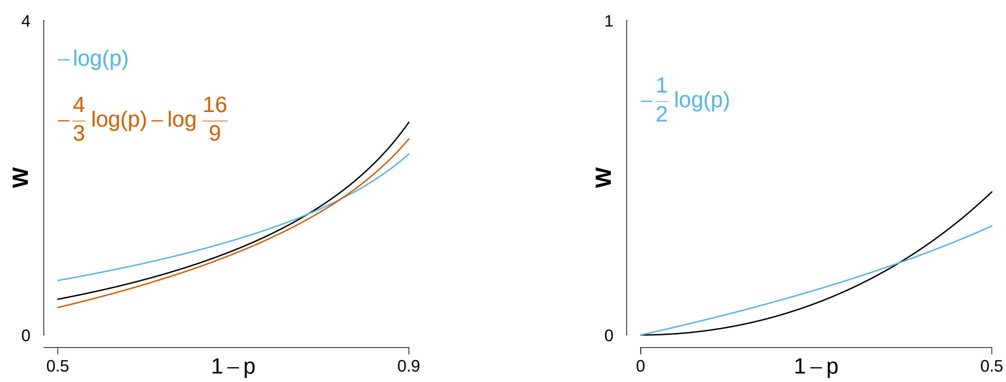
$$\mathrm{JAB}_{01} \approx \begin{cases} 3p\,\sqrt{n} & \text{if } p \leq .10 \\ \sqrt{p}\,\sqrt{n} & \text{if } .10 < p \leq .50 \text{ (simpler)} \\ \frac{4}{3}\,p^{2/3}\sqrt{n} & \text{if } .10 < p \leq .50 \text{ (more precise)} \\ p^{1/4}\sqrt{n} & \text{if } p > .50 \end{cases} \tag{9}$$

These transformation rules establish a direct link between two-sided p-values and Bayes factor tests of a point-null hypothesis, in contrast to the suggestion from previous work that such a link exists only for Bayes factor tests of a directional hypothesis (e.g., Berger & Delampady, 1987; Casella & Berger, 1987; Dickey, 1977; Edwards et



(a) χ^2_1 quantile function for Wald's W.

(b) Top segment approx., $p \leq .10$.



(c) Middle segment approx., $.10 < p \leq .50$.

(d) Bottom segment approx., $p > .50$.

Figure 2. Piecewise approximation to the χ^2_1 quantile function, chosen to obtain a simple transformation from the p-value to Jeffreys' approximate Bayes factor JAB_{01}. See text for details.

al., 1963; Hubbard & Lindsay, 2008; Lindley, 1965; Morey & Wagenmakers, 2014; Pratt, 1965; but see Good, 1982). Note that the JAB_{01} transformation rules also hold for BIC-based model comparison (i.e., to obtain $\text{BF}_{01}^{\text{BIC}}$), because –just as W for JAB_{01}– the critical BIC quantity λ_{LR} is asymptotically χ^2_1 distributed under $\mathcal{H}_0$.

Remark 4: Accuracy

The JAB_{01} transformation rules are approximate: their main purpose is not to achieve an exceptionally close fit to the quantile function of the χ^2_1 distribution; instead, their main purpose is simplicity. In practical application, high accuracy in determining JAB_{01} is generally not needed. Jeffreys explains:

> "We do not need K [BF_{01} – EWAL] with much accuracy. Its importance is that if $K > 1$ the null hypothesis is supported by the observations, while if K is very small the null hypothesis may be rejected. But it makes little difference to the null hypothesis whether the odds are 10 to 1 or 100 to 1 against it, and no difference at all whether they are 10^4 or 10^{4000} to 1; in any case, whatever alternative is most strongly supported will be set up as the hypothesis for use until further notice. I have gone as low as $K = 0.01$ to give a limit for unconditional rejection of the null hypothesis. $K = 10^{-1/2}$ represents only about 3 to 1 odds, and would be hardly worth mentioning in support of a new discovery; it is at $K = 10^{-1}$ and below that we can have strong confidence that a result will survive future investigation (...) minor correcting factors in K that do not reach 2 can be dropped; a decision that depends on them will be highly doubtful in any case." (Jeffreys, 1939, Appendix I, pp. 357-358)

Remark 5: Thresholds and Traffic Signs

In order to facilitate communication about grades of evidence, Jeffreys proposed a set of guidelines for interpreting Bayes factors. In the fragment below, K stands for BF_{01}, the evidence in favor of $\mathcal{H}_0$ over $\mathcal{H}_1$:

> "We may group the values into grades, as follows:
> Grade 0. $K > 1$. Null hypothesis supported.
> Grade 1. $1 > K > 10^{-1/2}$. Evidence against q [$\mathcal{H}_0$ – EWAL], but not worth more than a bare comment.
> Grade 2. $10^{-1/2} > K > 10^{-1}$. Evidence against q substantial.
> Grade 3. $10^{-1} > K > 10^{-3/2}$. Evidence against q strong.
> Grade 4. $10^{-3/2} > K > 10^{-2}$. Evidence against q very strong.
> Grade 5. $10^{-2} > K$. Evidence against q decisive." (Jeffreys, 1939, Appendix I, p. 357; for a discussion see Jeffreys, 1938a)

This scale is usually summarized by suggesting that Bayes factors $\in [1, 3)$ provide "weak" evidence, Bayes factors $\in [3, 10)$ provide "moderate" evidence, Bayes factors

$\in [10, 30)$ provide "strong" evidence, Bayes factors $\in [30, 100)$ provide "very strong" evidence, and Bayes factors larger than 100 provide "extreme" evidence (M. D. Lee & Wagenmakers, 2013; Wasserman, 2000; cf. Kass & Raftery, 1995).

Using the transformation rule from Eq. 9, these grades of evidence may be translated to their associated p-values and sample sizes. For instance, when $p \leq .10$ we have the relation $p = \left(3 \cdot \mathrm{JAB}_{10}\sqrt{n}\right)^{-1}$. Thus, when the evidence in favor of $\mathcal{H}_1$ is just on the boundary of being "worth more than a bare comment" (i.e., $\mathrm{JAB}_{10} = 3$), the associated p-value is $p = \left(9\sqrt{n}\right)^{-1}$. This critical value equals $p = .0351$ when $n = 10$, $p = .0111$ when $n = 100$, and $p = .0035$ when $n = 1000$. Other critical p-values are listed in Table 1. With high sample sizes and with strong evidence against $\mathcal{H}_0$, the corresponding p-values are markedly lower than the current standard level of .05 (Benjamin et al., 2018; Colquhoun, 2014; cf. Table 1 in Raftery, 1993 and Table 9 in Raftery, 1995).

Table 1
The p-values that correspond to threshold levels of Bayes factor evidence as approximated by JAB_{10}. The general expression is $p = (3 \cdot JAB_{10}\sqrt{n})^{-1}$, with the exception of the $JAB_{10} = 1$, $n = 10$ entry, for which the transformation rule is $p = 1/n$.

	p-value		
JAB_{10}	$n = 10$	$n = 100$	$n = 1000$
1	.1000	.0333	.0105
3	.0351	.0111	.0035
10	.0105	.0033	.0011
30	.0035	.0011	.0004
100	.0011	.0003	.0001

Based on the JAB transformation rules, we propose a Bayesian traffic light system that serves to temper the enthusiasm of researchers who wish to reject $\mathcal{H}_0$ after obtaining a p-value lower than or equal to $\alpha = .10$: whenever $p > (9\sqrt{n})^{-1}$ (i.e., $\mathrm{JAB}_{10} < 3$), the light is red, meaning that the evidence against $\mathcal{H}_0$ is only weak; whenever $p \in ((30\sqrt{n})^{-1}, (9\sqrt{n})^{-1}]$ (i.e., $\mathrm{JAB}_{10} \in [3, 10)$), the light is yellow, meaning that the evidence against $\mathcal{H}_0$ is moderate; and whenever $p \leq (30\sqrt{n})^{-1}$ (i.e., $\mathrm{JAB}_{10} \geq 10$), the light is green, meaning that the evidence against $\mathcal{H}_0$ is strong.[5]

Table 2 shows the critical sample sizes for threshold values of JAB_{10} and p-values at popular thresholds of .05, .01, and .005. The first row shows that $\mathrm{JAB}_{10} = 1$ is associated with $p = .05$ when $n = 44$; for $p = .05$, higher values of n will result in values of JAB_{10} lower than 1, that is, evidence in favor of $\mathcal{H}_0$. When $p = .005$, n needs to be over 4,444 in order for the data to provide evidence in favor of $\mathcal{H}_0$. Increasing the

[5]Note that the proposed flags are purely evidential; there is no implication that data with weak evidence ought to be suppressed (Shiffrin et al., 2021). Specifically, a red flag does not imply that a finding should not be published. It does mean that the the evidence against $\mathcal{H}_0$ is only weak.

threshold level of evidence on JAB_{10} decreases the critical sample sizes. For instance, $p = .05$ can only be associated with $\text{JAB}_{10} \geq 3$ if $n \leq 5$. There is no sample size for which $p = .05$ is associated with $\text{JAB}_{10} = 10$.

Table 2
The sample sizes that correspond to threshold levels of Bayes factor evidence as approximated by JAB_{10}. The general expression is $n = (3p \cdot JAB_{10})^{-2}$. 'NA' entries indicate that no sample size is consistent with the postulated pair of JAB_{10} and p-value.

	Sample size n		
JAB_{10}	$p = .05$	$p = .01$	$p = .005$
1	44	1111	4444
3	5	123	494
10	NA	11	44
30	NA	1	5
100	NA	NA	NA

Remark 6: Model-Averaging the MLE

Suppose one takes seriously the specification of the prior model probability $p(\mathcal{H}_0)$ and its complement $p(\mathcal{H}_1)$; then, upon obtaining the MLE $\hat{\theta}$ under $\mathcal{H}_1$, a better point estimate may be obtained using model averaging (e.g., Iverson, Wagenmakers, & Lee, 2010; Jeffreys, 1935, p. 222; Jeffreys, 1961, p. 365; Hoeting, Madigan, Raftery, & Volinsky, 1999; van den Bergh, Haaf, Ly, Rouder, & Wagenmakers, 2021; Wrinch & Jeffreys, 1921, p. 387). Specifically, one may use

$$\hat{\theta}_{\text{BMA}} = \theta_0 \, p(\mathcal{H}_0 \mid y^n) + \hat{\theta} \, p(\mathcal{H}_1 \mid y^n). \tag{10}$$

When $\theta_0 = 0$ we have

$$\hat{\theta}_{\text{BMA}} = \hat{\theta} \, (1 - p_{\text{JAB}}), \tag{11}$$

where p_{JAB} is the JAB-approximate posterior probability of $\mathcal{H}_0$. For instance, when $p \leq .10$ and with equal prior model probabilities we have the JAB-based model-averaged point estimate

$$\hat{\theta}_{\text{BMA}} = \frac{\hat{\theta}}{1 + 3p \sqrt{n}}, \tag{12}$$

which shows that the model-averaged shrinkage of the MLE toward $\theta_0 = 0$ depends on both the p-value and sample size. In other words, a p-value may be used in concert with sample size to achieve an approximate 'spike and slab' type regularization of parameter estimates.

Remark 7: Significance Seeking

There is a method guaranteed to produce statistically significant results: test a multitude of associations or differences, highlight the most compelling effects, and do not apply any kind of correction (for vivid demonstrations see Bennett, Baird, Miller, & Wolford, 2011; De Groot, 1956/2014; Vigen, 2015). Known as data dredging, *p*-hacking, significance-seeking, and, more poetically, a stroll in 'the garden of forking paths' (Gelman & Loken, 2014; Simmons, Nelson, & Simonsohn, 2011), this method inflates evidence and produces spurious results. The $3p\sqrt{n}$ transformation rule from Eq. 9 suggests that significance seeking also compromises the Bayes factor: for fixed sample size n, a spuriously low *p*-value is linearly related to a spuriously low Bayes factor (i.e., spurious evidence against $\mathcal{H}_0$). Indeed, Jeffreys proposed that when multiple hypotheses are in play, the prior model probabilities ought to be adjusted in favor of $\mathcal{H}_0$ (cf. Westfall, Johnson, & Utts, 1997):

> "The need for such allowances for selection of alternative hypotheses is serious. (...) In twenty trials we should therefore expect to find an estimate giving $K < 1$ [$\mathrm{BF}_{01} < 1$ – EWAL] even if the null hypothesis was correct (...) If we persist in looking for evidence against q [$\mathcal{H}_0$ – EWAL] we shall always find it unless we allow for selection." (Jeffreys, 1961, p. 255)

Thus, methods that produce spuriously low *p*-values will generally also produce spuriously low Bayes factors against $\mathcal{H}_0$. An important exception to this rule is the topic of the next remark.

Remark 8: The Likelihood Principle and Optional Stopping

The likelihood principle is repudiated by standard frequentist practice whereas it is respected by standard Bayesian practice (Berger & Wolpert, 1988; Edwards et al., 1963; Good, 1992). For instance, the same observed sequence of conditionally independent binary observations (e.g., six coin tosses: HHHHHT) yield one *p*-value for the binomial sampling plan ("toss the coin six times") and a different *p*-value for the negative binomial sampling plan ("toss the coin until the first tails"). In contrast, Bayesian inference is invariant to such chances in sampling plan, as they do not affect the kernel of the likelihood function and consequently divide out when applying Bayes' rule (Bernardo & Smith, 1994; Lindley, 1993; Pratt et al., 1995). The transformation rules from Eq. 9 hold for the usual frequentist *p*-value as defined for a fixed-N sampling plan.

The philosophical divide concerning the likelihood principle becomes practically relevant in the case of optional stopping in sequential medical trials. Because the *p*-value fluctuates randomly under $\mathcal{H}_0$, multiple successive tests will ensure its eventual rejection at any non-zero α-level. Therefore such successive tests are known as "sampling to a foregone conclusion" and require corrections to retain a particular Type I error rate. Bayesian inference offers a dramatically different perspective:

> "The likelihood principle emphasized in Bayesian statistics implies, among other things, that the rules governing when data collection stops are irrelevant to data interpretation. It is entirely appropriate to collect data until a point has been proven or disproven, or until the data collector runs out of time, money, or patience." (Edwards et al., 1963, p. 193)

The primary reason why the Bayesian does not sample to a foregone conclusion is that, in contrast to the p-value, the Bayes factor is generally consistent: under $\mathcal{H}_0$, the Bayes factor does not drift randomly, but instead gradually attains higher and higher values in favor of $\mathcal{H}_0$. Consequently:

> "if you set out to collect data until your posterior probability for a hypothesis which unknown to you is true has been reduced to .01, then 99 times out of 100 you will never make it, no matter how many data you, or your children after you, may collect." (Edwards et al., 1963, p. 239)

Details concerning optional stopping can be found elsewhere (e.g., Anscombe, 1963; Berger & Mortera, 1999; Rouder, 2014; Wagenmakers et al., 2018; for a discussion see de Heide & Grünwald, 2021; Hendriksen, de Heide, & Grünwald, 2021; Sanborn & Hills, 2014). Here we only demonstrate how a sequence of p-values can be transformed to a sequence of approximate Bayes factors, which results in qualitatively different behavior because of the inclusion of the $\sqrt{n}$ term.

Specifically, consider a synthetic data set subjected to three different one-sample t-tests. The first test is the standard frequentist test that produces a two-sided p-value. The second test is the "Jeffreys-Zellner-Siow" t-test that assigns a Cauchy distribution to the test-relevant effect size parameter δ (e.g., Gronau, Ly, & Wagenmakers, 2020; Jeffreys, 1961; Rouder, Speckman, Sun, Morey, & Iverson, 2009). Specifically, $\delta = \mu/\sigma \sim \text{Cauchy}(0, \gamma = 1/\sqrt{2} \approx 0.707)$, as has become the default specification in the field of psychology (Morey & Rouder, 2018).[6] For comparison purposes, we assume $p(\mathcal{H}_0) = p(\mathcal{H}_1) = 1/2$ and we focus on the posterior probability of $\mathcal{H}_0$ (i.e., p_{JZS}). The third test is based on JAB; again, we assume equal prior model probabilities and focus on the posterior probability of $\mathcal{H}_0$ (i.e., p_{JAB}). The data set under consideration is generated under $\mathcal{H}_0$, from a standard normal distribution: $y_i \sim N(0, 1)$. The three test are executed after every new observation is added, for a total length of 2500 consecutive observations.

The results of this demonstration are displayed in Figure 3. The black line shows that the frequentist fixed-N p-value fluctuates randomly (because $\mathcal{H}_0 : \delta = 0$ is true). A total of 75 observations have a fixed-N p-value lower than .05: at $N \in [6, 7]$, at $N \in [1623, 1647]$, at $N \in [1649, 1653]$, at $N \in [1655, 1660]$, and at $N \in [1741, 1777]$. This haphazard behavior of the p-value stands in marked contrast to the regular behavior of the Bayesian p-values. The red line gives p_{JZS}, and the other colors (barely visible because of the nearness in their values) refer to three different implementations of

[6] Parameter γ of the Cauchy distribution refers to its interquartile range.

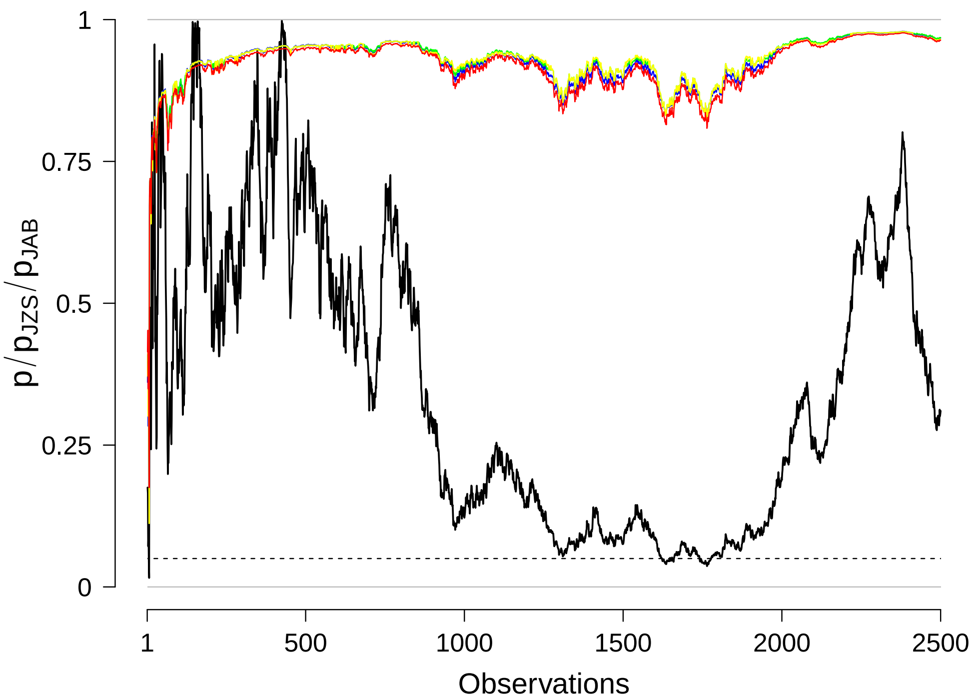


Figure 3. A comparison of p-values from a one-sample t-test. For $N = 2500$ consecutive observations drawn from a standard normal distribution, the black line indicates the corresponding sequence of frequentist fixed-N p-values. The red line indicates p_{JZS}, and the other colors correspond to three different instantiations of JAB: blue is p_{JAB} computed from Eq. 6, green is p_{JAB} computed from the precise form of Eq. 9, and yellow is p_{JAB} computed from the simpler form of Eq. 9. The fact that the different colors can hardly be discriminated attests to the closeness of their values. See text for details.

p_{JAB}: the blue line is computed from Eq. 6 using the quantile function of the χ^2_1 distribution, the green line is computed from the more precise form of Eq. 9, and the yellow line is computed from the simpler form of Eq. 9. After a brief initial period of uncertainty, the Bayesian p-values indicate support in favor of $\mathcal{H}_0$.

It is perhaps surprising how closely JAB approximates the default JZS t-test, as it was not designed for this purpose. This closeness can be understood and quantified as follows. With σ known and $\hat{\theta}$ near zero, the unit information prior that underlies JAB amounts to a standard normal prior on δ (i.e., $\mu \sim N(0, \sigma^2)$ equals $\delta = \mu/\sigma \sim N(0, 1)$). As sample size increases, the posterior distributions for δ under the Cauchy and the standard normal prior converge, and the difference in the Bayes factor is increasingly determined by the ratio of the prior ordinates at $\delta = 0$ (e.g., Dickey

& Lientz, 1970; Ly & Wagenmakers, 2021a). Specifically, as n increases, we have $\text{BF}_{01}(y^n; \text{Cauchy}(0, 1/\sqrt{2}))\,/\,\text{BF}_{01}(y^n; \text{Normal}(0,1)) \rightarrow \frac{1}{2}\sqrt{\pi} \approx 0.886$. This means that the Bayes factors from the two approaches will be relatively similar (Kass & Wasserman, 1995, pp. 930-931).[7] For the synthetic data from Figure 3, after 2500 observations we have $t = -1.01$, $p = 0.31$. At this point the default JZS t-test yields $\text{BF}_{01} \approx 26.51$; for JAB, Eq. 6 yields $\text{JAB}_{01} \approx 29.88$; the precise form of Eq. 9 yields $\text{JAB}_{01} \approx 30.55$, and the simpler form of Eq. 9 yields $\text{JAB}_{01} \approx 27.85$. The ratio of the JZS Bayes factor and JAB equals $26.51/29.88 \approx 0.887$, close to the theoretical limit of $\frac{1}{2}\sqrt{\pi} \approx 0.886$. Transformed to posterior probabilities, the difference in Bayes factors is barely noticeable: under the JZS specification, $p(\mathcal{H}_0 \mid y^n) \approx 26.51/27.51 \approx 0.964$, whereas under JAB we obtain $p(\mathcal{H}_0 \mid y^n) \approx 29.88/30.88 \approx 0.968$. Figure 3 also shows that the different instantiations of JAB yield results that are highly similar to one another.

This demonstration is consistent with the conjecture from Jeffreys (1977, p. 89) that his general form provides a good approximation to default hypothesis tests, and confirms the closeness of the p-value transformation rules that is already evident from Figure 2.

The pattern from Figure 3 is representative of other sequences generated under $\mathcal{H}_0$ in the sense that (1) the fixed-N p-value fluctuates wildly; (2) p_{JZS} and p_{JAB} are highly similar, and consistently indicate evidence in favor of $\mathcal{H}_0$; (3) the different implementations of p_{JAB} are highly similar to one another as well. It is remarkable, in our opinion, that the simple inclusion of the $\sqrt{n}$ term transforms a fixed-N p-value to a posterior probability of $\mathcal{H}_0$; these quantities are conceptually opposite in the sense that the posterior probability of $\mathcal{H}_0$ (but not the fixed-N p-value) applies without modification to a sequential design, obeys the likelihood principle, is consistent, is able to quantify evidence in favor of $\mathcal{H}_0$, and was designed to quantify relative rather than absolute performance.[8]

Remark 9: A One-sided Version of JAB_{01}

Jeffreys's general form was intended to approximate an objective Bayesian two-sided test, that is, a test in which the prior distribution $g(\theta)$ is symmetric around its mode θ_0. However, researchers often wish to entertain a one-sided specification of the alternative hypothesis. For instance, when $\mathcal{H}_0$ is "the vaccine is ineffective", a reasonable specification of $\mathcal{H}_1$ is "the vaccine offers protection against the disease" (i.e., $\mathcal{H}_+ : \theta > \theta_0$). Although it may be theoretically possible for a vaccine to make

[7] More generally, $\text{BF}_{01}(\text{Cauchy}(0, \gamma))\,/\,\text{BF}_{01}(\text{Normal}(0, \sigma)) \rightarrow (\gamma/\sigma)\sqrt{\pi/2}$. With $\gamma = \sigma$, this exactly equals Jeffreys's alternative approximate form based on invariance theory mentioned earlier (cf. Jeffreys, 1961, p. 277, Eq. 5; Kass & Wasserman, 1995, pp. 930).

[8] The approximation remains acceptable when the MLE is not close to zero and sample size is small, although it ultimately breaks down. In these cases an improvement is possible by adding to JAB_{01} the multiplicative term $\exp\{\frac{1}{2}(\bar{y}/s)^2\}$ that corrects the bias from centering the prior at the MLE (cf. Eq. 4).

it easier to contract a disease (i.e., $\mathcal{H}_- : \theta < \theta_0$), allowing for that possibility wastes prior mass on highly implausible values that, moreover, are not a proper statistical translation of the hypothesis under scrutiny.

Jeffreys (1961, pp. 277-278, p. 283) acknowledged that the general form requires adjustment if a one-sided test is desired, and sketched by what method such an adjustment can be obtained. Specifically, to construct a Bayesian one-sided test one may fold the symmetric prior distribution from the two-sided test around θ_0 such that it assigns mass only to values of $\theta > \theta_0$ (or $\theta < \theta_0$, depending on the direction expected under $\mathcal{H}_1$). Jeffreys further explained that the Bayes factor in favor of $\mathcal{H}_0$ is (1) approximately halved when the sample effect is clearly consistent with the direction postulated under $\mathcal{H}_1$; (2) unaltered when the sample effect equals θ_0; (3) greatly increased when the sample effect is clearly inconsistent with the direction postulated under $\mathcal{H}_1$. However, Jeffreys did not proceed to propose a specific adjustment to his general form.

Here we propose an approximate adjustment based on the method outlined in Morey and Wagenmakers (2014). This method takes advantage of the fact that for moderate sample sizes and relatively vague priors, the one-sided p-value is approximately equal to the posterior mass on one side of θ_0 (e.g., Marsman & Wagenmakers, 2017; Morey & Wagenmakers, 2014, and references therein). In a directional test we wish to compare, say, $\mathcal{H}_0 : \theta = \theta_0$ to $\mathcal{H}_+ : \theta > \theta_0$. The resulting approximate Bayes factor, JAB_{0+}, can be obtained by multiplying the non-directional Bayes factor JAB_{01} with a correction term that includes the one-sided p-value, p_1:

$$\text{JAB}_{0+} = \frac{1}{2(1-p_1)}\ \text{JAB}_{01}. \tag{13}$$

Note that this reproduces the qualitative behavior outlined by Jeffreys (1961, p. 283): (1) if $p_1 \to 0$ (i.e., the effect is unambiguously in the direction predicted under $\mathcal{H}_+$) then $\text{JAB}_{0+} \to \text{JAB}_{01}/2$, a halving of the evidence in favor of $\mathcal{H}_0$; (2) if $p_1 = 1/2$ (i.e., the posterior distribution for θ under $\mathcal{H}_1$ is approximately symmetric around θ_0), the Bayes factor is unaltered; and (3) if $p_1 \to 1$ (i.e., the effect is unambiguously in the direction opposite to that predicted under $\mathcal{H}_+$) then $\text{JAB}_{0+} \to \infty$.

To illustrate, consider the earlier one-sample t-test from Figure 3, which yielded $\text{JAB}_{01} \approx 29.88$ based on $t = -1.01$ after $n = 2500$ observations. The corresponding two-sided fixed-N p-value was approximately .31; under the hypothesis $\mathcal{H}_+$ that the effect is positive, the one-sided p-value is $p_1 \approx .84$ (i.e., $1 - .31/2$). Application of Eq. 13 then yields $\text{JAB}_{0+} \approx 93.32$: the effect goes in the direction opposite to that expected under the alternative hypothesis, and this increases the evidence in favor of $\mathcal{H}_0$ by a factor of about three. For comparison, application of the standard Bayesian t-test with $\mathcal{H}_+ : \delta \sim N^+(0, 1)$ (i.e., a folded, positive-only standard normal prior) yields $\text{BF}_{01} = 96.08$.

Remark 10: Bayes Factor Tests of Existence versus Direction

Consider the case where $p \leq .10$, such that $\text{JAB}_{01} = 3p\sqrt{n}$ and $\text{JAB}_{10} = [3p\sqrt{n}]^{-1}$. Instead of conducting a test of existence involving a point-null hypothesis $\mathcal{H}_0 : \theta = \theta_0$, one may also conduct a test of direction, contrasting $\mathcal{H}_+ : \theta > \theta_0$ against $\mathcal{H}_- : \theta < \theta_0$. The approximate relation between p-values and Bayes factor tests of direction entails that $\text{BF}_{+-} \approx \frac{1-p/2}{p/2} = \frac{2}{p} - 1$. Having expressed both Bayesian tests as a function of the p-value, their relation is given as $\text{BF}_{10} = [\sqrt{n}(6 - 3p)]^{-1}\,\text{BF}_{+-} \approx [6\sqrt{n}]^{-1}\,\text{BF}_{+-}$.

For instance, when $p = .01$ and $n = 100$, then $\text{JAB}_{10} = 3^1/_3$ (i.e., moderate evidence for the presence of an effect). In contrast, the associated test of direction yields $\text{BF}_{+-} \approx 199$ (i.e., extreme evidence that the effect is larger instead of smaller than θ_0). The adjustment factor that transforms the test for direction to a test of existence equals $[\sqrt{n}(6 - 3p)]^{-1} \approx .0017$. Note that this adjustment factor is always smaller than $^1/_6$.

In sum, when $p \leq .10$, an approximate objective Bayes factor test for direction is about $6\sqrt{n}$ times more compelling than an approximate Bayes factor test for existence. Consequently, whenever $\text{BF}_{+-} \in (1, 6\sqrt{n})$ the conclusion from the test of direction stands in apparent contradiction to the conclusion from the test of existence, thus yielding a Bayesian version of Jeffreys's paradox (cf. Ly & Wagenmakers, 2021b).

A similar relation holds when the test of existence is one-sided, such that $\mathcal{H}_1$ is replaced by, say, $\mathcal{H}_+ : \theta > \theta_0$. By Eq. 13 we have $\text{BF}_{0+} \approx [2 - p]^{-1}\, 3p\sqrt{n}$, from which we obtain the relation $\text{BF}_{+0} = [3\sqrt{n}]^{-1}\,\text{BF}_{+-}$. When testing $\mathcal{H}_+ : \theta > \theta_0$, it fundamentally matters whether the rival hypothesis is the point-null hypothesis $\mathcal{H}_0 : \theta = \theta_0$ or the negative-effect hypothesis $\mathcal{H}_- : \theta < \theta_0$; for the objective parameter priors under considerations here, the evidence that the data provide for $\mathcal{H}_+$ is $3\sqrt{n}$ times more compelling when it is contrasted against $\mathcal{H}_-$ instead of against $\mathcal{H}_0$.

This analysis underscores that it is vital for researchers to select the appropriate Bayesian test, that is, the test that best translates their substantive research question to statistical formalism. It may be tempting to misinterpret a test of direction as a test of existence, but doing so overestimates the evidence by a factor of about $6\sqrt{n}$ for a two-sided test and $3\sqrt{n}$ for a one-sided test.

Remark 11: The Edwards-Jeffreys Bayes Factor Band

As mentioned earlier, a long line of work has contrasted Bayes factors against p-values (e.g., Berger & Delampady, 1987; Berger & Sellke, 1987; Edwards et al., 1963; Held & Ott, 2018; Johnson, 2005, 2008, 2013; Sellke et al., 2001; Vovk, 1993). A key method is to derive a lower bound on BF_{01} by adopting a prior distribution $g(\theta)$ that is unreasonably biased against $\mathcal{H}_0$. These 'oracle' prior distributions are informed by the data to various degrees, and are generally incapable to quantify evidence in favor of $\mathcal{H}_0$ (i.e., they are inconsistent under $\mathcal{H}_0$). The most extreme example is the MLE prior, that is, the prior distribution that concentrates all mass on $\hat{\theta}$ (Edwards et al., 1963). Together with the p-value transformation rule this yields

$$\begin{aligned} \text{minBF}_{01}^{\text{MLE}} &= \exp\left(-\tfrac{1}{2}W\right) \\ &\approx 3p, \end{aligned} \tag{14}$$

where the second step assumes that $p \leq .10$. This is the same as JAB_{01} but without the $\sqrt{n}$ term, confirming the interpretation of $\sqrt{n}$ as a Bayesian correction for selection. When $p = .05$, $\text{minBF}_{01}^{\text{MLE}} = .15$ and the associated posterior probability for $\mathcal{H}_0$ equals $3p/(1 + 3p) \approx .13$.

A slightly more reasonable prior is obtained when $g(\theta)$ is restricted to be symmetric around θ_0. As shown by Berger and Sellke (1987) this is well-approximated by a two-point prior at $\hat{\theta}$ and $2\theta_0 - \hat{\theta}$ resulting in

$$\begin{aligned} \text{minBF}_{01}^{\text{sym}} &\approx 2\ \exp\left(-\tfrac{1}{2}W\right) \\ &\approx 6p, \end{aligned} \tag{15}$$

where the second step again assumes that $p \leq .10$. When $p = .05$, $\text{minBF}_{01}^{\text{sym}} = .30$ with a posterior probability for $\mathcal{H}_0$ of $6p/(1 + 6p) \approx .23$.

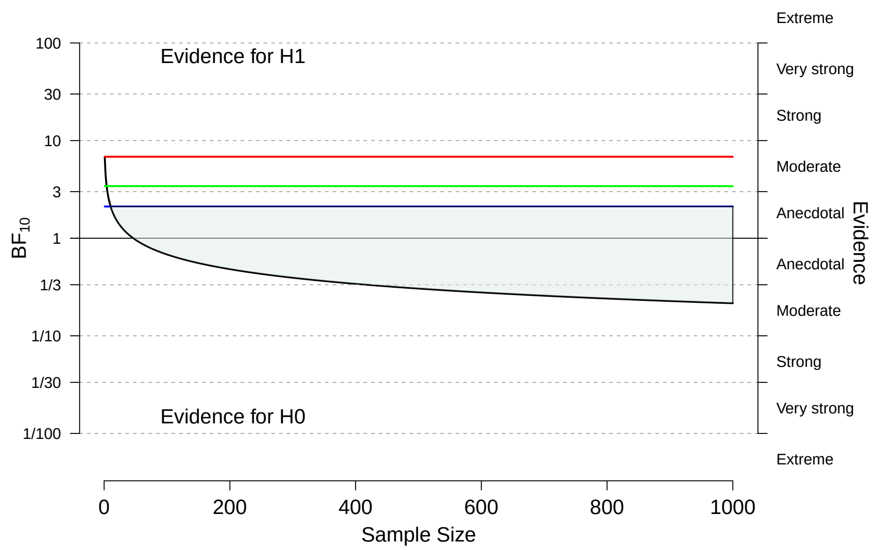
An even more reasonable prior –and a tighter bound– is obtained when $g(\theta)$ is restricted to be a normal distribution with mean θ_0 and a variance that is cherry-picked to provide the most evidence against $\mathcal{H}_0$. Edwards et al. (1963, p. 231) showed that when $\sqrt{W} > 1$ (i.e., $|t| > 1$, or $\delta = \hat{\theta}/\sigma > n^{1/2}$) this results in

$$\begin{aligned} \text{minBF}_{01}^{\text{nor}} &\approx \sqrt{e}\,\sqrt{W}\ \exp\left(-\tfrac{1}{2}W\right) \\ &\approx 3p\,\sqrt{e}\,\sqrt{W}, \end{aligned} \tag{16}$$

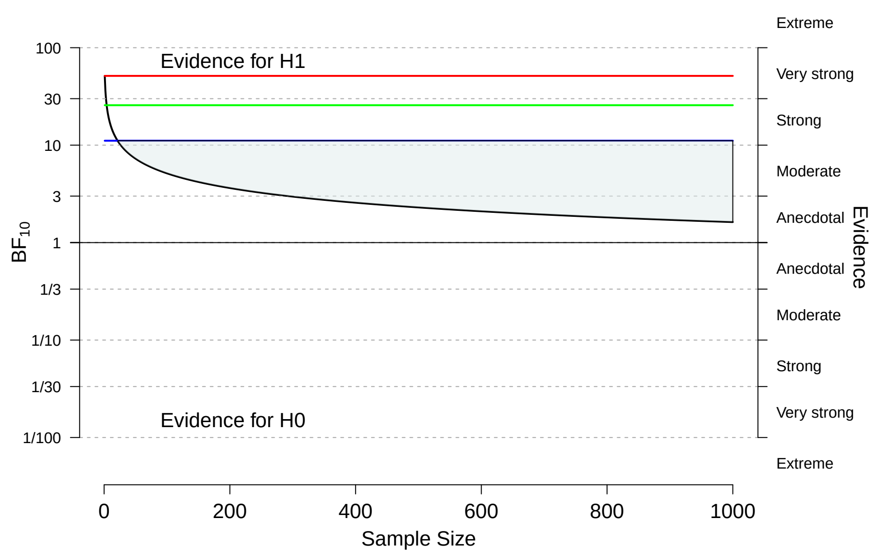
where the second step again assumes that $p \leq .10$. When $p = .05$, $\sqrt{W} \approx 1.96$, and $\text{minBF}_{01}^{\text{nor}} \approx 3p\,\sqrt{e}\,1.96 \approx 9.69p = 0.48$ with a posterior probability for $\mathcal{H}_0$ of $9.69p/(1 + 9.69p) \approx .33$. Because the mapping from $\sqrt{W}$ to the p-value does not depend on n, this bound does not fluctuate with sample size. Echoing concerns from the literature (e.g., Benjamin et al., 2018; Berger & Sellke, 1987; Edwards et al., 1963), this means that whenever $\mathcal{H}_0$ is rejected based on a p-value that is just below .05, the evidence against $\mathcal{H}_0$ is no greater than a Bayes factor of about 2 ("not worth more than a bare comment", Jeffreys, 1939, p. 357).

Figure 4 shows the Bayes factor bounds together with JAB_{10}, for the scenario where the p-value is fixed at .05 (top panel) or at .005 (bottom panel) and sample size is varied from 1 to 1000. In both panels the azure "Edwards-Jeffreys band" corresponds to Bayes factors lower than $\text{maxBF}_{10}^{\text{nor}}$ and higher than JAB_{10}.

Whenever $p \leq .10$ and sample size is not very small, the band provides a region that contains almost all Bayes factors that can be obtained from a reasonable prior specification for θ. The most optimistic Bayes factors are given by $\text{maxBF}_{10}^{\text{nor}}$, where the prior is directly informed by the data; such a prior would generally be



(a) Bayes factor bounds and Edwards-Jeffreys band for $p = .05$.



(b) Bayes factor bounds and Edwards-Jeffreys band for $p = .005$.

Figure 4. Bayes factor bounds and Edwards-Jeffreys band. The horizontal lines indicate three increasingly tight bounds on the Bayes factor. Red = $\text{maxBF}_{10}^{\text{MLE}}$, green = $\text{maxBF}_{10}^{\text{sym}}$, and blue = $\text{maxBF}_{10}^{\text{nor}}$. The black decreasing line is JAB_{10}, and the azure band is for Bayes factors lower than $\text{maxBF}_{10}^{\text{nor}}$ and larger than JAB_{10}. See text for details.

considered unacceptably biased against $\mathcal{H}_0$. The most pessimistic Bayes factors are given by JAB_{10}, which is based on the unit-information prior. This prior is wider and more vague than what can be obtained using a more informed specification; also, the behavior of JAB_{10} is similar to that of the BIC, which has been accused of underfitting the data (e.g., Burnham & Anderson, 2002). We would therefore argue that if a researcher believes the evidence falls outside the Edwards-Jeffreys bound the onus is on them to support this claim with a concrete alternative analysis.

These considerations suggest an ad-hoc compromise, namely to report as a single representative value the geometric mean of Edwards' bound and Jeffreys' approximation. When $p < .10$ this yields

$$\begin{aligned}\mathrm{BF}_{10}^{\mathrm{EJ}} &= \left[\mathrm{maxBF}_{10}^{\mathrm{nor}} \cdot \mathrm{JAB}_{10}\right]^{1/2} \\ &= \frac{1}{3p\,\left[e\,W\,n\right]^{1/4}} \\ &\approx \frac{1}{3.852p\,\left[W\,n\right]^{1/4}}.\end{aligned} \tag{17}$$

When $|t| < 1$, $\mathrm{maxBF}_{10}^{\mathrm{nor}}$ is defined to be 1 (Edwards et al., 1963), and simplicity suggests $\mathrm{BF}_{10}^{\mathrm{EJ}} = [\mathrm{JAB}_{10}]^{1/2}$ whenever $p > .10$. In addition to the single value $\mathrm{BF}_{10}^{\mathrm{EJ}}$, the Edwards-Jeffreys band can be provided in order to indicate the plausible range of alternative values. The compromise $\mathrm{BF}_{10}^{\mathrm{EJ}}$ value will still be consistent under $\mathcal{H}_0$, but support it less enthusiastically than JAB. For instance, when $p = .005$ and $n = 100$ (and consequently, $\sqrt{W} = 2.8$) we have $\mathrm{JAB}_{10} = 6^2\!/\!_3$ and $\mathrm{maxBF}_{10}^{\mathrm{nor}} = 14.44$. This may then be reported as $\mathrm{BF}_{10}^{\mathrm{EJ}} = 9.81[6.67, 14.44]$. A further advantage is that such a report emphasizes the fact that a single data set can give rise to various Bayes factors, depending on the details of the prior specification. Assessment of this proposal awaits further study.

Note that with very small sample sizes it is not true that $\mathrm{se}(\hat{\theta}) \ll \sigma_g$ and in this case JAB as formulated in Eq. 6 is biased against $\mathcal{H}_0$. The bias is visible from Figure 4 because when $eW > n$ (top panel: $n < 10$; bottom panel: $n < 22$), $\mathrm{JAB}_{10} > \mathrm{maxBF}_{10}^{\mathrm{nor}}$, which is anomalous. Nevertheless, the applications below demonstrate that JAB may perform adequately even when sample size is small, an issue that we will revisit in the Concluding Comments.

Application to Four Popular Hypothesis Tests

Up to now we have focused our attention on the simplified form of JAB_{01} and set $A = 1$ (cf. Eq. 2 vs. Eq. 6). Doing so was convenient, captured the qualitative behavior of the test, retained the correspondence to the BIC, and respected Jeffreys's claim that in his tests, the value of A is "usually not far from 1" (Jeffreys, 1977, p. 89). However, the derivation of JAB_{01} suggests that for specific tests with widely applied default prior distributions it may be possible to improve the approximation by using values of A other than 1.

The second complication that we have so far conveniently ignored is the definition of the $\sqrt{n}$ term outside the exponential. This term stems from the standard error of the MLE, which equals $\text{se}(\hat{\theta}) = \sigma/\sqrt{n}$ in the case of n i.i.d. observations contributing, say, to the estimate of a population mean. However, for other scenarios it may be less obvious what the value of n should be. Relevant past work on the correct definition of n has been conducted mostly in the context of the BIC (e.g., Bayarri et al., 2019; Berger, Bayarri, & Pericchi, 2014; Berger, Ghosh, & Mukhopadhyay, 2003; Jones, 2011; Kass & Raftery, 1995, p. 779; Masson, 2011; Nathoo & Masson, 2016; Pauler, 1998; Raftery, 1995). The same considerations play a role in the definition of $\sqrt{n}$ for JAB_{01} as well.

Below are concrete examples featuring four of the most popular hypothesis tests across the empirical sciences: the two-sample t-test, the binomial test, the comparison of two proportions, and the correlation test. It will become apparent that unexpected complications lurk even in these well-studied, seemingly trivial scenarios, both with respect to the definition of A and of n (cf. Weakliem, 1999). We demonstrate that when A and n are defined judiciously, the JAB_{01} approximation can be surprisingly accurate.

Example 1: The Two-Sample t-Test

In the one-sample t-test the definition of n is unambiguous, and an application of the JAB_{01} approximation as per Eq. 6 and Eq. 9 leads to results that are highly similar to those provided by standard objective Bayesian tests (cf. Figure 3).

For the two-sample t-test, the definition of n is slightly less straightforward. Let the number of observations in group 1 and group 2 be denoted n_1 and n_2, respectively. Let s_p denote the pooled standard deviation. Then the standard error for the difference between the two group means, $\text{se}(\Delta\mu)$ is given by

$$\text{se}(\Delta\mu) = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}. \tag{18}$$

When $n_1 = n_2$, this simplifies to

$$\text{se}(\Delta\mu) = \frac{s_p}{\sqrt{\frac{1}{2}n_1}}, \tag{19}$$

which suggests that for the balanced two-sample t-test, the general $\sqrt{n}$ term outside of the exponential in Eq. 6 ought to be defined as $\sqrt{n_1/2}$.

As a first demonstration we consider a two-sample t-test with $n_1 = n_2 = 200$ and an observed t-value of 2.163. For these data, the Bayes factor that contrasts $\mathcal{H}_0 : \delta = 0$ with $\mathcal{H}_1 : \delta \sim N(0, 1)$ yields $\text{BF}_{01} = 0.999$ (cf. Gronau et al., 2020), meaning that the data are almost perfectly non-diagnostic. The associated p-value equals .0311. Applying JAB with an outside term of $\sqrt{n_1/2} = 10$ yields the close approximation $\text{JAB}_{01} = 0.979$, whereas JAB with an outside term of $\sqrt{n_1} = \sqrt{200}$ yields a poorer approximation: $\text{JAB}_{01} = 1.385$.

As a second demonstration we change only the t-value to 3.066, such that the analytical result now yields $\text{BF}_{10} = 10.004$, and the associated p-value equals .0023. Applying JAB with an outside term of $\sqrt{n_1/2} = 10$ again yields a close approximation (i.e., $\text{JAB}_{10} = 10.423$), whereas JAB with an outside term of $\sqrt{n_1} = \sqrt{200}$ yields a poorer approximation: $\text{JAB}_{01} = 7.370$.

Returning to the more general case of an unbalanced two-sample t-test, Eq. 18 can also be expressed as

$$\text{se}(\Delta\mu) = \frac{s_p}{\sqrt{\frac{n_1 \cdot n_2}{n_1 + n_2}}}, \tag{20}$$

which implies that the outside $\sqrt{n}$ term in JAB be defined as $\sqrt{(n_1 \cdot n_2)/(n_1 + n_2)}$. This corrects a "provisional suggestion" by Good (1984a, p. 174), who suggested that the outside term should equal $\sqrt{n_1 \cdot n_2}$ instead.[9]

To demonstrate the unbalanced two-sample t-test we adjust the example above such that $n_1 = 350$, $n_2 = 50$, and $t = 1.975$. The analytical result yields $\text{BF}_{01} = 0.999$, and the associated p-value equals .0490. When we apply JAB with the outside term equal to $\sqrt{(350 \cdot 50)/400} \approx 6.61$, this yields $\text{JAB}_{01} = 0.953$. Changing the t-value to $t = 2.945$ yields $\text{BF}_{10} = 10.005$ and $p = .0034$, and the associated JAB approximation gives 11.028.

The JAB approximation can be made even more accurate by including the multiplicative term $\exp\{\frac{1}{2}t^2(1/n_1 + 1/n_2)\}$ that corrects the bias from centering the prior at the MLE instead of at $\Delta\mu = 0$ (cf. Eq. 4). In the above scenario, the corrected approximation yields $\text{JAB}_{01} \approx 0.996$ when $t = 1.975$ and $\text{JAB}_{10} \approx 9.987$ when $t = 2.945$.

Example 2: The Binomial Test

The binomial test features a comparison of $\mathcal{H}_0 : \theta = \theta_0$ versus $\mathcal{H}_1 : \theta \sim g(\theta)$, where $\theta \in [0, 1]$ is a binomial chance parameter. With s successes and $f = n - s$ failures, the MLE is $\hat{\theta} = s/n$ with its standard error equal to

$$\text{se}(\hat{\theta}) = \sqrt{\frac{\hat{\theta}\,(1 - \hat{\theta})}{n}}. \tag{21}$$

[9] The correct outside term was already given in Jeffreys (1961, p. 252). This underscores the validity of the epigraph, as Good was highly familiar with Jeffreys's statistical work.

Upon substituting, the term $S_{\hat{\theta}}$ (S stands for scale) outside the exponential of Eq. 2 becomes

$$\begin{aligned} S_{\hat{\theta}} &= \frac{1}{\sqrt{2\pi}} \frac{1}{g(\hat{\theta})} \frac{1}{\sqrt{\hat{\theta}\,(1-\hat{\theta})}} \sqrt{n} \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{g(\hat{\theta})} \frac{1}{\pi\sqrt{\hat{\theta}\,(1-\hat{\theta})}} \sqrt{n}. \end{aligned} \tag{22}$$

Note, first, that the $\sqrt{\pi/2}$ term reoccurs in the binomial test as it did in the t-test. Second, note that $\left[\pi\sqrt{\hat{\theta}\,(1-\hat{\theta})}\right]^{-1}$ is the Jeffreys prior for θ (Jeffreys, 1946); hence, if $g(\hat{\theta})$ is defined as the height of the Jeffreys prior at the MLE, $S_{\hat{\theta}}$ reduces to $\sqrt{\pi/2}\,\sqrt{n}$, not too far away from $\sqrt{n}$. This result would be in line with that provided by BIC. However, Jeffreys (1961, pp. 275-277) argued that the Jeffreys prior ought to be used for estimation, and that for testing a uniform distribution on θ is more appropriate, such that $g(\hat{\theta}) = 1$. With this uniform prior the outside factor would then equal

$$S_{\hat{\theta}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\hat{\theta}\,(1-\hat{\theta})}} \sqrt{n}. \tag{23}$$

This form returns in the analytical derivation of the Bayes factor for the binomial. Specifically, with a uniform prior on θ under $\mathcal{H}_1$ we have

$$\mathrm{BF}_{01} = \frac{(n+1)!}{s!f!} {\theta_0}^s (1-\theta_0)^f. \tag{24}$$

As indicated by Jeffreys (1961, p. 256), with s and f large the Stirling approximation yields

$$\mathrm{BF}_{01} \approx \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\theta_0\,(1-\theta_0)}} \sqrt{n}\, \exp\left\{ -\tfrac{1}{2} \frac{(s-n\theta_0)^2}{n\theta_0(1-\theta_0)} \right\}, \tag{25}$$

the term outside of the exponential being identical to $S_{\hat{\theta}}$ save for the fact that θ_0 has been replaced with $\hat{\theta}$.

Example 3: Comparison of Two Proportions

Kass and Vaidyanathan (1992) proposed a log-odds-ratio test for the equivalence of two proportions (see also Gronau, Raj, & Wagenmakers, 2021; Hoffmann, Hofman, & Wagenmakers, 2021). Under $\mathcal{H}_1$, the log odds ratio ψ is assigned a prior distribution directly, thus accounting for the fact that two proportions are usually dependent – if we learn that one proportion is high, this affects our belief about the value of the other proportion (e.g., Howard, 1998). Denoting the two groups by A and B, the

Kass and Vaidyanathan (1992) approach is specified as follows:

$$\begin{aligned} y_A &\sim \text{Binomial}(n_A, \theta_A) \\ y_B &\sim \text{Binomial}(n_B, \theta_B) \\ \log\left(\frac{\theta_A}{1-\theta_A}\right) &= \gamma - \psi/2 \\ \log\left(\frac{\theta_B}{1-\theta_B}\right) &= \gamma + \psi/2, \end{aligned}$$

where γ is the grand mean and ψ is the log odds ratio that quantifies the difference between the two proportions. We have that $\mathcal{H}_0 : \psi = 0$ whereas $\mathcal{H}_1 : \psi \sim g(\psi)$; a default prior distribution is the standard normal (Gronau et al., 2021; Hoffmann et al., 2021). This test is a logistic regression with group a dummy predictor.

In the 2×2 contingency table, we denote the observed frequencies by n_{11}, n_{12}, n_{21}, and n_{22}, as per usual. The MLE for ψ then equals $\log\left[(n_{11} \cdot n_{22})/(n_{12} \cdot n_{21})\right]$; here we follow Haldane (1956) and use a modified version of the MLE that takes finite values when one or more cell counts equal zero:

$$\hat{\psi} = \log\left[\frac{(n_{11} + 1/2) \cdot (n_{22} + 1/2)}{(n_{12} + 1/2) \cdot (n_{21} + 1/2)}\right]. \tag{26}$$

The associated standard error is then given by (e.g., Anscombe, 1956; Gart, 1966; Haldane, 1956; cf. Agresti, 1999):

$$\text{se}(\hat{\psi}) = \sqrt{\frac{1}{n_{11} + 1/2} + \frac{1}{n_{12} + 1/2} + \frac{1}{n_{21} + 1/2} + \frac{1}{n_{22} + 1/2}}. \tag{27}$$

Assume that under $\mathcal{H}_0$, $\psi = 0$, and that under $\mathcal{H}_1$, ψ is assigned a normal prior: $\psi \sim N(\mu_g, \sigma_g^2)$. Furthermore, set $\sigma_g^2 = 1$ consistent with the unit-information concept. Application of the general form of JAB then yields:

$$\begin{aligned} \text{JAB}_{01} &\approx \frac{p(\hat{\psi} \mid \mathcal{H}_0)}{p(\hat{\psi} \mid \mathcal{H}_1)} \\ &\approx \frac{p(\hat{\psi} \mid \hat{\psi} \sim N(0, \text{se}(\hat{\psi})^2))}{p(\hat{\psi} \mid \hat{\psi} \sim N(\mu_g, \sigma_g^2))} \\ &= \frac{\left[\sqrt{2\pi}\,\text{se}(\hat{\psi})\right]^{-1} \exp\left(-\frac{1}{2}\left[\hat{\psi}/\text{se}(\hat{\psi})\right]^2\right)}{\left[\sqrt{2\pi}\,\sigma_g\right]^{-1} \exp\left(-\frac{1}{2}\left[(\hat{\psi} - \mu_g)/\sigma_g\right]^2\right)} \\ &= \frac{\text{se}(\hat{\psi})^{-1} \exp\left(-\frac{1}{2}\left[\hat{\psi}/\text{se}(\hat{\psi})\right]^2\right)}{\exp\left(-\frac{1}{2}\left[(\hat{\psi} - \mu_g)\right]^2\right)}. \end{aligned} \tag{28}$$

We consider two cases. In the first, $\mu_g = \hat{\psi}$, which eliminates the denominator and yields

$$\text{JAB}_{01} = \text{se}(\hat{\psi})^{-1}\ \exp\left(-\tfrac{1}{2}\left[\hat{\psi}/\text{se}(\hat{\psi})\right]^2\right). \tag{29}$$

In the second, $\mu_g = \psi_0 = 0$, and this yields

$$\text{JAB}_{01} = \exp\left(\tfrac{1}{2}\hat{\psi}^2\right)\ \text{se}(\hat{\psi})^{-1}\ \exp\left(-\tfrac{1}{2}\left[\hat{\psi}/\text{se}(\hat{\psi})\right]^2\right), \tag{30}$$

which adds a modest correction factor. This is the form that we will compare against the full Bayesian results using the methodology implemented by Gronau et al. (2021).

The results of this comparison are displayed in Figures 5a and 5b. Synthetic data are simulated under the null hypothesis. In the top panel, as in Figure 3, the black line shows that the frequentist fixed-N p-value fluctuates randomly (because $\mathcal{H}_0 : \psi = 0$ is true). In contrast, the Bayesian p-values constantly favor $\mathcal{H}_0$ over $\mathcal{H}_1$, although the extent of this preference is only modest. The different Bayesian p-values are so close that the colors can barely be distinguished – this signals that the full Bayesian result is well-approximated by JAB (cf. Eq. 30), which in turn is well-approximated by the p-value transformation rules (cf. Eq. 9).

The bottom panel of Figure 5 shows that the JAB-style approximation from Eq. 30 appears adequate even for lower sample sizes. Other explorations (not shown here) indicate that when the unit-information prior is centered on the MLE instead of on $\psi_0 = 0$ this worsens the approximation only slightly, with a difference noticeable only when sample size is small or $\hat{\psi}$ is large.

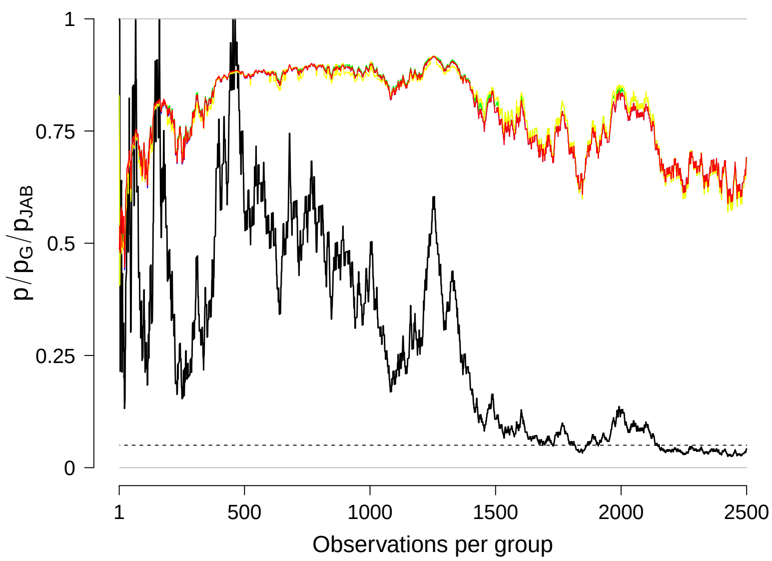
JAB fails when $n_{11} = n_{21} = 0$ (i.e., no successes are observed in either group). The full Bayesian result indicates that this scenario yields slight evidence in favor of $\mathcal{H}_0$, a preference that increases very slowly with the group sizes. Application of Eq. 30, however, yields $\text{JAB}_{01} = 1/\text{se}(\hat{\psi})$. With high group sizes, this means $\text{JAB}_{01} \to 1/2$, falsely indicating weak support in favor of $\mathcal{H}_1$ rather than $\mathcal{H}_0$. The culprit in this case is the definition of the standard error, which is only approximate and becomes unreliable when cell counts are near zero.

Note that Eq. 28 and Eq. 30 suggest that in BIC, the definition of effective sample size for logistic regression or contingency tables is more complicated than simply the total number of cases or counts as has been suggested previously (e.g., Raftery, 1986; Selig, Shaw, & Ankerst, in press).

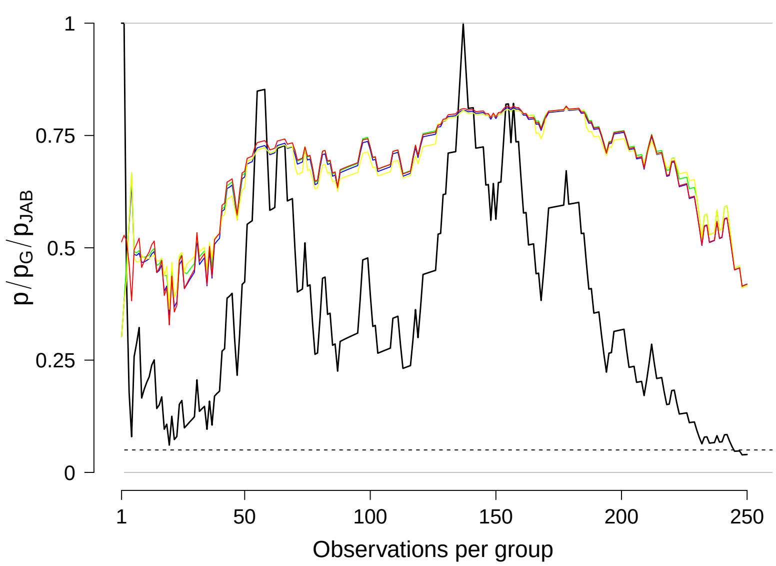
Example 4: The Pearson Correlation Test

In the Pearson correlation test the test-relevant parameter ρ quantifies the degree to which two normally distributed variables are linearly related. We have that $\mathcal{H}_0 : \rho = 0$ and $\mathcal{H}_1 : \rho \sim g(\rho) \in [-1, 1]$. The default prior distribution is uniform (Jeffreys, 1961, p. 290). The standard error of the Pearson correlation coefficient equals:

$$\text{se}(\hat{\rho}) = \frac{\sqrt{1-\hat{\rho}^2}}{\sqrt{n-2}}. \tag{31}$$



(a) Test of two proportions for $N = 2500$ pairs of observations.



(b) Test of two proportions for $N = 250$ pairs of observations.

Figure 5. A comparison of p-values from a test of two proportions under $\mathcal{H}_0$. For consecutive pairs of observations generated by independent Bernoulli distributions with common $\theta = 1/2$ (i.e., $\psi = 0$), the black line indicates the corresponding sequence of frequentist fixed-N p-values. The red line indicates p_{G} (i.e., based on $\mathcal{H}_1 : \psi \sim N(0, 1)$), the blue line is p_{JAB} from Eq. 6, the green line is p_{JAB} from the precise form of Eq. 9, and the yellow line is p_{JAB} from the simpler form of Eq. 9. See text for details.

Upon substituting, the term outside the exponential of Eq. 2 becomes

$$\begin{aligned} S_{\hat{\rho}} &= \frac{1}{\sqrt{2\pi}} \frac{1}{g(\hat{\rho})} \frac{1}{\sqrt{1-\hat{\rho}^2}} \sqrt{n-2} \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{g(\hat{\rho})} \frac{1}{\pi\sqrt{1-\hat{\rho}^2}} \sqrt{n-2}. \end{aligned} \tag{32}$$

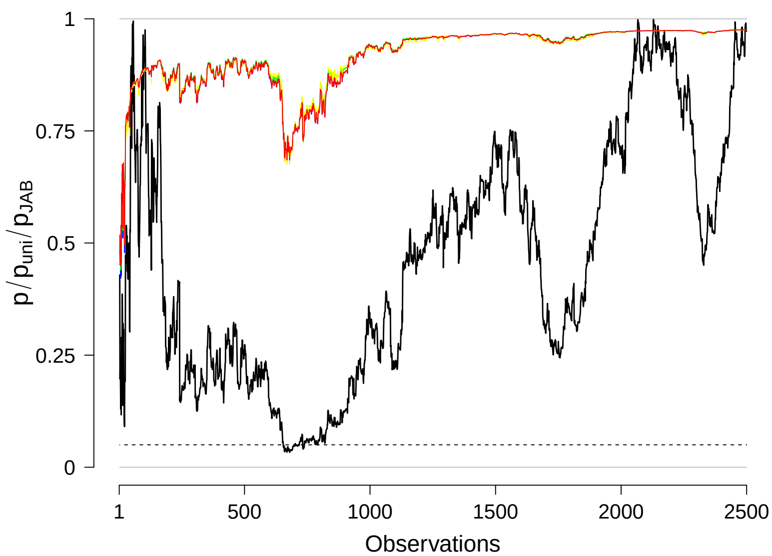
Note that the term $\left[\pi\sqrt{1-\hat{\rho}^2}\right]^{-1}$ is the Jeffreys prior for a binomial chance θ scaled to the $[-1, 1]$ interval, that is, a stretched beta($1/2, 1/2$) distribution (cf. Jeffreys, 1961, p. 82). Thus, similar to the binomial test example discussed earlier, if $g(\hat{\rho})$ is defined as the height of the stretched beta($1/2, 1/2$) distribution at the MLE, $S_{\hat{\rho}}$ reduces to $\sqrt{\pi/2}\,\sqrt{n-2}$, not too far away from $\sqrt{n-2}$. With a uniform distribution on ρ, however, $g(\hat{\rho}) = 1/2$. With this uniform prior the outside factor equals

$$\begin{aligned} S_{\hat{\rho}} &= \sqrt{2\pi} \frac{1}{\pi\sqrt{1-\hat{\rho}^2}} \sqrt{n-2} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-\hat{\rho}^2}} \sqrt{n-2}. \end{aligned} \tag{33}$$

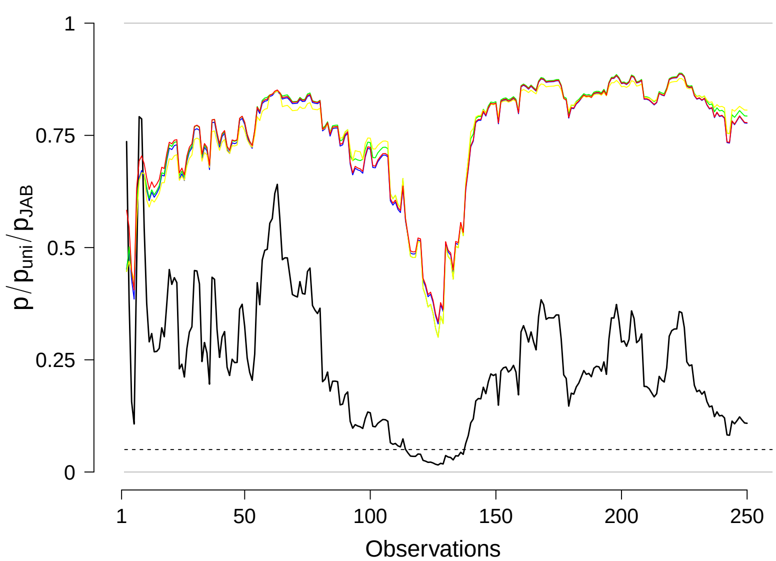
This is the form that we will compare against the full Bayesian solution (Ly et al., 2016). Synthetic data are simulated under the null hypothesis. The results, displayed in Figures 6a and 6b, are similar to those reported above for the test between two proportions. In the top panel, as before, the black line shows that the frequentist fixed-N p-value fluctuates randomly (because $\mathcal{H}_0 : \rho = 0$ is true). In contrast, the Bayesian p-values again constantly favor $\mathcal{H}_0$ over $\mathcal{H}_1$, although the extent of this preference is not always strong. The different Bayesian p-values are again so close that the colors can barely be distinguished, signaling that the full Bayesian result is well-approximated by JAB (cf. Eq. 33), which in turn is well-approximated by the p-value transformation rules (cf. Eq. 9). The bottom panel of Figure 6 shows that the JAB-style approximation from Eq. 33 appears adequate even for lower sample sizes.

Concluding Remarks

Our goals were threefold. Firstly we wanted to draw attention to Jeffreys's general approximate Bayes factor JAB, which we believe has been underappreciated if not largely ignored. Secondly, we presented a piecewise transformation that directly relates p-values to an approximate objective Bayes factors for the test of a point null hypothesis against a composite alternative. Thirdly, we demonstrated how the JAB approximation can be improved by carefully specifying the individual components involved in the derivation of JAB. This results in simple yet relatively accurate approximation for the t-test, the binomial test, the comparison of two proportions, and the correlation test. The same strategy –essentially just an application of Eq. 3–



(a) Correlation test for $N = 2500$ pairs of observations.



(b) Correlation test for $N = 250$ pairs of observations.

Figure 6. A comparison of p-values from a Pearson correlation test under $\mathcal{H}_0$. For consecutive pairs of observations drawn from a bivariate standard normal distribution with $\rho = 0$, the black line indicates the corresponding sequence of frequentist fixed-N p-values. The red line indicates p_{uni} (i.e., based on $\mathcal{H}_1 : \rho \sim \mathrm{Uniform}[-1, 1]$), the blue line is p_{JAB} from Eq. 6, the green line is p_{JAB} from the precise form of Eq. 9, and the yellow line is p_{JAB} from the simpler form of Eq. 9. See text for details.

can be adopted to construct approximations for other tests, for instance the test of a first-order autocorrelation, the test of equality for two variances, the comparison of two correlations, and so on.

The transformation from p-value to JAB is straightforward; its central component is the $\sqrt{n}$ term, or more generally the sample size involved in the standard error of the MLE. Adding this term changes the quantity of interest in fundamental ways. For instance, in contrast to the p-value, JAB is able to quantify evidence in favor of the null hypothesis, may be monitored until it is sufficiently compelling, and does not reject the null when the evidence is ambiguous. It is seen that the p-value is both fundamentally incompatible with JAB but also tightly related: the difference is in the $\sqrt{n}$ scaling, which exerts a surprisingly profound effect.

Hiding in plain sight, one complication with JAB is the specification of A and $\sqrt{n}$. These terms become particularly influential when the observed effect size is close to zero, in which case the exponential term is near 1 and the evidence in favor of $\mathcal{H}_0$ is driven solely by the outside factor $A\sqrt{n}$. Blindly applying the simple form of JAB yields an approximation that is essentially identical to BIC, and worse than what can be achieved with a little more care.

Another complication with JAB is that the approximation is valid only when $\text{se}(\hat{\theta}) \ll \sigma_g$; thus, as was discussed in relation to the results shown in Figure 4, JAB may be biased against $\mathcal{H}_0$ when sample size is low. In the extreme case that $n = 1$, JAB_{01} reduces to $\exp\{-\frac{1}{2}W\}$, which equals the result from the oracle point prior at the MLE (Edwards et al., 1963; the same issue is evident for the BIC). Nevertheless, the applications showed that JAB provided an accurate approximation for default tests even when sample size is low. We suggest this is due to two factors: first, JAB can be improved by adding a correction factor for centering the prior at θ_0 rather than $\hat{\theta}$ (cf. Eq. 4 vs. 5 and Eq. 29 vs. 30); second, in the applications we did not compute W directly but instead inferred W from the p-value. This benefits the JAB approximation because the p-value takes into account the fact that $\text{se}(\hat{\theta})$ is estimated with error. A final drawback of JAB is that it can break down in extreme cases. An example concerns the comparison of two proportions, where JAB fails when the 2×2 contingency table contains zero cell counts.

It is important to have available simple expressions for the Bayes factor, as this lowers the bar for their practical application, offers insight into the relation with p-values, and allows a quick Bayesian evaluation of a frequentist report. We hope that this work will stimulate statisticians to reexamine Jeffreys's *Theory of Probability* (cf. Lindley, 1980; Ly et al., 2016; Robert, Chopin, & Rousseau, 2009), and we hope that JAB will encourage empirical researchers to asses their hypotheses with a statistical methodology that complements the ubiquitous p-value in informative ways.

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Appendix A: Jeffreys's Derivation and Explanation

Throughout his work on hypothesis testing, Jeffreys repeatedly discussed the general approximate form of his Bayes factor (i.e., Eq. 2; for references and a historical overview see Ly & Wagenmakers, 2021b). The general form was introduced in Jeffreys (1936, p. 417); a particularly straightforward derivation was presented in the addenda of the 1937 reprint of the 1931 first edition of Jeffreys's book *Scientific Inference*:

> "Suppose we consider as a serious possibility that a quantity x may be zero; denote this proposition by q, with prior probability $\frac{1}{2}$. The proposition that x is not zero is denoted by $\sim q$, also with prior probability $\frac{1}{2}$; but if x is not zero it may be anywhere in a range of length m. An actual determination from data θ suggests a value of $x_0 \pm \sigma$. Now, if x is really 0, the probability of finding a mean in a range dx_0 about dx_0 is $\frac{1}{\sqrt{(2\pi)}\sigma} \exp\left(-\frac{x_0^2}{2\sigma^2}\right) dx_0$. But if x is not 0, the probability that it would be in such a range is dx_0/m. Given then that x_0 has actually been found in such a range, the posterior probabilities of q and $\sim q$ are in the ratio of these two expressions, namely
>
> $$\frac{P(q \mid \theta h)}{P(\sim q \mid \theta h)} = \frac{m}{\sqrt{(2\pi)}\sigma} \exp\left(-\frac{x_0^2}{2\sigma^2}\right).$$
>
> When x_0 is large compared with σ, this is small, q has a small posterior probability, and we can assert with confidence that x is different from zero. But σ, the standard error of the mean, is proportional to $n^{-\frac{1}{2}}$, where n is the number of observations; hence if n is large the first factor is large of order $\sqrt{n}$, and the ratio will be large if x_0 is less than σ. Thus a discrepancy less than a certain amount increases the probability that the parameter sought is zero; one more than this amount decreases it and indicates that the parameter is needed. In the cases examined the critical value, with ordinary numbers of observations, ranges from about 1.5 to 3 times the standard error, increasing with the number of observations. The larger the number of observations the stronger the support for the simple law $x = 0$ if the empirical value turns out to be within its standard error. To put the argument in words, if x_0 is of order σ, this is what we should expect if x is zero, but if x might be anywhere in a range m it is a remarkable coincidence that it should be in just this one. On the other hand, if x_0 is substantially more than σ, we should not expect it if x is zero, but we should expect it if x is not zero; in both cases we adopt the less remarkable coincidence." (Jeffreys, 1937, pp. 250-251)

A more general derivation can be found in Jeffreys (1961, pp. 246-247), which we now paraphrase in an extended form using modern notation (for an intuitive presentation see Cousins, 2017, p. 399):

"We shall call $\mathcal{H}_0$ the *null hypothesis*, following Fisher, and $\mathcal{H}_1$ the *alternative hypothesis*. To say that we have no information initially as to whether the new parameter is needed or not we must take

$$p(\mathcal{H}_0) = p(\mathcal{H}_1) = \tfrac{1}{2}.$$

But $\mathcal{H}_1$ involves an adjustable parameter, α say, and

$$p(\mathcal{H}_1) = \sum p(\mathcal{H}_1, \alpha)$$

over all possible values of α. We take α to be zero on $\mathcal{H}_0$. Let the prior probability of $\mathrm{d}\alpha$, given $\mathcal{H}_1$, be $f(\alpha)\,\mathrm{d}\alpha$, where

$$\int f(\alpha)\,\mathrm{d}\alpha = 1,$$

integration being over the whole range of possible values when the limits are not given explicitly. Then

$$p(\mathcal{H}_1\,\mathrm{d}\alpha) = \tfrac{1}{2}\int f(\alpha)\,\mathrm{d}\alpha.$$

We can now see in general terms that this analysis leads to a significance test for α. For if the maximum likelihood solution for α is $\hat{\alpha} \pm \mathrm{se}(\hat{\alpha})$, the chance of finding $\hat{\alpha}$ in a particular range, given $\mathcal{H}_0$, is nearly

$$p(\mathrm{d}\hat{\alpha} \mid \mathcal{H}_0) = \frac{1}{\sqrt{(2\pi)}\,\mathrm{se}(\hat{\alpha})}\,\exp\left(-\tfrac{1}{2}\frac{\hat{\alpha}^2}{\mathrm{se}(\hat{\alpha})^2}\right)\,\mathrm{d}\hat{\alpha},$$

and the chance, given $\mathcal{H}_1$ and a particular value of α, is

$$p(\mathrm{d}\hat{\alpha} \mid \mathcal{H}_1) = \frac{1}{\sqrt{(2\pi)}\,\mathrm{se}(\hat{\alpha})}\,\exp\left(-\tfrac{1}{2}\frac{(\hat{\alpha}-\alpha)^2}{\mathrm{se}(\hat{\alpha})^2}\right)\,\mathrm{d}\hat{\alpha}.$$

Hence by the principle of inverse probability

$$p(\mathcal{H}_0 \mid \hat{\alpha}) \propto \frac{1}{\sqrt{(2\pi)}\,\mathrm{se}(\hat{\alpha})}\,\exp\left(-\tfrac{1}{2}\frac{\hat{\alpha}^2}{\mathrm{se}(\hat{\alpha})^2}\right),$$

$$p(\mathcal{H}_1\,\mathrm{d}\alpha \mid \hat{\alpha}) \propto \frac{1}{\sqrt{(2\pi)}\,\mathrm{se}(\hat{\alpha})}\,f(\alpha)\,\exp\left(-\tfrac{1}{2}\frac{(\hat{\alpha}-\alpha)^2}{\mathrm{se}(\hat{\alpha})^2}\right)\,\mathrm{d}\alpha.$$

It is to be understood that in pairs of equations of this type the sign of proportionality indicates the same constant factor, which can be adjusted to make the total probability 1.

Consider two extreme cases. There will be a finite interval of α such that $\int f(\alpha)\,\mathrm{d}\alpha$ through it is arbitrarily near unity. If $\hat{\alpha}$ lies in this range and $\mathrm{se}(\hat{\alpha})$ is so large that the exponent in the last equation above is small over most of this range, we have on integration, approximately,

$$p(\mathcal{H}_1 \mid \hat{\alpha}) = p(\mathcal{H}_0 \mid \hat{\alpha}) \propto \frac{1}{\sqrt{(2\pi)}\mathrm{se}(\hat{\alpha})}.$$

In other words, if the standard error of the maximum likelihood estimate is greater than the range of α permitted by $\mathcal{H}_1$, the observations do nothing to decide between $\mathcal{H}_0$ and $\mathcal{H}_1$.

If, however, $\mathrm{se}(\hat{\alpha})$ is small, so that the exponent can take large values, and $f(\alpha)$ is continuous, the integral will be nearly $f(\hat{\alpha})$, and

$$\frac{p(\mathcal{H}_0 \mid \hat{\alpha})}{p(\mathcal{H}_1 \mid \hat{\alpha})} \approx \frac{1}{\sqrt{(2\pi)}\mathrm{se}(\hat{\alpha})f(\hat{\alpha})} \exp\left(-\frac{1}{2}\frac{\hat{\alpha}^2}{\mathrm{se}(\hat{\alpha})^2}\right)."$$

For related expositions see Edwards et al. (1963, p. 228), Edwards (1965), and Cousins (2017).

Based on this result Jeffreys draws a number of general conclusions:

"We shall in general write

$$\mathrm{BF}_{01} = \frac{p(\mathcal{H}_0 \mid y)}{p(\mathcal{H}_1 \mid y)}\Big/\frac{p(\mathcal{H}_0)}{p(\mathcal{H}_1)}.$$

This is independent of any particular choice of $p(\mathcal{H}_0)/p(\mathcal{H}_1)$. If the number of observations, n, is large, $\mathrm{se}(\hat{\alpha})$ is usually small like $n^{-1/2}$. Then if $\hat{\alpha} = 0$ and n large, BF_{01} will be large of order $n^{1/2}$, since $f(\alpha)$ is independent of n. Then the observations support $\mathcal{H}_0$, that is, they say that the new parameter is probably not needed. But if $|\hat{\alpha}|$ is much larger than $\mathrm{se}(\hat{\alpha})$ the exponential factor will be small, and the observations will support the need for the new parameter. For given n, there will be a critical value of $\hat{\alpha}/\mathrm{se}(\hat{\alpha})$ such that $\mathrm{BF}_{01} = 1$ and no decision is reached.

The larger the number of observations the stronger the support for $\mathcal{H}_0$ will be if $|\hat{\alpha}| < \mathrm{se}(\hat{\alpha})$. This is a satisfactory feature; the more thorough the investigation has been, the more ready we shall be to suppose that if we have failed to find evidence for α it is because α is really 0. But it carries with it the consequence that the critical value of $|\hat{\alpha}/\mathrm{se}(\hat{\alpha})|$ increases with n (though that of $|\hat{\alpha}|$ of course diminishes); the increase is very slow, since it depends on $\sqrt{(\log n)}$, but it is appreciable. The test does not draw the line at a fixed value of $|\hat{\alpha}/\mathrm{se}(\hat{\alpha})|$." (Jeffreys, 1961, p. 248)

Appendix B, Online: Prior-Independent Insights on Evidence, Replications, and Sequential Planning

This appendix contains remarks on the general version of JAB_{01} (Eq. 2) in which A can take on any finite value. As shown below, the general form of JAB_{01} may be used (1) to quantify the difference in the strength of evidence between an x-sigma result and an $(x+d)$-sigma result; (2) to quantify the evidential impact of an ideal replication experiment; and (3) to assist in sequential planning, where data collection can be continued until a threshold of evidence is reached or until resources are depleted. Importantly, these results are independent of the prior distribution $g(\theta)$ and therefore hold in considerable generality.

Remark B1. Difference in Evidence Between a x-Sigma Result and an $(x+d)$-Sigma Result

For arbitrary but fixed prior distribution $g(\theta)$, for arbitrary but fixed sample size n, and for arbitrary but fixed single-sample standard deviation σ, the difference in evidence $\log \text{JAB}_{10}^{\Delta}$ between an x-sigma result and an $(x+d)$-sigma result is

$$\log \text{JAB}_{10}^{\Delta} = \tfrac{1}{2}d^2 + dx, \tag{34}$$

which is obtained by dividing two versions of Eq. 2 such that the factor outside the exponential cancels. For instance, consider the results of a fictitious experiment of sample size n, with a certain single-sample standard deviation σ, subjected to a Bayes factor test that contrasts $\mathcal{H}_0 : \theta = \theta_0$ to $\mathcal{H}_1 : \theta \sim g(\theta)$. The difference in the log Bayes factor between a 2-sigma result and a 3-sigma result then equals 2.5, meaning that whatever the Bayes factor is for the 2-sigma outcome, the result for the 3-sigma outcome is $\exp(2.5) \approx 12.18$ times more compelling, or about an order of magnitude. Other numbers are listed in Table 3.

Remark B2. Evidential Impact of an Exact Replication

Let E_1 denote an original experiment, with evidence $\text{JAB}_{01}(E_1)$ given by Eq. 2. Let E_2 be a replication experiment with the same design (i.e., same n, same σ). Assume that the data from both experiments are exchangeable, and that the same MLE $\hat{\theta}$ obtains. One may consider this an ideal replication, where the data from E_1 and E_2 provide exactly the same statistical information. For the complete data set, the evidence is given by $\text{JAB}_{01}(E_1, E_2)$, obtained by replacing n by $2n$ in Eq. 2 (this occurs in two places, as $\text{se}(\hat{\theta}) = \sigma/\sqrt{n}$). The change in evidence brought about by observing E_2 is known as the "replication Bayes factor" (Ly, Etz, Marsman, & Wagenmakers, 2019; see also Verhagen & Wagenmakers, 2014). When E_1 yields an x-sigma result, the replication Bayes factor is given by

$$\begin{aligned} \text{JAB}_{01}(E_2 \mid E_1) &= \frac{\text{JAB}_{01}(E_1, E_2)}{\text{JAB}_{01}(E_1)} \\ &= \sqrt{2}\ \exp\left(-\tfrac{1}{2}x^2\right). \end{aligned} \tag{35}$$

Table 3
The difference in the log Bayes factor JAB_{10} (and its exponent, in brackets and rounded to whole numbers) between an x-sigma finding and an $(x+d)$-sigma finding, for two experiments with equal sample sizes, equal single-sample standard deviation, and analyzed with the same Bayes factor hypothesis test.

	$(x+d)$-sigma					
x-sigma	1	2	3	4	5	6
0	0.5 (2)	2 (7)	4.5 (90)	8 (2981)	12.5 (268,337)	18 (65,659,969)
1	–	1.5 (4)	4 (55)	7.5 (1808)	12 (162,755)	17.5 (39,824,784)
2	–	–	2.5 (12)	6 (403)	10.5 (36,316)	16 (8,886,111)
3	–	–	–	3.5 (33)	8 (2981)	13.5 (729,416)
4	–	–	–	–	4.5 (90)	10 (22,026)
5	–	–	–	–	–	5.5 (245)

Notably, this result depends only on x and is independent of the number of observations n, the sampling variability σ, and the prior distribution $g(\theta)$. As an example, assume that E_1 yields a p-value that is just significant at the .05 level, such that $x \approx 2$. The observation of an ideal replication E_2 then increases the evidence in favor of $\mathcal{H}_1$ by a multiplicative factor of $\text{JAB}_{10}(E_2 \mid E_1) = \exp(2)/\sqrt{2} \approx 5.22$. For $x = 1$, $\text{JAB}_{10}(E_2 \mid E_1) = \sqrt{e/2} \approx 1.17$: replicating a one-sigma result has almost no evidential impact. Finally, for $x = 0$ (i.e., data perfectly consistent with $\mathcal{H}_0$), $\text{JAB}_{01}(E_2 \mid E_1) = \sqrt{2}$: regardless of how large the sample may be, replicating a perfect null-result increases the evidence in favor of $\mathcal{H}_0$ only by a factor of about 1.41. This is also evident from Eq. 2, as when $\hat{\theta} - \theta_0 = 0$, the evidence for $\mathcal{H}_0$ is proportional to $\sqrt{n}$ for E_1, and proportional to $\sqrt{2n}$ for E_1 and E_2 together.

Consider now the case of medical clinical trials, where the common requirement for approval of a drug is that two experiments both need to find $p < .05$ (i.e., the *two-trials rule*; Kay, 2015, Section 9.4). The minimal requirement is therefore that two experiments, E_1 and E_2, both achieve a 2-sigma result. We again assume that the data from E_1 and E_2 are exchangeable, and that the replication is perfect. From Eq. 2 it follows that two experiments that each yield a 2-sigma result yield a $2\sqrt{2} \approx 2.83$-sigma result when analyzed jointly. In other words, two 2-sigma results roughly translate to a single 3-sigma result. For concreteness, consider two experiments with $n = 100$ and p-values just significant at .05, so $x = 2$. Applying Eq. 2 with $A = 1$ yields $\text{JAB}_{01}(E_1) = \sqrt{100} \cdot \exp(-2) \approx 1.35$, a smidgen of evidence in favor of $\mathcal{H}_0$ in fact. Eq. 35 then yields $\text{JAB}_{01}(E_2 \mid E_1) \approx 1/5.22 \approx 0.19$, such that the total evidence in favor of $\mathcal{H}_1$ equals $\text{JAB}_{10}(E_1, E_2) = 5.22/1.35 \approx 3.87$ (which can also obtained by applying Eq. 2 with $n = 200$). This level of evidence is generally not considered compelling – Jeffreys described a Bayes factor of $16/3 \approx 5.33$ as "odds that would interest a gambler, but would be hardly worth more than a passing mention in a scientific paper" (Jeffreys, 1939, p. 196, repeated in Jeffreys, 1961, pp. 256-257).

One way to obtain more evidence is to demand that both experiments yield significance at a lower threshold, say $\alpha = .005$ (e.g., Benjamin et al., 2018) which is approximately a 3-sigma result. Consider again two experiments with $n = 100$ but now with $x = 3$, so p-values somewhat lower than .005. Applying Eq. 2 with $A = 1$ yields $\text{JAB}_{10}(E_1) = \sqrt{100} \cdot \exp(-4.5) \approx 9.00$, which is moderate evidence for $\mathcal{H}_1$ (Jeffreys, 1939; M. D. Lee & Wagenmakers, 2013; Wasserman, 2000). Now Eq. 35 yields $\text{JAB}_{10}(E_2 \,|\, E_1) \approx 63.65$, and the total evidence in favor of $\mathcal{H}_1$ equals $\text{JAB}_{10}(E_1, E_2) = 9.00 \cdot 63.65 \approx 572.97$, which is compelling and would increase a prior model probability of 0.50 to a posterior model probability of $572.97/573.97 \approx 0.998$.

In order to obtain more evidence, we may also consider additional replication experiments: $E_3, E_4, ..., E_k$. As before, we assume that these replication experiments yield the same statistical information as the original, and that the data are exchangeable. Assuming that $E_1, ..., E_k$ all yield an x-sigma result, the replication Bayes factor associated with the addition of experiment $k+1$ is given by

$$\begin{aligned}\text{JAB}_{01}(E_{k+1} \mid E_1, ..., E_k) &= \frac{\text{JAB}_{01}(E_1, ..., E_{k+1})}{\text{JAB}_{01}(E_1, ..., E_k)} \\ &= \sqrt{\frac{k+1}{k}}\; \exp\left(-\tfrac{1}{2}x^2\right).\end{aligned} \tag{36}$$

For instance, consider the scenario where two $n = 100$ experiments have each produced a 2-sigma result, and a third experiment is conducted that also yields a 2-sigma result. With $k = 2$ and $x = 2$, application of Eq. 2 yields $\text{JAB}_{10}(E_1, E_2) \approx 3.86$ and $\text{JAB}_{10}(E_1, E_2, E_3) \approx 23.29$. The addition of the third experiment therefore increases the evidence in favor of $\mathcal{H}_1$ by a factor of $23.29/3.86 \approx 6.03$, which can also be obtained directly from Eq. 36 as $\left[\sqrt{(3/2)} \cdot \exp(-2)\right]^{-1}$.

With a sequence of four perfect-replication experiments, the evidence factors associated with the progression of each new experiment are: (1) when $x = 3$: $\text{JAB}_{10}(E_2 \,|\, E_1) = 63.65$, $\text{JAB}_{10}(E_3 \,|\, E_1, E_2) = 73.50$, $\text{JAB}_{10}(E_4 \,|\, E_1, E_2, E_3) = 77.96$. As k increases, the evidence factors increase towards $\exp(4.5) \approx 90.02$; (2) when $x = 2$: $\text{JAB}_{10}(E_2 \,|\, E_1) = 5.22$, $\text{JAB}_{10}(E_3 \,|\, E_1, E_2) = 6.03$, $\text{JAB}_{10}(E_4 \,|\, E_1, E_2, E_3) = 6.40$. As k increases, the evidence factors increase towards $\exp(2) \approx 7.39$; (3) when $x = 1$: $\text{JAB}_{10}(E_2 \,|\, E_1) = 1.17$, $\text{JAB}_{10}(E_3 \,|\, E_1, E_2) = 1.35$, $\text{JAB}_{10}(E_4 \,|\, E_1, E_2, E_3) = 1.43$. As k increases, the evidence factors increase towards $\exp(0.5) \approx 1.65$; (4) when $x = 0$, a perfect null result, Eq. 36 simplifies to $\sqrt{(k+1)/k}$ and the evidence factors in favor of $\mathcal{H}_0$ are: $\text{JAB}_{01}(E_2 \,|\, E_1) = 1.41$, $\text{JAB}_{01}(E_3 \,|\, E_1, E_2) = 1.22$, $\text{JAB}_{01}(E_4 \,|\, E_1, E_2, E_3) = 1.15$. As k increases, the evidence factors decrease towards 1. Thus, in contrast to the situation where $x \neq 0$, perfect replications of null-results carry increasingly less diagnostic value. Intuitively, this happens because the prior distribution $g(\theta)$ under $\mathcal{H}_1$ concentrates near θ_0 as the replication experiments accumulate, and as a consequence the predictions from $\mathcal{H}_1$ will increasingly mimic those of $\mathcal{H}_0$, making the models more difficult to discriminate. However, it should be noted that the procedure is nevertheless consistent under $\mathcal{H}_0$, as $\prod_{k=1}^{n} \sqrt{(k+1)/k} = \sqrt{n+1}$.

Remark B3. Sequential Planning

Suppose an experiment E_1 is conducted that yields an x-sigma finding, and an associated Bayes factor $\text{JAB}_{01}(E_1)$. This Bayes factor may be insufficiently compelling, necessitating the collection of additional observations, denoted by E_2^*. How many additional observations n^* can the experimenter expect to collect in order achieve a target level of evidence? And given that resources allow only a particular number of additional observations to be collected, what is the expected increase in the level of evidence? Let the total amount of evidence (desired or anticipated) be given by $\text{JAB}_{01}(E_1, E_2^*)$, and let the required change in evidence brought about by E_2^* be given by $\text{JAB}_{01}(E_2^* \mid E_1)$. Let n denote the number of observations in E_1, and $m \cdot n$, $m > 1$ the total number of observations across E_1 and E_2^* together, such that the required number of additional observations equals $n^* = n(m-1)$. We make the assumption –surely false, but useful to obtain an approximate lower bound result– that the to-be-collected data in E_2^* yield the same MLE and the same single-observation sampling uncertainty σ. Then each m is associated with a particular change in evidence $\text{JAB}_{01}(E_2^* \mid E_1)$, as follows:

$$\begin{aligned} \text{JAB}_{01}(E_2^* \mid E_1) &= \frac{\text{JAB}_{01}(E_1, E_2^*)}{\text{JAB}_{01}(E_1)} \\ &= \sqrt{m}\ \exp\left(-\tfrac{1}{2}x^2(m-1)\right). \end{aligned} \tag{37}$$

This equation can be used to address the two questions above. For concreteness, we revisit an earlier example and assume that E_1 with $n = 100$ yielded a 2-sigma result (i.e., $x = 2$). Applying Eq. 2 with $A = 1$ yielded $\text{JAB}_{01} = 1.35$. First, suppose the existing resources allow for another $n^* = 100$ observations to be collected. Hence $m = 2$ and Eq. 37 simplifies to Eq. 35, with $\text{JAB}_{01}(E_2^* \mid E_1) \approx 0.19$, such that the total expected evidence is $\text{JAB}_{01}(E_1, E_2^*) \approx 5.22/1.35 \approx 3.87$.

Second, suppose that m is not given, but instead we are given $\text{JAB}_{01}(E_2^* \mid E_1)$, the desired additional evidence coming from E_2^*. For comparison purposes, we use $\text{JAB}_{01}(E_2^* \mid E_1) = .19$, implying that the target level of evidence for E_1 and E_2^* combined is $5.22/1.35 \approx 3.87$. Solving Eq. 37 for m yields $m = 2.00$, as it should. A special case arises when E_1 yields $x = 0$, a perfect null result. Then $\text{JAB}_{01}(E_2^* \mid E_1) = \sqrt{m}$; in order to double the evidence, $m = 4$ and when $n = 100$, this means that n^* needs to be at least 300. Further note that for $x \neq 0$, increasing sample size allows any desired level of evidence for $\mathcal{H}_1$ to be obtained, as

$$\lim_{m \to \infty} \sqrt{m}\ \exp\left(-\tfrac{1}{2}x^2(m-1)\right) = 0.$$

It is not trivial to solve Eq. 37 for m. We have

$$m = \frac{-\mathcal{W}_{-1}(-\left[\text{JAB}_{01}(E_2^* \mid E_1)\right]^2\ e^{-x^2}\ x^2)}{x^2}, \tag{38}$$

where $\mathcal{W}_{-1}(z)$ denotes the non-principal branch of the Lambert-W product-log function, which is defined for $-1/e \leq z < 0$ (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996). For Eq. 38 the lower bound entails the restriction $\exp(x^2) \geq e\,x^2\,[\mathrm{JAB}_{01}(E_2^* \mid E_1)]^2$, which is violated when $x \neq 0$ and $[\mathrm{JAB}_{01}(E_2^* \mid E_1)]^2$ is larger than $\exp(x^2-1)/x^2$; in other words, when there exist a true non-zero effect, it is impossible to obtain arbitrarily large evidence in favor of $\mathcal{H}_0$. For instance, when $x = 1$ then $\mathrm{JAB}_{01}(E_2^* \mid E_1)$ cannot exceed 1.

The result can be easily obtained in R, for instance using the `lamW` package (Adler, 2021):

```
library(lamW); B <- .19; x <- 2;
m <- -(lambertWm1(-B^2*exp(-x^2)*x^2))/x^2
```

An approximation to Eq. 38 (Corless et al., 1996) is given by

$$m \approx \frac{-\{\ln([\mathrm{JAB}_{01}(E_2^* \mid E_1)]^2\; e^{-x^2}\, x^2) - \ln(-\ln(([\mathrm{JAB}_{01}(E_2^* \mid E_1)]^2\; e^{-x^2}\, x^2)))\}}{x^2}. \tag{39}$$

For the example above, this approximation yields $m = 1.93$.