Theoretical evaluation of partial credit scoring of the multiple-response test item

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ABSTRACT

We compute and compare statistics for the validity, the precision and the reliability of five different scoring rules for the selected-response type of test items where the number of keys is an arbitrary integer and the test-takers are perfectly rational agents. We consider a hypothetical test of factual recognition, in which the underlying ability that we seek to measure is the fraction of the item options that the test-taker truly recognizes (and not only guesses correctly), assumed directly proportional the test-taker's domain knowledge.

KEYWORDS

Error variance; multiple-response test item; partial credits model

1. Introduction

In a previous contribution (Persson, 2021, hereonafter designated “Paper I”), we made a theoretical comparison of the ideal properties of a number of scoring functions for the multiple-choice test item; that is, the type of selected-response item where there is one and only one correct option (“key”) among any number of incorrect ones (“distractors”). Here, we report a similar analysis but for the type of selected-response item where there is an arbitrary integer number k of keys among c options in total; following Ma (2004), we call this type of test item a “multiple-response” one. Although there is only a single completely correct response to such an item (when all k keys are indicated and none of the $c - k$ distractors), there is a large number of non-equivalent incorrect responses which we take to be ordered along a scale of “partial knowledge”.

This type of test item has not received the same broad attention as the corresponding multiple-choice test one (for which $k = 1$ invariably) and much of earlier work is in the inaccessible gray literature. The impetus for the present investigation was a recent paper which called for “further studies [...] with different populations to see how robust the results are as general scoring methods” (Betts, Muntean, Kim, & Kao, 2021, p. 21), although the study of this type of test item stretches back to at least the 1930s under different names (Bauer, Holzer, Kopp, & Fischer, 2011; Cronbach, 1941; Dressel & Schmid, 1953; Hsu, Moss, & Khampalikit, 1984; Pomplun & Omar, 1997; Scheidemann, 1933). As regards non-empirical contributions in particular, Duncan and

Milton (1978) offer a completely theoretical study of scoring functions for multiple-response test items that is in many ways complementary to this one. Their study is focused on formal equivalence between different scoring rules and on formalizations of rational strategies of the test-taker (and test-maker), including cases where the test-taker is misinformed (and not merely ignorant). In this contribution, on the other hand, we aim to compare different proposed scoring rules theoretically so as to ascertain which ones, if any, are superior. Our aims are hence similar to those of the empirical literature just cited – *i. e.* pragmatic – but our approach is different as we eschew empirical data and aim to obtain results which are population-independent (in the sense that they can be verified and reproduced exactly by other investigators).

The theoretical undergirding of the present contribution is in all respects identical with the one used in Paper I, but we will offer a brief recapitulation in the next section for the benefit of the reader. The most basic underlying assumption is that the test-taker is a rational agent who strives to maximize her expected score on the test, and while this is unlikely to hold true for all real test-takers, we consider the results as mathematical limiting cases. In the actual testing situation, the cognitive effort of behaving perfectly rationally need not be excessive, and does not require decision-theoretical calculations, provided the test instructions clearly outline the rational strategy to pursue under all different types of situations. If the test instructions do not cover strategy, there is a clear risk that construct-irrelevant variance is introduced in the observed scores. Furthermore, it is *a priori* unlikely that this assumption will disproportionately affect different scoring rules to the extent that the comparisons are completely invalidated. In sum, theoretical modeling of rational test takers offers the following advantages over empirical studies:

- The theoretical perfectly rational test-taker may be a specific case that is not entirely representative, but empirical studies with real people are never perfectly representative either.
- There is no measurement uncertainty in the results, unlike empirical studies.
- Real people can be instructed on rational strategies, and hence approach the behavior modeled with increased training. This is also desirable from the point of view of test validity.

Let us at this point stress that we are concerned primarily with the issues of small-scale classroom testing, and hence we focus on the scoring function of the test, as it is both the most important instrument at the time of test administration to influence test-taker behavior and is responsible for the most direct and tangible result of the measurement: the observed score. Statistical estimates of unknown parameters *a posteriori* from the sampled data, as in item response theory, is more suitable for large-scale assessments with greater statistical power and is also likely to be perceived as less transparent and “fair” by the students.

2. Theoretical model

We assume that when the test-taker is faced with an item, she uses her knowledge of the answer options to decide whether they are keys or distractors, and guesses if advantageous among the ones for which her knowledge is incomplete. We consider the actual latent score to be the number of answer options for which she is certain about their status as either distractor or key, and this latent score may be reflected to different degrees in the observed score depending on the scoring function used. This

presupposes that the item stem is comprehensible to the test-taker; in case it is not, we may simply consider the latent score to be zero for that item. For simplicity, we shall always assume that the item stem is comprehensible and thus avoid this theoretical hurdle.

To illustrate the point of the preceding paragraph, consider the following example test item intended to test geography knowledge (the actual number of answer options, keys or distractors, is immaterial).

Which of the following are African countries?

- (1) Algeria
- (2) Abu Dhabi
- (3) Dubai
- (4) Sierra Leone
- (5) Egypt
- (6) South Sudan
- (7) Suriname
- (8) Manitoba
- (9) ...

The test-taker that knows the names of all of the countries in Africa will answer the item fully correctly with certainty because she knows for each of the options whether it is a distractor or key regardless of which distractors and keys are chosen by the test-maker; for everyone else, the completely correct answer may only be obtained with some amount of luck: either “luck of the draw” (the set of answer options on the item just happen to be known by a given test-taker) or fortunate guessing.

We shall represent the latent score not as an integer value which will depend on test length, but as the fraction f of the total number of answer options that the test-taker knows. In this way, our representation of the latent score becomes independent of test length and we shall for convenience, but without loss of generality, consider a test composed of a single item. In regard to the single item, we interpret the value of f as the probability that the test-taker knows an arbitrary answer option. In this way, we treat f as a real-valued variable that reflects the domain knowledge of the test-taker.

This model applies only to tests of factual recognition; it is supposed that there are no higher-order cognitive demands, such as reasoning, imposed by the test item. This seems a reasonable assumption for tests of vocabulary or general knowledge, but will not work for assessments of problem solving, or conceptual thinking. However, the selected-response type of test item is arguably ill-suited for those kinds of assessments anyway, which is why we do not consider it a strong limitation of the present work.

3. Scoring functions

The scoring functions that we will investigate are simple and have probably been independently conceived countless of times throughout the history of multiple-response tests, and hence tracing their first occurrence in the literature would be a senseless task. We shall use the nomenclature of Betts et al. (2021), but we start our investigation within a general linear scoring function of which most of the scoring functions considered by Betts et al. (2021) are special cases, at least in the high-ability end of the test-taker ability distribution.

3.1. Preliminaries

Let c be the total number of answer options of a multiple-response test item and let us assume that q options are correctly identified as keys, that d options are correctly identified as distractors, that Q options are incorrectly identified as keys and that D ones are misidentified as distractors. Naturally, the expected contribution to the score from guessing (rather than from knowledge) is very small if the item is scored dichotomously (zero total points awarded whenever $Q \neq 0$ or $D \neq 0$). For instance, the *a priori* probability of providing a completely correct response when unsure about $c - q - d$ answer options is 2^{d+q-c} which rapidly approaches zero as c increases. However, so does the probability of knowing the fully correct answer, which is simply f^c . Dichotomous scoring thus seemingly increases test reliability at the expense of an increase in test difficulty, leading to poorer discrimination and likely reduced validity. Scoring functions with appropriate penalties may discourage guessing while at the same time not increasing the difficulty of the items unreasonably. In the following, we shall formalize the ones discussed by Betts et al. (2021) within the framework of the underlying f -score.

To facilitate the discussion, we consider the observed score on the total item to be a sum over scores for the individual options and write the general score function for the item as,

$$S(q, d, Q, D) = qa + db - QA - DB \quad (1)$$

where a, b, A and B are constants chosen appropriately to ensure that the score is an increasing function of the number of correctly identified options. Since the parameters and S are linearly dependent, in the following we always consider $a = 1$ without loss of generality and usually leave it out of the expressions for brevity except where extra clarity is sought.

The observed q and d are the sum of two contributions: a “deterministic” one from the test-taker’s knowledge and a “stochastic” one from chance. The probability of correctly identifying a single answer option correctly is $1/2 + f/2$ (the complementary probability to that of failing to know, $1 - f$, and failing to guess, $1/2$, the correct option status). We may hence regard the process of correctly identifying each key as a Bernoulli process with this probability. The probability that the test-taker correctly indicates q' of the k keys is hence binomially distributed,

$$P\left(q', k, \frac{1}{2} + \frac{f}{2}\right) = \binom{k}{q'} \left(\frac{1}{2} + \frac{f}{2}\right)^{q'} \left(\frac{1}{2} + \frac{f}{2}\right)^{k-q'} \quad (2)$$

The expected value of q is hence,

$$\langle q \rangle = \sum_{q'=0}^k q' P\left(q', k, \frac{1}{2} + \frac{f}{2}\right) = k \left(\frac{1}{2} + \frac{f}{2}\right) \quad (3)$$

and using the completely analogous argument for the $c - k$ distractors, we have that

$$\langle d \rangle = (c - k) \left(\frac{1}{2} + \frac{f}{2}\right) \quad (4)$$

As for D and Q , they correspond to errors over the k keys and $c - k$ distractors,

respectively, with the probability $(1 - f)/2$ in the Bernoulli trials, and their expected values are

$$\langle D \rangle = \sum_{k'}^k k' P \left(k', k, \frac{1}{2} - \frac{f}{2} \right) = \frac{k(1 - f)}{2} \quad (5)$$

and

$$\langle Q \rangle = \sum_{k'}^{c-k} k' P \left(k', k, \frac{1}{2} - \frac{f}{2} \right) = \frac{(c - k)(1 - f)}{2} \quad (6)$$

respectively.

Now since eq. (1) is linear in all parameters, the expected score of a test-taker of ability f is obtained by merely substituting the expectation values above into the equation. We then have, after some rearrangement,

$$\langle S \rangle(f) = kfa + (c - k)fb + \frac{k(1 - f)}{2}(a - A) + \frac{(c - k)(1 - f)}{2}(b - B) \quad (7)$$

where the first two terms correspond to actual knowledge (they disappear as $f \rightarrow 0$), and the last two terms correspond to luck (they disappear as $f \rightarrow 1$). One may of course choose $a = A$ and $b = B$ to keep only the first two terms. In this case, luck does not impact the expected score, but the variance is not similarly affected, as seen next.

The variance in S is given by,¹

$$\sigma_S^2 = a^2\sigma_q^2 + b^2\sigma_d^2 + A^2\sigma_Q^2 + B^2\sigma_D^2 + aB(\sigma_q^2 + \sigma_D^2) + bA(\sigma_d^2 + \sigma_Q^2) \quad (8)$$

To see this, note that q and D , on the one hand, and d and Q on the other hand, are defined over disjunct sets of options: q and D are defined over the set of true keys, whereas d and Q are defined over the set of true distractors. Consequently, the covariances σ_{qd}^2 , σ_{QD}^2 , σ_{qQ}^2 and σ_{dD}^2 all vanish. Moreover, since for the same reason, $q + D = k$ and $d + Q = c - k$,² we have that

$$2\sigma_{qD}^2 = -\sigma_q^2 - \sigma_D^2 \quad (9)$$

$$2\sigma_{dQ}^2 = -\sigma_d^2 - \sigma_Q^2 \quad (10)$$

because the sums $q + D$ and $d + Q$ are without variance. Explicit expressions for the f -dependence of the variances involved can be obtained from the underlying binomial distributions, but are unnecessary to reproduce in print here.

¹ Given the explicit function for the variance, a natural question to ask is how to choose the coefficients b , A , B to minimize either the variance itself, or the relative uncertainty, $\sigma_S/\langle S \rangle$. However, the coefficients defined from this criterion will be f -dependent, which makes them impossible to specify *a priori* but have to be extracted *a posteriori* from the observed scores in the test-taking population. This pushes the analysis away from that which is appropriate for the small-scale classroom assessment. A compromise could be to derive coefficients for an arbitrary but fixed f , say $f = 1/2$, but that idea will not be pursued here.

² Note that there is no possibility to record “blanks”; allowing for a blank response to each option is the equivalent of turning the test item into c separate “true-or-false test items”.

3.2. Multiple true/false scoring

This linear scoring function is obtained by simply setting $A = B = 0$ and $b = a$ in eq. (1) which yields,

$$S_{\text{MTF}}(q, d, Q, D) = q + d \quad (11)$$

Guessing is not penalized and is, in fact, impossible to avoid (q and d are counted on disjunct sets of options and there are only two possibilities for each option), so the expected score follows immediately from the same substitutions in eq. (7),

$$\langle S_{\text{MTF}} \rangle(f) = \frac{c(1+f)}{2} \quad (12)$$

The variance follows analogously from eq. (8),

$$\sigma_{\text{MTF}}^2 = \sigma_q^2 + \sigma_d^2 \quad (13)$$

where the individual variances can be found from the binomial distributions.

We shall need in our comparisons later also the probability of attaining at least a given minimum score s , denoted by $\pi_{\text{MTF}}(s; f, c, k)$. This probability is given by

$$\pi_{\text{MTF}}(s; f, c, k) = \sum_{\substack{q'+d' \geq s \\ q', d' \geq 0}} P\left(q', k, \frac{1}{2} + \frac{f}{2}\right) P\left(d', c-k, \frac{1}{2} + \frac{f}{2}\right) \quad (14)$$

where P is the binomial distribution of eq. (2).

3.3. Ripkey scoring

In this non-linear scoring function (Ripkey, Case, & Swanson, 1996, the “pick N ” rule), points are awarded for each correctly identified key, but a penalty of a total zero score is incurred if more than k options are indicated as keys. From eq. (1), we obtain it by setting $A = B = b = 0$, and letting $S = 0$ if $q + Q > k$:

$$S_{\text{R}}(q, d, Q, D) = \begin{cases} q, & \text{if } q + Q \leq k \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

We shall consider two cases: k known and k unknown to the test-taker, corresponding to how this scoring function is implemented in Ripkey et al. (1996) and in Betts et al. (2021), respectively. To keep the notation simple, we will use a prime to distinguish between the two scoring functions when necessary.

3.3.1. Number of keys is known

We shall start by assuming that k is known to the test-taker beforehand. The rational test-taker will in this case indicate the q' keys of which she is certain, and then guess among the remaining $c - q' - d'$ options, where d' is the number of known distractors, until k options are marked as keys. Taking into account that for each key correctly guessed, the probability of guessing the next one correctly decreases, the probability

of correctly identifying q keys out of k is given by,

$$F'_R(q, c, k, f) = \sum_{q'=0}^q P(q', k, f) \sum_{d'=0}^{c-k} P(d', c-k, f) \mathcal{P}(q-q', k-q', \{p_{ij}\}) \quad (16)$$

where $\mathcal{P}(q, n, \{p_{ij}\})$ designates the n -trial success distribution of q , given success rates that vary as

$$p_{ij} = \frac{k - q' - i}{c - q' - d' - i - j} \quad (17)$$

when having chosen i keys successfully and j ones mistakenly among the unknown options. The first sum in eq. (16) corresponds to the probability that the test-taker recognizes (“knows”) each of the k keys, the second one to blind guessing among the remaining unknown options to provide an answer with exactly k keys indicated. We provide some details on the distribution function $\mathcal{P}$ in the Appendix.

Using this probability distribution, the expectation for q for the rational test-taker of ability f under this scoring rule becomes

$$\langle S'_R \rangle(f) = \sum_{q'=0}^k q' F'_R(q', c, k, f) \quad (18)$$

but does not reduce to a simple short-form expression in the elementary functions that would be worth reproducing here explicitly, contrary to the case of $\langle S_{\text{MTF}} \rangle$. The item score variance is a function of σ_q^2 alone, and may be found from eq. (16) in the same manner as eq. (41). Likewise, the probability of obtaining at least a given minimum score is obtained after obvious substitutions in eq. (42) further below.

3.3.2. Number of keys is unknown

We consider now the case where k is unknown to the test-taker.

With there being no bias of any one unknown option being either key or distractor, the binomial distribution provides the *a priori* probability of there being k' keys among the $c - q' - d'$ unknown options as

$$P\left(k', c - q' - d', \frac{1}{2}\right) = 2^{q'+d'-c} \binom{c - q' - d'}{k'} \quad (19)$$

and so the gross expected gain, from test-taker’s perspective, of picking one more key among the unknown options is

$$g(q', d', c, 1) = 2^{q'+d'-c} \sum_{k'=1}^{c-q'-d'} \binom{c - q' - d'}{k'} \frac{k'}{c - q' - d'} \quad (20)$$

or, more generally, when picking κ unknown keys,

$$g(q', d', c, \kappa) = \sum_{k'=\kappa}^{c-q'-d'} P(k', c - q' - d', 1/2) \sum_{m=1}^{\kappa} m \mathcal{P}(m, \kappa, \{p'_{ij}\}) \quad (21)$$

where the p'_{ij} are given by (note the difference with respect to eq. [17], reflecting the test-taker's state of knowledge),

$$p'_{ij} = \frac{k' - i}{c - q' - d' - i - j} \quad (22)$$

Similarly, the gross expected loss of the same action is

$$l(q', d', c, \kappa) = q' \left(1 - \sum_{k'=\kappa}^{c-q'-d'} P(k', c - q' - d', 1/2) \right) = q' \sum_{k'=0}^{\kappa-1} P(k', c - q' - d', 1/2) \quad (23)$$

Hence, guessing is advantageous if $g(q', d', c, 1) > l(q', d', c, 1) = q'/2^{c-q'-d'}$, which yields the inequality,

$$\sum_{k'=1}^{c-q'-d'} \binom{c-q'-d'}{k'} \frac{k'}{c-q'-d'} = 2^{c-q'-d'-1} > q' \quad (24)$$

Thus, we see that guessing is rational for quite a broad range of q' , being irrational only when there are few unknown options. This establishes the rationality of guessing at least once under “unknown Ripkey scoring” but the important question is how many keys to assign by guesswork.

Let us answer this last question by the following analysis, in which we consider the difference,

$$\begin{aligned} g(q', d', c, \kappa + 1) - l(q', d', c, \kappa + 1) - g(q', d', c, \kappa) + l(q', d', c, \kappa) = \\ \sum_{k'=\kappa+1}^{c-q'-d'} P(k', c - q' - d', 1/2) \sum_{m=1}^{\kappa+1} m \mathcal{P}(m, \kappa + 1, \{p'_{ij}\}) \\ - \sum_{k'=\kappa}^{c-q'-d'} P(k', c - q' - d', 1/2) \sum_{m=1}^{\kappa} m \mathcal{P}(m, \kappa, \{p'_{ij}\}) \\ - q' P(\kappa, c - q' - d', 1/2) \end{aligned} \quad (25)$$

which is the rate of change of the expected net gain as a function of the number of keys supposed among the unknown options. Setting this equation equal to zero, we may determine the optimal value of κ for any given combination of c, q' and d' . In what follows, we denote this value simply κ for the sake of notational brevity in the equations. In general, when $c = 6$ (our test case later), $\kappa \leq 4$ (with the maximum obtained for the state of maximum ignorance $q' = d' = 0$).

The expected score, from the test-maker's perspective, is

$$\langle S_R \rangle(f) = \sum_{q'=0}^k P(q', k, f) \sum_{d'=0}^{c-k} P(d', c - k, f) \Theta(k - q' - \kappa) \sum_{q''=0}^{\kappa} \mathcal{P}(q'', \kappa, \{p_{ij}\}) (q' + q'') \quad (26)$$

where Θ is Heaviside's step function (which is zero for negative arguments and unity

otherwise) which corrects for overassignment of keys. The variance is

$$\sigma_R^2(f) = \sum_{q'=0}^k P(q', k, f) \sum_{d'=0}^{c-k} P(d', c-k, f) \Theta(k-q'-\kappa) \sum_{q''=0}^{\kappa} \mathcal{P}(q-q', \kappa, \{p_{ij}\}) (q'+q'')^2 - [\langle S_R \rangle(f)]^2 \quad (27)$$

The cumulative probability function of a minimum score s is given by,

$$\pi_R(s; f, c, k) = \sum_{\substack{q'+q'' \geq s \\ k-q'-\kappa \geq 0 \\ q', q'', d' \geq 0}} P(q', k, f) P(d', c-k, f) \mathcal{P}(q'', \kappa, \{p_{ij}\}) \quad (28)$$

3.4. Plus/minus scoring

In this scoring function, $b = B = 0$ and $A = a$ in eq. (1). It corresponds to points awarded for correctly identified keys and equal points subtracted for incorrectly identified keys. Moreover, putative negative item scores are taken as zero which introduces a non-linearity for small f . In symbols, we may write this scoring function as,

$$S_{\pm}(q, d, Q, D) = \max\{q - Q, 0\} \quad (29)$$

Indicating n unknown options as keys leads to an expected gain that depends on the number of known keys, as any series of guesses can never lead to a loss greater than this number. In other words, if q' are the number of keys known, guessing on $n > q'$ keys leads to an expected gain, assuming unbiased priors. Hence, under this scoring rule, the rational test-taker will surely guess if there are more unknown options than known keys, and if risk-neutral she may guess as long as there are unknown options even if fewer than known keys. A lower bound for the expected score follows immediately from eq. (7), that is

$$\langle S_{\pm} \rangle(f) \geq kf \quad (30)$$

The equality holds as $f \rightarrow 1$, but in general the expected score will exceed kf on account of negative item scores being forbidden.

The exact analysis, which underlies all of the reported numerical comparisons, is nevertheless straightforward. Let q' keys be known beforehand and q'' the number of correct guesses so that $q = q' + q''$. If n guesses are made in total, then the number of distractors marked as keys is $Q = n - q''$.³ Then $q - Q = 2q'' + q' - n$, and the expected score is

$$\begin{aligned} \langle S_{\pm} \rangle(f) = & \sum_{q'=0}^k \sum_{d'=0}^{c-k} \sum_{q''=0}^{n(c, q', d')} P(q', k, f) P(d', c-k, f) \mathcal{P}(q'', n(c, q', d'), \{p_{ij}\}) \times \\ & \times \max\{0, 2q'' + q' - n(c, q', d')\} \end{aligned} \quad (31)$$

³There is a subtle but important distinction between the process we consider here and the analogous one in section 3.1. There, the process considered was a decision on each option of distractor or key. Here, the test-taker picks a random selection of presumed keys from the unknown options.

and the set of probabilities $\{p_{ij}\}$ is given by eq. (17).

We have highlighted through notation that n is not constant but depends to some extent on the total number of options known, $q' + d'$, d' being the number of known distractors. For instance, if there are no unknown options, there can be no guessing and so it follows that $n(c, k, c - k) = 0$. The variance is given by a similar formula,

$$\sigma_{\pm}^2 = \sum_{q'=0}^k \sum_{d'=0}^{c-k} \sum_{q''=0}^{n(c,q',d')} P(q', k, f) P(d', c - k, f) \mathcal{P}(q'', n(c, q', d'), \{p_{ij}\}) \times \\ \times \max\{0, 2q'' + q' - n(c, q', d')\}^2 - [\langle S_{\pm} \rangle(f)]^2 \quad (32)$$

and the probability $\pi_{\pm}(s; c, k, f)$ of obtaining at least s points is given by

$$\pi_{\pm}(s; c, k, f) = \sum_{\substack{2q'' + q' - n(c, q', d') \geq s \\ q'', q', d' \geq 0}} P(q', k, f) P(d', c - k, f) \mathcal{P}(q'', n(c, q', d'), \{p_{ij}\}) \quad (33)$$

Finally, we need to determine the value of $n(c, q', d')$ that a perfectly rational test-taker would find. As stated earlier, $n > q'$ is necessary to ensure an expected gain (for an unbiased prior). Moreover, if k were known to the test-taker, n would be bounded by $k - q'$ for obvious reasons. However, from the test-taker's perspective, being completely ignorant of the distribution of keys and distractors and hence assuming a uniform prior, it is clear that as the number of guesses increases, the expected gain decreases as the distribution narrows by the central limit theorem. Therefore, the most rational choice is

$$n(c, q', d') = \begin{cases} q' + 1, & 2q' + 1 \leq c - d' \\ 0, & \text{otherwise} \end{cases} \quad (34)$$

In other words, eschew guessing unless there are enough unknown options for $q' + 1$ guesses.

3.5. Subset scoring

In this scoring function, the item score increases with each correctly identified key, but indicating any distractor erroneously as a key results in zero total points for the item. The scoring function is obtained from eq. (1) by setting, for instance, $b = B = 0$, $A = q$, at all times, but since $S = 0$ whenever $Q > 0$, we may write,

$$S_S(q, d, Q, D) = \max\{q(1 - Q), 0\} \quad (35)$$

For the honest test-taker, who indicates only the keys of which she is certain and leaves the rest as distractors, the expected score is

$$\langle S_S \rangle(f) = kf \quad (36)$$

assuming an unbiased distribution of distractors and keys over the test items, in line with Laplace's principle of indifference. However, since the expected loss of a blind guess becomes smaller than the expected gain as f decreases, with the exception of

the completely knowledgeable test-taker (corresponding to $f = 1$), the rational one will guess and eq. (36) represents merely a limiting special case.

From the test-taker's perspective, the probability of a correct guess is independent of the number of options that the test-taker recognizes. Guessing is hence only rational when the test-taker knows at most one of the keys, in which case the *a priori* probability of a single correct guess is $1/2$; otherwise, or for more than one guess, the expected loss exceeds the expected gain. Like for Ripkey scoring, assuming q' keys are known and d' distractors, the frequency of correctly guessing a key is $(k - q')/(c - q' - d')$. The probability of marking q correct keys and no incorrect ones is hence

$$F_S(q = 1, c, k, f) = P(0, c, f) \frac{k}{c} \quad (37)$$

$$F_S(q = 2, c, k, f) = P(1, k, f) \sum_{d'=0}^{c-k} P(d', c - k, f) \frac{k - 1}{c - d' - 1} + P(2, k, f) \quad (38)$$

$$F_S(q > 2, c, k, f) = P(q, k, f) \quad (39)$$

As always, the expected score is

$$\langle S_S \rangle(f) = \sum_{q'=0}^k q' F_S(q', c, k, f) \quad (40)$$

and the variance follows as

$$\sigma_S^2 = \sum_{q'=0}^k (q')^2 F_S(q', c, k, f) - [\langle S_S \rangle(f)]^2 \quad (41)$$

The probability of obtaining s or more points is

$$\pi_S(s; f, c, k) = \sum_{q' \geq s} F_S(q', c, k, f) \quad (42)$$

with $\pi_S(0; f, c, k) = 1$.

3.6. Dichotomous scoring

For this scoring rule, which does not fit at all into the framework of eq. (1), points are awarded for the item only if all keys and all distractors are correctly identified. In symbols, we may write it as

$$S_D = \begin{cases} 1, & q + d = c \\ 0, & q + d \neq c \end{cases} \quad (43)$$

There is no penalty for guessing, so the exact formulae for the expectation and variance are particularly simple for this scoring rule, being those of the Bernoulli distribution

with expectation value

$$\langle S_D \rangle(f) = \left(\frac{1}{2} + \frac{f}{2} \right)^c \quad (44)$$

corresponding to the probability of c successful identifications, one for each option. Likewise, the variance in S_D is given by,

$$\sigma_D^2 = \left(\frac{1}{2} + \frac{f}{2} \right)^c \left[1 - \left(\frac{1}{2} + \frac{f}{2} \right)^c \right] \quad (45)$$

The probability $\pi_D(s; f, c, k)$ of obtaining at least s points is

$$\pi_D(s; f, c, k) = \begin{cases} (f+1)^c / 2^c, & s > 0 \\ 1, & s = 0 \end{cases} \quad (46)$$

and is discontinuous at $s = 0$.

4. Numerical comparisons

The expectation values on which we base all our numerical results may be interpreted through the central limit theorem as the average of an infinite population of test-takers on a single test item, or as that of a single test-taker confronted with an infinite number of equivalent test items. We shall compare the different scoring functions on the following number of different statistics.

The first one is the linearity between the observed score S and the latent ability f . Having a linear relation in this case is appealing because it ensures that the scores constitute a true interval scale, and makes comparisons between test subjects comparable at both ends of the ability distribution. Moreover, the extent of linearity could also be taken as a measure of the construct validity (since we want the score to reflect the proportion of the test-taker's domain knowledge).

The second one is the dispersion in test scores (for a given ability) expected from the scoring functions. We use the standard deviation normalized by either the expected score⁴ or by the scoring range R (the difference between the maximum and minimum points that can be theoretically awarded for a response to the test item). In both cases, this is a measure of precision related to, but distinct from, the reliability of the test (Mellenbergh, 1996) and should be as small as possible to enable consistently fair rankings of test-takers and reproducible measurements. The normalization by the expectation is suitable because all of the investigated scoring functions share a common minimum score of zero; it is necessary (as opposed to using the standard deviation directly) to account for the difference in range of allowed scores and hence to ensure that the computed precision increases with increasing number of test items. Since these two goals are also achieved when normalizing by the range, we include this measure also for completeness. Note that for an honest test-taker, all of the scoring functions simply represent a count of correct identifications and it is for this ratio scale that we compute the precision, not the underlying f -score (the agreement with which is a matter of validity).

⁴Normalizing the uncertainty by the value of the measurement is the standard way of computing the relative error in, for instance, analytical chemistry (Harris, 2010).

	Pearson's r
$\langle S_{\text{MTF}} \rangle$	perfect
$\langle S'_{\text{R}} \rangle$	0.995
$\langle S_{\text{R}} \rangle$	0.987
$\langle S_{\pm} \rangle$	0.941
$\langle S_{\text{S}} \rangle$	0.999
$\langle S_{\text{D}} \rangle$	0.931

Table 1. Extent of linearity with respect to latent trait f as measured by the Pearson correlation coefficient rounded to three decimals for the different scoring functions considered. The reported results are for $c = 6$ and $k = 3$. Note that $\langle S_{\text{MTF}} \rangle$ is perfectly linear under all conditions.

The third one is the reliability, computed as the proportion of the test score variance in the population that is given by the expectation of the score. While this, like the precision, is also related to the repeatability of the score, it has a different mathematical definition and yields as we shall see a different ranking among the scoring functions. It is not obvious which of the precision and reliability is to be prioritized, and this choice may depend on the purpose of the test score.

The fourth one is a measure of the discriminative power of the scoring function using the discriminatory index of Paper I (denoted as d there, but we use D here to avoid a namespace conflict). In other words, how highly and accurately resolved the ability f is reflected in the observed score. This measure reflects how evenly the different levels of ability are distributed among the possible scores.

The last one is the coefficient of validity introduced in Paper I. What this coefficient measures is closely related to the linearity above, but unlike the correlation coefficient, it is not population-independent; *i. e.* changing the underlying ability distribution, changes the value of this statistic. Because of this, it is of practical significance since the non-linear regimes need not necessarily affect the score if these fall outside the ability range tested.

4.1. Linearity between observed and latent score

Obviously, for $\langle S_{\text{MTF}} \rangle$, being perfectly linear in f , no numerical calculations are necessary. Likewise, for $\langle S_{\text{D}} \rangle$ it is clear that it is of power-law form in f , and hence highly non-linear. However, for a quantitative comparison of the non-linear scoring functions considered, the Pearson correlation coefficient was used as a measure of linearity between $\langle S \rangle$ and f . In these calculations, as well as all that follow, we have arbitrarily set $c = 6$ and $k = 3$. The numerical results are given in Table 1 and the following qualitative ordering from worst to best is evident

$$S_{\text{D}} < S_{\pm} < S_{\text{R}} < S'_{\text{R}} < S_{\text{S}} < S_{\text{MTF}}$$

However, apart perhaps from the S_{D} and $S_{\pm}$ scoring rules, these correlation coefficients are all too close to unity to be of any decisive importance when choosing a scoring function.⁵ We should look elsewhere to differentiate decisively between them.

⁵Recall that there is no uncertainty in the calculated expectation values, being as they are obtained from analytical closed-form expressions, and therefore the deviation of Pearson's r from unity is a measure solely of the deviation from linearity.

	Case I	Case II	Case III
$\sigma_{\text{MTF}}/\langle S_{\text{MTF}} \rangle$	0.408	0.193	0.063
$\sigma'_{\text{R}}/\langle S'_{\text{R}} \rangle$	0.447	0.198	0.064
$\sigma_{\text{R}}/\langle S_{\text{R}} \rangle$	∞	0.451	0.098
$\sigma_{\pm}/\langle S_{\pm} \rangle$	1.000	0.618	0.292
$\sigma_{\text{S}}/\langle S_{\text{S}} \rangle$	1.000	0.458	0.114
$\sigma_{\text{D}}/\langle S_{\text{D}} \rangle$	7.937	0.959	0.376

Table 2. Relative dispersions of the observed scores rounded to three decimal places for three different f -distributions. The reported results are for $c = 6$ and $k = 3$. The relative uncertainty of S_{R} for Case I diverges since eq. (25) prescribes four guesses in this case, leading to a vanishing score for this particular test item.

	Case I	Case II	Case III
$\sigma_{\text{MTF}}/R_{\text{MTF}}$	0.204	0.144	0.050
$\sigma'_{\text{R}}/R'_{\text{R}}$	0.224	0.155	0.054
$\sigma_{\text{R}}/R_{\text{R}}$	0.000	0.262	0.067
$\sigma_{\pm}/R_{\pm}$	0.167	0.270	0.131
$\sigma_{\text{S}}/R_{\text{S}}$	0.167	0.254	0.093
$\sigma_{\text{D}}/R_{\text{D}}$	0.124	0.272	0.105

Table 3. Dispersions normalized by the scoring range R of the observed scores rounded to three decimal places for three different f -distributions. The reported results are for $c = 6$ and $k = 3$.

4.2. Precision

The dispersion of the different scoring functions was computed under three conditions, all with $c = 6$ and $k = 3$. In the first condition (Case I), the perfectly homogeneous case with $f \equiv 0$ as a constant was considered, corresponding to a Borelian *Gedanken* experiment of an infinite number of “monkeys” taking the test.⁶ In the second case, the uniform distribution in $f \in [0, 1]$ was used. In the last one (Case III), a normal distribution with mean $\langle f \rangle = 0.6$ and standard deviation $\sigma_f = 0.1$ was used. In this way, a measure of the sensitivity of the results to the underlying ability distribution is obtained, and a connection with the results reported in Paper I is also made.

The results are summarized in Table 2, using the relative dispersion (standard deviation normalized by the expectation), and in Table 3, using the dispersion normalized by the scoring range R . From Table 2, we have the following ranking in terms of precision (after excluding Case I which is more of a curiosity):

$$S_{\text{D}} < S_{\pm} < S_{\text{S}} < S_{\text{R}} < S'_{\text{R}} \approx S_{\text{MTF}}$$

The two end points are the same as for the linearity comparison in the previous section, with only some slight reordering in the middle. The differences among the scoring functions are somewhat more pronounced this time, however, and favor the “multiple true/false” scoring function. From Table 3, we have instead,

$$S_{\text{D}} \approx S_{\pm} < S_{\text{R}} \approx S_{\text{S}} < S'_{\text{R}} < S_{\text{MTF}}$$

after amalgamating the rankings of Case II and III, which is more or less the same ranking as in Table 2.

⁶The analogy is not perfect, as these “monkeys” are rational agents that only guess when theoretically advantageous. In other words, they are able to read and follow the instructions.

	Case II	Case III
S_{MTF}	0.429	0.087
S'_R	0.422	0.084
S_R	0.480	0.061
$S_{\pm}$	0.306	0.058
S_S	0.490	0.096
S_D	0.369	0.054

Table 4. Reliability coefficients computed for rational, and thus guessing, test-takers for two different ability distributions (one broad and one narrow). Results are for test items with $c = 6$ and $k = 3$ and are rounded to three decimal places.

4.3. Reliability

Since the expected values of the observed scores by definition correspond to the “true score” in the sense of classical test theory (Klein & Cleary, 1969), it follows that for Case I (the case where f is strictly constant), the reliability computed according to the classical test theory prescription as the proportion of variance that is true variance vanishes completely. This example highlights one inadequacy of classical test theory as arguably the reliability is not strictly zero in this case; or at least not identical between all scoring functions.

For Cases II and III, the total score variance is given by

$$\Sigma_S^2 = \int_0^1 \sigma_S^2(f) \phi(f) df + \int_0^1 [\langle S \rangle(f)]^2 \phi(f) df - \left[\int_0^1 \langle S \rangle(f) \phi(f) df \right]^2 \quad (47)$$

where $\phi(f)$ is the probability density function for f in the test-taker population (normal distribution for Case III and uniform distribution for Case II; *vide supra*). The last two terms correspond to the “true score” variance; the first term to the “error variance”. The reliability coefficient is hence,

$$\rho = \frac{1}{\Sigma_S^2} \left(\int_0^1 [\langle S \rangle(f)]^2 \phi(f) df - \left[\int_0^1 \langle S \rangle(f) \phi(f) df \right]^2 \right) \quad (48)$$

The results are presented in Table 4.

The computed reliability coefficients are much smaller for Case III than for Case II, because the underlying ability distribution is much more narrow, leading to a smaller contribution of true score variance to the total variance.⁷ Overall, the reliability coefficients are very low, as expected from test-takers that use guessing to their maximum advantage, but it should be stressed that the computed coefficient is not in principle from a test-retest, but from an infinite number of parallel forms with non-identical items. An element of chance is hence introduced for all but the perfectly knowledgeable test-taker, and this is reflected in the computed reliability coefficient.

In terms of performance, the overall ranking differs somewhat between Case II,

$$S_{\pm} < S_D < S'_R < S_{\text{MTF}} < S_R < S_S$$

⁷As noted above, if the distribution is narrowed to the extent that f is constant, the reliability coefficient vanishes.

	Case II	Case III
S_{MTF}	0.297	0.236
S'_R	0.511	0.461
S_R	0.505	0.686
$S_{\pm}$	0.523	0.375
S_S	0.557	0.788
S_D	0.001	0.000

Table 5. Discriminative index D of the different scoring functions under two different f -distributions. See text for details.

and Case III,

$$S_D < S_{\pm} < S_R < S'_R < S_{\text{MTF}} < S_S$$

but, unarguably, the dichotomous scoring rule is the overall worst performer, whereas S_S is unequivocally the best. In passing, we note that the computed reliability coefficients are numerically fairly close to the ones computed in Paper I for the multiple-choice test item. There is hence little if any inherent advantage in the multiple-response test item as regards the reliability of the test.

4.4. Discriminatory power

Following the analysis of Paper I, we consider the discriminatory power of a test as the reflection of the average probability that a test-taker of ability f receives a score reflecting f , the average in question being taken over the f -distribution. Other indices of discrimination, like Ferguson's (1949), are not as informative since they take into account only the observed, and not the latent score, which we know by stipulation within the model.

In the present case, the discriminative index D is computed as,

$$D = \int_0^1 df \phi(f) \frac{\pi(f\delta p; f, c, k) - \pi(f\delta p + \Delta; f, c, k)}{\Delta} \quad (49)$$

where $\Delta = \delta p/10$ is an arbitrary sensitivity parameter, δp is the range of the scoring function and $\pi(s; f, c, k)$ denotes the probability that a test-taker of ability f attains a score of s or higher on the item with c options and k keys. As stated, D is computed as a weighted average over the distribution $\phi(f)$. Results are given in Table 5.

For all of the scoring functions except S_R and S_S , the discriminative power is reduced when the ability distribution is narrowed (Case III). This means that S_R and S_S are more discriminative in the region around $f \approx 0.6$ than they are on average over the entire interval $f \in [0, 1]$, while the opposite holds true for the other scoring functions. Notably, the dichotomous scoring function is virtually unable to discriminate at all for any f . Looking at Case II, when it comes to discrimination, the ranking obtained is

$$S_D < S_{\text{MTF}} < S_R < S'_R < S_{\pm} < S_S$$

For Case III, only some minor readjustments occur in the middle with respect to that of Case II.

	Case II	Case III
S_{MTF}	perfect	perfect
S'_R	0.990	0.999
S_R	0.964	0.984
$S_{\pm}$	0.951	0.991
S_S	0.995	0.999
S_D	0.923	0.997

Table 6. Validity coefficients computed for rational test-takers for $c = 6$ and $k = 3$ computed for two different ability distributions.

4.5. Validity

We use the coefficient of validity introduced in Paper I. To some extent this comparison is redundant with that of the correlation coefficients reported above, but we include it for completeness.

In short, given the variance in S , $\sigma_S^2(f)$, for test-takers of a fixed ability f – which is an error variance since it does not reflect variation in ability – we calculate its expectation as

$$\int_0^1 \sigma_S^2(f) \phi(f) df$$

that is, the first term of eq. (47). We then define the expected “Platonic error” as

$$\langle E \rangle(f) = \langle S \rangle(f) - \langle S \rangle(0) - f[\langle S \rangle(1) - \langle S \rangle(0)] \quad (50)$$

that is, the deviation of the expected score from that which is expected if linearity holds, and compute its variance

$$\sigma_E^2 = \int_0^1 [\langle E \rangle(f)]^2 \phi(f) df - \left[\int_0^1 \langle E \rangle(f) \phi(f) df \right]^2 \quad (51)$$

This error variance is due to variation of the error in the “Platonic true score” which measures the underlying ability directly on an interval scale.

With these two quantities, we now define the validity coefficient as

$$\rho = 1 - \frac{\sigma_E^2}{\sigma_E^2 + \int_0^1 \sigma_S^2(f) \phi(f) df} \quad (52)$$

that is, as the proportion of the error variance which is not due to variation in predicted scores from the underlying ability. The results are tabulated in Table 6.

The ranking of the scoring functions is more-or-less the same as that obtained from the linearity between observed and latent score for obvious reasons, but for this statistic, there is a sensitivity also to the underlying ability distribution. The restriction of range from the narrow distribution leads to improvements for all scoring functions where an improvement is possible, and for dichotomous scoring this improvement is particularly marked. Any continuous function will approach linearity as the interval is decreased, and if the test-takers differ but very little in ability, then all scoring functions become equally valid. For now we note that for Cases II and III, $S_{\pm}$ is the worst and second worst performer, respectively, among the partial credits scoring

functions (that is, excluding S_D). The significance that I attach to this will become clear in the Conclusion next.

5. Conclusion

In this paper, we have derived the statistical properties of a number of different scoring functions for the multiple-response test item, under the assumption of rational test-takers being tested on factual recall (that is, not reasoning or other higher-order cognitive functions) and used these in a number of numerical experiments to compare them with each other. Together with Paper I, this covers virtually the whole domain of selected-response type test items with, nonetheless, the notable exception of confidence scoring rules (Dirkzwager, 2003; Finetti, 1965; Shuford, Albert, & Massengill, 1966).

The only firm conclusion is that S_D is the worst scoring function of the ones investigated, it being the least discriminatory, the least linear (the least valid), the least reliable, and the least precise and it is difficult to envision any use for it at all. It is not as unambiguously clear, however, which scoring function investigated is the best. Whereas S_{MTF} is the only scoring function considered that, perfectly linear, provides a true interval score for its expectation while being the most precise, and the third most reliable, it comes in merely second-to-last for item-level discrimination. Another strong contender is S_S , it being the second most linear, the most reliable, the third most precise and the most discriminative. While the results are also mixed in the analysis of Betts et al. (2021), they nevertheless conclude that the plus/minus scoring rule, $S_{\pm}$, is preferable, while in our investigation this scoring function turned out to be the most unreliable, the least precise and the most nonlinear of all the scoring functions that assign partial credits (only dichotomous scoring is worse) with its one possible merit being its item-level discrimination. The conclusion of Betts et al. (2021) is hence at odds with ours, and the reader is well advised to reach his or her own conclusion after comparing methodologies.

Unlike S_S or S_R which also count only the correctly identified keys for points, $S_{\pm}$ does not discourage guessing as forcefully and this ultimately is the reason for its poor performance in our comparisons. Although we have not reported any results for a such a modification, if the plus/minus scoring rule would be changed so as not to disallow negative item scores, guessing would become less theoretically advantageous for the test-taker. Since guessing unavoidably leads to reduced precision and reliability of the score, such a simple change would improve the computed statistics of this scoring rule. It is easy to see that the good results of S_S in our comparisons is due to the severe penalty (unconditional zero item score) that it attaches to erroneous answers.

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Appendix A. On the probability distribution $\mathcal{P}$

When picking keys randomly from the unknown options, the trials cannot be considered independent; in other words, the probability of success on any given guess depends on the past number of successful picks. We let the function $\mathcal{P}(q, n, \{p_{ij}\})$ denote the probability that in n attempts, there are q successfully identified keys, with the probability of success p_{ij} in each attempt being a function of the prior number of successes, i , and failures, j . The set $\{p_{ij}\}$ must be specified according to the circumstances (governing test item parameters). For the results reported in the paper, $\mathcal{P}$ has been computed and tabulated numerically. This is an arduous task that involves summation over all of the relevant probability space, but can be readily automated with a computer.

While exact algebraic expressions exist and can be derived for all cases (although quite long for $n > 3$), a simple closed-form expression for $\mathcal{P}$ cannot be provided in general but some particular cases are

$$\mathcal{P}(q, n, \{p_{ij}\}) = 0, \quad q > n \tag{A1}$$

$$\mathcal{P}(n, n, \{p_{ij}\}) = \prod_{i=0}^{n-1} p_{i0} \quad (\text{A2})$$

and

$$\mathcal{P}(0, n, \{p_{ij}\}) = \prod_{j=0}^{n-1} (1 - p_{0j}) \quad (\text{A3})$$

For illustrative purposes, the closed-form algebraic expressions of $\mathcal{P}$ for $n = 1, 2, 3$ are provided below.

$$\begin{aligned} \mathcal{P}(0, 1, \{p_{ij}\}) &= (1 - p_{00}) \\ \mathcal{P}(1, 1, \{p_{ij}\}) &= p_{00} \\ \mathcal{P}(0, 2, \{p_{ij}\}) &= (1 - p_{00})(1 - p_{01}) \\ \mathcal{P}(1, 2, \{p_{ij}\}) &= p_{00}(1 - p_{10}) + (1 - p_{00})p_{01} \\ \mathcal{P}(2, 2, \{p_{ij}\}) &= p_{00}p_{10} \\ \mathcal{P}(0, 3, \{p_{ij}\}) &= (1 - p_{00})(1 - p_{01})(1 - p_{02}) \\ \mathcal{P}(1, 3, \{p_{ij}\}) &= p_{00}(1 - p_{10})(1 - p_{11}) + (1 - p_{00})p_{01}(1 - p_{11}) + (1 - p_{00})(1 - p_{01})p_{02} \\ \mathcal{P}(2, 3, \{p_{ij}\}) &= p_{00}p_{10}(1 - p_{20}) + p_{00}(1 - p_{10})p_{11} + (1 - p_{00})p_{01}p_{11} \\ \mathcal{P}(3, 3, \{p_{ij}\}) &= p_{00}p_{10}p_{20} \end{aligned} \quad (\text{A4})$$