₁ LATENT CLASS CLUSTER ANALYSIS

₂ ALI ÜNLÜ

₃ TECHNICAL UNIVERSITY OF MUNICH, MUNICH, GERMANY

February 12, 2019

Correspondence please address to:

Prof. Dr. Ali Ünlü
TUM School of Education
Technical University of Munich
Arcisstrasse 21
80333 Munich
Germany

aligalibuenlue@gmail.com
ORCID 0000-0002-6941-5074
www.meb.edu.tum.de

Abstract

This paper describes the technique of exploratory latent class cluster analysis. The classical analysis is a model-based statistical approach for identifying unobserved subgroups from observed categorical data and for classifying cases into the identified subgroups based on membership probabilities estimated directly from the statistical model.

In the first part on mathematical modeling of the paper, we introduce the data and the sampling distribution for the data as required in the analysis of latent classes, the fundamental model assumptions are reviewed, and the general unrestricted latent class model is presented. Classification of cases into the clusters using modal assignment is discussed. In the second part on inferential statistics of the paper, we briefly review the classical maximum likelihood methodology related to parameter estimation and model testing, and the information criteria AIC and SIC for model selection. In the third part on case study of the paper, the General Social Survey data are analyzed using the software Latent

21 GOLD[®]. We present the Latent GOLD[®] profile plot and tri plot
22 options for the graphical representation of the results. The Latent
23 GOLD[®] classification output illustrating the assignment of
24 respondents to the latent survey respondent types is also shown.

25 Key words: Latent class analysis, model-based clustering,
26 model-based classification, General Social Survey data, Latent
27 GOLD[®] software

1. Introduction

Latent class analysis (LCA) is a statistical approach for examining latent categorical variables (Andersen, 1982; Clogg, 1995; Dayton, 1998; Formann, 1984; Goodman, 1974a,b, 1978; Langeheine and Rost, 1988; Lazarsfeld and Henry, 1968; McCutcheon, 1987; Vermunt and Magidson, 2004). Applications of LCA are numerous (Hagenaars and McCutcheon, 2002; Rost and Langeheine, 1997); for example in medical research, when the accuracy (sensitivity and specificity) of a diagnostic test for screening patients for a disease and disease prevalence have to be estimated (for reviews, see Enøe et al., 2000; Hui and Zhou, 1998; Walter and Irwig, 1988).

The basic idea underlying the exploratory LCA approach to cluster analysis (Vermunt and Magidson, 2002) can be summarized as follows. The conditional probabilities of the item responses given the unobserved group memberships of a postulated latent class (LC) model (LCM) differ across unobserved subgroups, called the latent classes, clusters, or types. These subgroups form the categories of a discrete latent variable representing concepts such as attitude toward abortion, Alzheimer disease status, or knowledge state in Euclidean geometry. Having estimated these parameters and assessed the fit of the LCM (including the identification of the number of latent classes), the parameters can be compared across the subgroups to

show how they differ from each other. This enables to characterize the unobserved subgroups or the latent variable, based on what is only observable, the data. In a next step, as the classification part of LCA, cases can be classified into the identified clusters based upon membership probabilities estimated directly from the fitted LCM.

These steps will be worked out in detail for this paper. In doing so, we want to embed our elaborations into a general modeling scheme, shown in Figure 1. Figure 1 presents a schematic representation of the basic modeling process in the quantitative behavioral or psychological sciences and the essentials of what is common to the majority of modeling endeavors.

[Figure 1 about here]

Addressed are the aspects of data, mathematical modeling, and inferential statistics in the overall modeling process. An experiment yields the encoded data (top black arrow), which are first modeled mathematically, presupposing hypothetically known population parameters (bottom black arrow). After the model has been fully developed mathematically, the, in fact, empirically unknown parameters of the derived mathematical model and hence the model itself are estimated and fully specified, respectively, from the real data using inferential statistics (full-dotted red circle). The ultimate goal then is to manufacture a

67 theoretically validated and empirically fitted mathematical model that can
68 be utilized to eventually describe and draw conclusions about the
69 experiment or empirical phenomenon under consideration, from which the
70 whole process initiated (green arrow).

71 This paper is structured as follows. In Section 2, we motivate the
72 General Social Survey example test items and data, and the general
73 problem of exploratory latent class cluster analysis. In Section 3, the
74 mathematical modeling is discussed, including the data and sampling
75 distribution for the data, and leading to the formulation of the general
76 latent class model and classification problem. In Section 4, we outline the
77 inferential statistics required for parameter estimation and model testing,
78 and model selection criteria. The LCA analysis of the General Social
79 Survey data based on the software Latent GOLD® is presented in Section
80 5. Finally, we conclude with summarizing the basic idea and a schematic
81 overview of the overall technical procedure of exploratory latent class
82 cluster analysis in Section 6.

83 2. 1982 General Social Survey and the Problem

84 To demonstrate a real LCA application, we take as starting point part
85 of the empirical data from the 1982 General Social Survey (GSS). For
86 illustration purposes, we restrict ourselves to example GSS test items and

87 data, but for more details, refer to McCutcheon (1987).

88 Two indicators ascertain the respondents' opinions regarding the
89 purpose of surveys (the PURPOSE item) and how accurate the results of
90 surveys are (ACCURACY). Two additional indicators cover the evaluations
91 made by the interviewers of the respondents' levels of understanding of the
92 interview questions (UNDERSTANDING) and cooperation shown in
93 answering the questions (COOPERATION). The original item formulations
94 and corresponding response categories are summarized in Table 1.

[Table 1 about here]

95 The corresponding data set, for the subsample of sample size
96 $N = 1,202$ white respondents, analyzed in the paper is shown in Table 2.
97 Table 2 gives the contingency table representation of the GSS data (as
98 compared to the frequency table representation reported later in Table 3).

[Table 2 about here]

99 A remark is in order. Latent GOLD[®] (see Section 5) is commercial
100 software distributed by Statistical Innovations,
101 www.statisticalinnovations.com. The demo version of Latent GOLD[®] is
102 freely available and can be downloaded at
103 www.statisticalinnovations.com/latent-gold-5-1/, after registration. The
104 demo version only works with the sample data files downloaded with the

105 software. The data set that we analyze in this paper is also contained in the
106 SPSS system file gss82white.sav accompanying the Latent GOLD® demo
107 version. This demo version works with the sample data file gss82white.sav,
108 so the contents of this paper can be rerun by the reader, at no cost.

109 We consider the above four items of the GSS study. In our example,
110 the PURPOSE variable has the response categories “Good”, “Depends”,
111 and “Waste”. The ACCURACY variable has the categories “Mostly True”
112 and “Not True”. The UNDERSTANDING item has two categories “Good”
113 and “Fair, Poor”. The item COOPERATION has the categories
114 “Interested”, “Cooperative”, and “Impatient, Hostile”. Thus, we have two
115 dichotomous and two trichotomous example items.

116 Given the four items, what types of respondent attitudes toward
117 surveys may we expect in this example? As an initial and intuitive guess,
118 for instance there may be the type of “Ideal” respondents, who endorse
119 surveys, and who are assessed to have understood the interview questions,
120 in contrast to the type of “Skeptic” respondents, with a high understanding
121 level as well, but who do not endorse surveys. We see that these types are
122 hidden, not observable, since they group together people with similar latent
123 characteristics. Are these the only two respondent types that are possible,
124 or stated differently, how many clusters can prevail? For example, there
125 may be a third type of “Believers”, who positively respond toward surveys

126 in the interview, without having really understood the interview questions.

127 We can ask further pertinent questions. What is the share or

128 percentage each of those types may take in the population of interest?

129 According to Table 2, we may expect the “Ideal” attitude type toward

130 surveys to be the dominant one in the target population. How can we detect

131 and characterize such latent classes? How can we assign individuals of our

132 sample to those groups, meaning predict their attitudes toward surveys?

133 These questions can be addressed by and answered through the use of

134 the LCA technique. The goals of the exploratory LCA approach to cluster

135 analysis are:

- 136 1. Identify the unobservable (e.g., survey attitude) types, clusters, or latent
137 classes, which group together subjects or cases that share similar latent
138 characteristics (e.g., different attitudes toward surveys). In other words,
139 the goal of the analysis is to construct a latent typology (e.g., of survey
140 attitude types).

141 More specifically, this goal of the classical LCA approach can be

142 subdivided into the following three subtasks:

143 a. How many clusters are underlying the data?

144 b. What sizes or proportions do the latent classes occupy in the
145 population of reference?

146 c. What are the characteristics of the identified types; that is, what do
147 they mean, or how do we interpret them?

148 2. Classify subjects or cases of a sample into the (e.g., survey attitude)
149 types, clusters, or latent classes. That is, practically identify for each
150 sample unit the unobserved subgroup it may belong or be assigned to.

151 **3. Mathematical Modeling: Formalization of the Data, General** 152 **Latent Class Model, and Classification by Modal Assignment**

153 In this section, we describe a formal representation of the data and the
154 naturally resulting multinomial sampling distribution for the data. The
155 basic model assumptions underlying LCA and the general unrestricted LC
156 model are introduced. The classification of cases into the clusters using the
157 modal assignment rule is recapitulated.

158 To simplify presentation, we will consider the LC model for three
159 manifest variables or indicators, I_1 , I_2 , and I_3 , which are assumed to be
160 dichotomous. All of the discussion can easily be extended to more than
161 three or polytomous indicators, with two or more than two answer
162 categories.

3.1. Data and Sampling Distribution for the Data

Specifying, and mathematically formalizing, what the data are is not a trivial task sometimes. We do this in our context, where the data arise as follows. We consider $N \in \mathbb{N}$ (set of natural numbers with 0) subjects randomly sampled from a population of reference, who respond to any of the three indicators I_1 , I_2 , and I_3 . The indicators are assumed to be dichotomously scored; score 1 for a positive response (e.g., solve or agree), and score 2 for a negative response (e.g., fail or disagree). In our data set (Table 3), for example, the dichotomous variable ACCURACY is scored 1 for “Mostly True” and 2 for “Not True”.

Let $\mathbf{I} = (I_1, I_2, I_3)$ denote the random vector obtained by viewing I_j for $j = 1, 2, 3$ as random variables that assume the scores 1 or 2. A realization $\mathbf{i} = (i_1, i_2, i_3)$ of $\mathbf{I}$ is called a response pattern. The set of all response patterns is the Cartesian product $\{1, 2\}^3 = \{1, 2\} \times \{1, 2\} \times \{1, 2\}$, which we write as $\mathcal{R}$.

The data are represented by the observed absolute counts $N(\mathbf{i}) \in \mathbb{N}$ of the response patterns $\mathbf{i} \in \mathcal{R}$. This can be summarized in a frequency table shown in Table 3, which lists all observed response patterns along with their absolute frequencies for our example data set.

[Table 3 about here]

182 Having mathematically defined what the data are, namely the pair
 183 $(\mathbf{i} = (i_1, i_2, i_3) \in \mathcal{R}, N(\mathbf{i}) \in \mathbb{N})$ of, both, a combination of the indicators’
 184 answer categories and the absolute count of the combination, the sampling
 185 distribution for the data can be derived.

186 We assume that the subjects give their response patterns independent
 187 of each other. The true probability of occurrence $\rho(\mathbf{i}) > 0$ of any response
 188 pattern $\mathbf{i} \in \mathcal{R}$ is assumed to stay constant across the subjects. This is an
 189 “iid” (independent and identically distributed) replication of the same
 190 random experiment, with at least two outcomes, repeated sample size N
 191 many times, and as a result, it leads to the binomial or multinomial
 192 distribution, for two or more possible outcomes, respectively.

193 More precisely, the data $\mathbf{x} = \{N(\mathbf{i})\}_{\mathbf{i} \in \mathcal{R}}$ are the realization of a random
 194 vector $\mathbf{X} = \{X_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{R}}$, which is multinomially distributed over $\mathcal{R}$, and has
 195 the following formula (for the probability of the data):

$$\begin{aligned}
 196 \quad \mathbb{P}(\mathbf{X} = \mathbf{x}) &= \mathbb{P}(X_{(2,2,2)} = N((2, 2, 2)), \dots, X_{(1,1,1)} = N((1, 1, 1))) \\
 197 \quad &= \frac{N!}{\underbrace{\prod_{\mathbf{i} \in \mathcal{R}} N(\mathbf{i})!}_{\text{multinomial coefficient}}} \prod_{\mathbf{i} \in \mathcal{R}} \rho(\mathbf{i})^{N(\mathbf{i})}.
 \end{aligned}$$

198 Here, $\rho(\mathbf{i}) > 0$ for any $\mathbf{i} \in \mathcal{R}$ with $\sum_{\mathbf{i} \in \mathcal{R}} \rho(\mathbf{i}) = 1$, and $0 \leq N(\mathbf{i}) \leq N$ for any
 199 $\mathbf{i} \in \mathcal{R}$ with $\sum_{\mathbf{i} \in \mathcal{R}} N(\mathbf{i}) = N$.

200 It is interesting to note that the multinomial distribution of the data
 201 arises naturally, without any reference to some psychometric or latent

202 variable model. That is, this data distribution holds “of its own volition”.

203 It is universally presumed in latent variable modeling.

204 3.2. Model Assumptions

205 Three Assumptions A1, A2, and A3 are pertinent to the classical LCA
206 approach, which we summarize next.

207 A1. We assume that the population of reference can be partitioned into
208 $T \in \mathbb{N}$ mutually exclusive and exhaustive subpopulations $K_1, K_2, \dots, K_T$,
209 also called types, clusters, or classes, with unknown proportions
210 $p(K_t) > 0$ for $1 \leq t \leq T$. Then, in particular, any element of the
211 population belongs to exactly one of these subpopulations, and we have
212 the natural restriction

$$213 \sum_{t=1}^T p(K_t) = 1.$$

214 Since these types, clusters, or classes generally group together cases (e.g.,
215 persons) who share similar unobservable characteristics (e.g., values),
216 they are called latent. In other words, we assume a latent typology
217 underlying the population of reference. Figure 2 gives a pictorial
218 representation of this assumption for four latent classes.

[Figure 2 about here]

219 Technically, the latent classes can be viewed as the realizations of a

latent random variable X , and denoted by $X = K_t$, $t = 1, \dots, T$.

In the GSS data example, a latent typology may consist of the survey attitude types of, for instance, “Ideal” and “Skeptic” respondents ($T = 2$), and we have $X = K_1$ and $X = K_2$, respectively.

A2. Conditional probabilities are the key quantities that provide the link between the observable level and latent level in latent variable models. Consider the following comparison of observable versus latent components:

Observable	Latent
$\mathbf{i}$	K_t
$\rho(\mathbf{i})$	$p(K_t)$
$\mathbf{I}$	X

Going from the latent level to the observable involves use of the conditional probabilities $r(\mathbf{I} = \mathbf{i} \mid X = K_t)$ (r stands for “response”), whereas the transition from the observable to the latent level is accomplished by the conditional probabilities $\mathbb{P}(X = K_t \mid \mathbf{I} = \mathbf{i})$.

For the LCM, a special sort of latent variable model, we also do need conditional probabilities. We assume that, for any of the T latent classes, there is a set of conditional probabilities for each of the observed indicators. For example, for the latent class K_1 , we have the set of

237 conditional probabilities (for notation, see below)

238 $\{r(I_1 = 1|X = K_1), r(I_1 = 2|X = K_1)\}$ for I_1 , and

239 $\{r(I_2 = 1|X = K_1), r(I_2 = 2|X = K_1)\}$ for I_2 .

240 We also assume that the latent classes are uniquely characterized by
 241 these conditional probabilities, both intra and inter classes. Here, “intra
 242 classes” means that any class is fully specified by its conditional
 243 probabilities. That is, there are no characteristics other than the
 244 conditional probabilities of the class that could be of relevance. “Inter
 245 classes” means that the latent classes can be uniquely distinguished from
 246 one another, solely based on their corresponding sets of conditional
 247 probabilities. That is, any two classes cannot have exactly the same
 248 conditional probabilities, across the items; there must be one item such
 249 that the conditional probabilities for one category of this item differ for
 250 the two latent classes.

251 Thus, we have conditional probabilities

$$252 \quad 0 \leq r(I_l = i_l|X = K_t) \leq 1$$

253 for any $1 \leq l \leq 3$, $i_l \in \{1, 2\}$, and $1 \leq t \leq T$, and it holds the following

254 second natural restriction (beside $\sum_{t=1}^T p(K_t) = 1$)

$$255 \quad \sum_{i_l=1}^2 r(I_l = i_l|X = K_t) = 1$$

256 for any $1 \leq l \leq 3$ and $1 \leq t \leq T$.

257 In the GSS data example, we can have the conditional probability for
 258 an “Ideal” respondent (type “Ideal”) to express a belief that the results
 259 of surveys are “Mostly True” (indicator ACCURACY).

260 A3. How can we compute such joint conditional probabilities as

261 $r(I_1 = 1, I_2 = 2, I_3 = 1|X = K_1)$ or $r(I_1 = 2, I_2 = 1, I_3 = 2|X = K_2)$?

262 The following third assumption allows to express these joint probabilities
 263 by means of the item-wise probabilities introduced in Assumption A2.

264 The assumption states that if the latent variable is held fixed, the
 265 relationships between the item responses are random and not anymore
 266 systematic.

267 Thus, we assume that, within any of the T latent classes, the observed
 268 scores to the indicators are independent. That is,

$$269 \quad r(I_1 = i_1, I_2 = i_2, I_3 = i_3|X = K_t) = \prod_{l=1}^3 r(I_l = i_l|X = K_t)$$

270 for any $i_l \in \{1, 2\}$, $1 \leq l \leq 3$, and $1 \leq t \leq T$.

271 This is the assumption of local independence, which is fundamental in
 272 classical LCA.

273 Two remarks are in order. First, the assumption of local independence
 274 can also be viewed as a parameter reduction. In the example with the

275 three dichotomous indicators, there are seven independent joint
 276 conditional probabilities $r(I_1 = i_1, I_2 = i_2, I_3 = i_3 | X = K_t)$ for any latent
 277 class K_t , $t = 1, \dots, T$. Under the assumption of local independence, this
 278 is reduced to three independent individual conditional probabilities
 279 $r(I_l = i_l | X = K_t)$ for any latent class K_t , $t = 1, \dots, T$. Second, the
 280 assumption of local independence is restrictive, from an empirical
 281 viewpoint. This assumption requires that the latent class does not
 282 change, that is, remains the same, when answering all items of the test.
 283 This precludes transfer (e.g., learning) effects that may empirically occur
 284 (and may alter the latent class) during testing.

285 3.3. The General Unrestricted Latent Class Model

286 Assumptions A1, A2, and A3 suffice to define the general unrestricted
 287 LC model. The general unrestricted (T -class) latent class model is a
 288 multinomial probability model (cf. Bishop et al., 2007), parametrizing the
 289 multinomial cell probabilities $\rho(\mathbf{i})$ for $\mathbf{i} = (i_1, i_2, i_3) \in \mathcal{R}$ (see Section 3.1) as
 290 follows:

$$\begin{aligned}
 291 \quad \rho(\mathbf{i}) &\stackrel{\text{A1}}{=} \sum_{t=1}^T \left\{ p(K_t) r(I_1 = i_1, I_2 = i_2, I_3 = i_3 | X = K_t) \right\} \\
 292 \quad &\stackrel{\text{A2, A3}}{=} \sum_{t=1}^T \left\{ p(K_t) \prod_{l=1}^3 r(I_l = i_l | X = K_t) \right\}.
 \end{aligned}$$

293 The LCM with T latent classes for three dichotomous items contains

$$294 \quad \underbrace{(T-1)}_{\text{independent class parameters (natural restriction)}} + \underbrace{T}_{\text{per class}} \cdot \underbrace{3}_{\text{per item}} \cdot \underbrace{(2-1)}_{\text{independent item parameter (natural restriction)}}$$

295 independent model parameters that need to be estimated from the data.

296 These parameters are:

$$\begin{aligned} 297 \quad & p(K_1), p(K_2), \dots, p(K_{T-1}), \\ 298 \quad & r(I_1 = 1|X = K_1), r(I_2 = 1|X = K_1), r(I_3 = 1|X = K_1), \\ 299 \quad & r(I_1 = 1|X = K_2), r(I_2 = 1|X = K_2), r(I_3 = 1|X = K_2), \\ 300 \quad & \vdots \\ 301 \quad & r(I_1 = 1|X = K_T), r(I_2 = 1|X = K_T), r(I_3 = 1|X = K_T). \end{aligned}$$

302 In the sequel, we assume that all model parameters are summarized in

303 a parameter vector

$$304 \quad \theta = (p(K_1), \dots, p(K_T), r(I_1 = 1|X = K_1), \dots, r(I_3 = 2|X = K_T)),$$

305 which ranges over the parameter space $\Theta = (0, 1]^T \times [0, 1]^{6T}$. In particular,

$$306 \quad \rho(\mathbf{i}) \stackrel{\text{LCM}}{=} \sum_{t=1}^T \left\{ p(K_t) \prod_{l=1}^3 r(I_l = i_l | X = K_t) \right\} \equiv f(\theta)$$

307 depends on θ , and we therefore also write $\rho_\theta(\mathbf{i})$. Mathematically, we treat

308 (known) θ to be fixed, but arbitrary. Statistically, we need to estimate

309 (unknown) θ based on the data (see Section 4). According to the equation

310 $\rho_\theta(\mathbf{i}) = f(\theta)$, each θ specifies its own LCM, and we can identify both a
 311 LCM and its parameter vector θ , and simply speak of the “LCM θ ”. So,
 312 what is a LCM? A LCM is a parameter vector $\theta \in \Theta$, which is linked to the
 313 sampling distribution of the data via $\rho_\theta(\mathbf{i}) = f(\theta)$. This procedure of
 314 parametrizing the multinomial cell probabilities through a function of the
 315 model parameters is not specific to LCA. In latent variable models in
 316 general, $\rho(\mathbf{i})$ is assumed to be a function $\Psi(\eta)$ of model parameters η , and
 317 it is the choices of Ψ and η that define the respective psychometric model.

318 *3.4. Classification of Cases by Modal Assignment*

319 How can we assign individuals to the unobservable classes? What
 320 parameters can we use for this purpose? Note that, first and foremost,
 321 $r(\mathbf{I} = \mathbf{i} | X = K_t)$ are theoretical parameters. Their direct practical value is
 322 limited because the conditioning is on the latent variable $X = K_t$, which we
 323 cannot observe. In contrast, directly more useful especially for classification
 324 are the parameters $\mathbb{P}(X = K_t | \mathbf{I} = \mathbf{i})$, in which the roles of the manifest and
 325 latent variables are interchanged, because in this case we condition on the
 326 observable variable $\mathbf{I} = \mathbf{i}$.

327 Beside the characterization of the latent classes and latent variable
 328 using the conditional probabilities postulated in Assumption A2, LC cluster
 329 analysis also allows for the assignment of cases to the latent classes based

upon posterior membership probabilities. For $1 \leq t \leq T$ and $\mathbf{i} = (i_1, i_2, i_3) \in \mathcal{R}$, the posterior membership probabilities (as opposed to the prior probabilities $p(K_t)$) are

$$\mathbb{P}(X = K_t | \mathbf{I} = \mathbf{i}) \stackrel{\text{Bayes' rule}}{=} \frac{p(K_t) r(\mathbf{I} = \mathbf{i} | X = K_t)}{\rho_\theta(\mathbf{i})} \stackrel{\text{A1, A2, A3, W}}{=} \frac{p(K_t) \prod_{l=1}^3 r(I_l = i_l | X = K_t)}{\sum_{t'=1}^T \left\{ p(K_{t'}) \prod_{l=1}^3 r(I_l = i_l | X = K_{t'}) \right\}}.$$

Note that the posterior membership probabilities $\mathbb{P}(X = K_t | \mathbf{I} = \mathbf{i})$ solely depend on the model parameters $p(K_t)$ and $r(I_l = i_l | X = K_t)$ of a LCM. In particular, these posterior probabilities can be directly estimated from the fitted statistical model. This is why we speak of model-based classification, as opposed to distance-based methods (e.g., Kaufman and Rousseeuw, 2005).

Having introduced the relevant probabilities, the most common classification rule is modal assignment. Assign all cases with a given response pattern $\mathbf{i} \in \mathcal{R}$ to a latent class K_{t_0} ($1 \leq t_0 \leq T$) for which the posterior membership probability $\mathbb{P}(X = K_{t_0} | \mathbf{I} = \mathbf{i})$ is largest. That is,

$$\mathbb{P}(X = K_{t_0} | \mathbf{I} = \mathbf{i}) = \max_{1 \leq t \leq T} \mathbb{P}(X = K_t | \mathbf{I} = \mathbf{i}).$$

An example may help to illustrate the modal assignment. Suppose we have the following posterior membership probabilities $\mathbb{P}(K_1 | \mathbf{i}) = 0.18$, $\mathbb{P}(K_2 | \mathbf{i}) = 0.27$, and $\mathbb{P}(K_3 | \mathbf{i}) = 0.55$ (these numbers must sum up to 1), for

349 $T = 3$ latent classes. According to modal assignment, all subjects of the
350 sample with the response pattern $\mathbf{i}$ are put into cluster K_3 , because this
351 cluster has the largest value (0.55).

352 We could also classify cases into clusters proportionally, with the
353 sampling weights for the clusters given by their posterior membership
354 probabilities. In the example, we would draw K_1 with probability 0.18, K_2
355 with probability 0.27, and K_3 with probability 0.55. That is, there is some
356 fractional chance specified by the posterior membership probability of
357 landing in each of the clusters.

358 In both cases, modal assignment and proportional assignment, we have
359 instances of model-based classification.

360 **4. Inferential Statistics: Parameter Estimation and Model Testing**

361 Reconsider Figure 1. So far, we have treated the mathematics with
362 presupposed fixed, but arbitrary parameters of the LCM. Now, we deal
363 with the statistics, the problem of specifying empirically plausible values for
364 the parameters based on the data. The goals are threefold. We discuss
365 parameter estimation by maximum likelihood, choosing plausible values for
366 the parameters; goodness-of-fit testing based on the deviance, whether a
367 particular choice of estimated parameters provides an adequate fit; and
368 model selection using the criteria AIC and SIC, among competing sets of

369 estimated parameters that do fit the data. Three classic references for the
 370 statistical techniques and results discussed in this section are Bishop et al.
 371 (2007), Cramér (1999), and Lehmann and Romano (2005), and for model
 372 selection by Burnham and Anderson (2002).

373 *4.1. Maximum Likelihood Estimation*

374 In Section 3.1, we have derived the multinomial formula for the
 375 probability $\mathbb{P}(\mathbf{X} = \mathbf{x})$ of the data $\mathbf{x} = \{N(\mathbf{i})\}_{\mathbf{i} \in \mathcal{R}}$. According to Section 3.3,
 376 we have modeled the multinomial cell probabilities $\rho(\mathbf{i}) = \rho_\theta(\mathbf{i})$. Thus, the
 377 probability of the data can be viewed as a function of the model parameters
 378 θ , with the data held fixed. Then, in maximum likelihood estimation, the
 379 parameter values are selected in such a way that with this choice of values
 380 the probability or likelihood of the given data is maximized. That is, we
 381 choose those values for the parameters that render the available data most
 382 probable.

383 More precisely, the kernel of the likelihood function (omitting the data
 384 dependent, constant multinomial coefficient), as a function of θ , for fixed
 385 data $\mathbf{x} = \{N(\mathbf{i})\}_{\mathbf{i} \in \mathcal{R}}$, is

$$386 \quad \mathcal{L}(\theta; \mathbf{x}) := \prod_{\mathbf{i} \in \mathcal{R}} \rho_\theta(\mathbf{i})^{N(\mathbf{i})} = \prod_{\mathbf{i} \in \mathcal{R}} \left\{ \sum_{t=1}^T \left\{ p(K_t) \prod_{l=1}^3 r(I_l = i_l | X = K_t) \right\} \right\}^{N(\mathbf{i})}.$$

387 This function $\mathcal{L}(\cdot; \mathbf{x})$ of θ , for constant data $\mathbf{x}$, must be globally maximized

388 over the parameter space Θ . This is called the maximum likelihood (ML)
389 estimation problem.

390 The maximum of the kernel of the likelihood function cannot, in
391 general, be obtained by analytical methods. For instance, setting equal to
392 zero the first-order partial derivatives of this function with respect to the
393 parameters results in a non-linear system of equations. Numerical
394 optimization methods are required. Popular iterative methods for solving
395 the ML estimation problem are Expectation-Maximization (EM) and
396 Newton-Raphson (NR) type algorithms (e.g., McLachlan and Krishnan,
397 2008). In this paper, we use the program Latent GOLD[®] to do ML
398 estimation. This program implements a hybrid EM-NR algorithm (for
399 details, see Vermunt and Magidson, 2016).

400 We want to note that one application of a local optimization routine
401 does not ensure the achievement of a global maximum. There may be
402 several local maxima. On the contrary, the local optimization routine must
403 be repeated multiple times, with different start values for the parameters,
404 until a global maximum has been achieved.

405 In the sequel, let $\hat{\theta} \in \Theta$ be the ML estimate of the parameter vector θ ,
406 that is,

407
$$\mathcal{L}(\hat{\theta}; \mathbf{x}) = \max_{\theta \in \Theta} \mathcal{L}(\theta; \mathbf{x}).$$

408 Viewed as a random (variable) vector, $\hat{\theta}$ is also called the ML estimator for
 409 θ .

410 4.2. Goodness-of-Fit Testing

411 The goodness-of-fit of an estimated LCM can be tested based on the
 412 log-likelihood ratio statistic (deviance)

$$413 \quad L^2(\theta; \mathbf{x}) := 2 \sum_{\mathbf{i} \in \mathcal{R}} \left\{ N(\mathbf{i}) \ln \left(\frac{N(\mathbf{i})}{N \rho_{\theta}(\mathbf{i})} \right) \right\}.$$

414 We test the null hypothesis that some LCM θ can be used to explain the
 415 data, that is, $\rho(\mathbf{i}) = \rho_{\theta}(\mathbf{i})$ for all $\mathbf{i} \in \mathcal{R}$, against the alternative hypothesis
 416 that no LCM fits the data. For this test problem, we hope for
 417 non-significance, that is, we do not want to reject the null hypothesis
 418 (albeit the type II error can be high and is not controlled). If the model is
 419 correct (i.e., under the null hypothesis), given certain regularity conditions
 420 are satisfied (Birch, 1964; Bishop et al., 2007; Read and Cressie, 1988), the
 421 random variable $L^2(\hat{\theta}; \mathbf{x})$, with $\hat{\theta}$ the ML estimator, is asymptotically (as
 422 $N \rightarrow \infty$) chi-squared with the number df of degrees of freedom
 423 $df = (|\mathcal{R}| - 1) - \{(T - 1) + T \cdot [3 \cdot (2 - 1)]\}$. That is, the number of
 424 degrees of freedom of the asymptotic chi-square distribution equals the
 425 number of possible response patterns minus one (i.e., number of degrees of
 426 freedom in the data), $|\mathcal{R}| - 1$, minus the number of independent model

427 parameters that were estimated, $(T - 1) + T \cdot [3 \cdot (2 - 1)]$.

428 The name of the statistic $L^2(\hat{\theta}; \mathbf{x})$ stems from the fact that it can also
 429 be written in the form of, for large N ($\stackrel{d}{\approx}$ means “approximately distributed
 430 as” and χ^2 is the chi-square distribution),

$$431 \quad L^2(\hat{\theta}; \mathbf{x}) = -2 \ln \left(\frac{\prod_{\mathbf{i} \in \mathcal{R}} \rho_{\hat{\theta}}(\mathbf{i})^{N(\mathbf{i})}}{\prod_{\mathbf{i} \in \mathcal{R}} \left(\frac{N(\mathbf{i})}{N} \right)^{N(\mathbf{i})}} \right) \stackrel{d}{\approx} \chi_{df}^2.$$

432 The log of the ratio of two (globally) maximized likelihoods is computed, in
 433 the numerator for the LCM with the ML estimate $\hat{\theta}$, and in the
 434 denominator for the saturated general multinomial distribution with the
 435 ML estimates $\widehat{\rho}(\mathbf{i}) = N(\mathbf{i})/N$ for $\mathbf{i} \in \mathcal{R}$.

436 4.3. Model Selection

437 It is not valid to compare LC models with different numbers of latent
 438 classes by general likelihood ratio tests, which would mean restricting
 439 parameters (class proportions) to boundary values (to be zero). This would
 440 violate a regularity condition, for the true parameter vector must be an
 441 interior point of the parameter space (e.g., Birch, 1964; Bishop et al., 2007).

442 Therefore, we select among competing LC models using the Akaike
 443 information criterion (AIC; Akaike, 1973), with the ML estimate $\hat{\theta}$,

$$444 \quad \text{AIC} := -2 \ln \left(\mathcal{L}(\hat{\theta}; \mathbf{x}) \right) + 2 \cdot \text{number of independent model parameters},$$

445 and the Schwarz information criterion (SIC; Schwarz, 1978)

446 $\text{SIC} := -2 \ln \left(\mathcal{L}(\hat{\theta}; \mathbf{x}) \right) + \ln(N) \cdot \text{number of independent model parameters.}$

447 We select a LC model with the smallest value of AIC or SIC.

448 For further reading, a comprehensive analysis and comparison of model
449 selection criteria for LCA are discussed in Lin and Dayton (1997), Lin
450 (2006), and Lin (2015).

451 5. Application of Latent GOLD® to GSS Data

452 We analyze the 1982 GSS example data set described in Section 2
453 using the Windows-based software Latent GOLD® for LC cluster analysis.
454 Latent GOLD® is commercial software and can be obtained from Statistical
455 Innovations at www.statisticalinnovations.com. They are the makers of
456 Latent GOLD®. There is a freely available demo version of this software.
457 The GSS data set used in the analyses of this section is also distributed as
458 the SPSS system file `gss82white.sav` with the demo version.

459 The first thing to know is how you can get your data into Latent
460 GOLD®. Via the menu, choose “File > Open...”, which brings up the
461 dialog box shown in Figure 3 and where we select our data set, the file
462 `gss82white.sav`. If necessary, change to the appropriate directory of the
463 data file that you want to open.

[Figure 3 about here]

464 After having loaded the data into Latent GOLD[®], via the menu,
465 choose “Model > Cluster”. This brings up the analysis dialog box for LC
466 cluster analysis displayed in Figure 4. You can also right or double click
467 “Model1” and select “Cluster” to open up the analysis dialog box.

[Figure 4 about here]

468 In Figure 5, we select the variables for the analysis. For this, we use
469 the mouse to highlight the four variables in the left-hand side variable box,
470 and then press the “Indicators” button to place the variables in the
471 right-hand side indicator box. To scan the data file, click on the “Scan”
472 button. Now, the variables or data can be inspected. For example, double
473 clicking the PURPOSE variable allows to view its categories, labels, scores,
474 and counts. We have a case weight variable “frq” in our SPSS data file,
475 which automatically appears in the “Case Weight” box in Latent GOLD[®].

[Figure 5 about here]

476 For the general LCM, we change the scale type of each variable from
477 default “Ord-Fixed” (ordinal) to “Nominal”, by right click on each variable
478 (or all variables simultaneously if jointly highlighted) in the indicator box.
479 This is shown in Figure 6.

[Figure 6 about here]

480 Latent GOLD[®] allows to estimate LC models with different numbers
481 of clusters in one run. Simply specify the number of latent classes to be
482 estimated in the “Clusters” box, shown in Figure 7. In our example, “1-4”
483 clusters are typed, that is, we estimate a 1-class LCM, a 2-class LCM, a
484 3-class LCM, and a 4-class LCM, all in one go. To start the estimation
485 process, we have to click the “Estimate” button at the bottom of the dialog
486 box.

[Figure 7 about here]

487 After the models have been estimated, click on the data file name
488 gss82white.sav on the left-hand side of the display (Outline Pane) to obtain
489 a summary of all models fitted on that data on the right-hand side of the
490 display (Contents Pane). Right click in the Contents Pane to access the
491 Model Summary Display, containing further statistics that can be displayed
492 if selected. Your screen should then look like in Figure 8.

[Figure 8 about here]

493 We can use the output shown in Figure 8 to determine the number of
494 clusters. The complete independence model (“Model1”) and the 2-class
495 model (“Model2”) are rejected right away. They yield larger BIC (SIC) and
496 AIC values or have highly significant p -values (cf. Figure 16). “Model3”
497 (three clusters) and “Model4” (four clusters) are the competing LC models.

498 We apply the heuristic rule to select among the two (cf. Figure 16).
 499 “Model3” and “Model4” have non-significant p -values of 0.11 and 0.58,
 500 respectively. That is, we cannot reject these models, for example, at an α
 501 level of 0.05. “Model3” has the minimum number of LCs ($T = 3$ versus 4)
 502 such that non-significance holds and is the one that is more parsimonious
 503 ($N_{\text{par}} = 20$ versus 27). Using this heuristic decision rule, “Model3”, the
 504 3-class LC model, is the favored model. In the sequel, we will use this model
 505 to exemplify the software. Nonetheless, “Model4” could also be considered,
 506 but for illustration purposes we will restrict our attention to “Model3”.

507 We can view the estimated model parameters of the 3-class LC model,
 508 see Figure 9. In the Outline Pane, click the expand icon “+” next to
 509 “Model3” and highlight the list element “Profile”. Now, the estimated
 510 model parameters are displayed in the Contents Pane. You can also click
 511 the expand icon next to “Profile” to see the subordinate entry “Prf-Plot” in
 512 the Outline Pane in Figure 9.

[Figure 9 about here]

513 We observe that the estimated cluster sizes are $p(K_1) = 0.6168$,
 514 $p(K_2) = 0.2039$, and $p(K_3) = 0.1793$ for “Cluster1”, “Cluster2”, and
 515 “Cluster3”, respectively. The conditional item probabilities
 516 $r(I_l = i_l \mid X = K_t)$, that is, the numbers between the two red arrows, can

517 be used to characterize or interpret, and distinguish the unobserved
518 clusters. We can see that latent class K_1 is the type of “Ideal” respondents,
519 who endorse surveys (PURPOSE = “Good”, 0.8905; ACCURACY =
520 “Mostly True”, 0.6148; COOPERATION = “Interested”, 0.9452), and who
521 are assessed to have understood the interview questions
522 (UNDERSTANDING = “Good”, 0.9957). In contrast, latent class K_3
523 constitutes the type of “Skeptic” respondents, with a high understanding
524 level (UNDERSTANDING = “Good”, 0.7532), but who do not endorse
525 surveys (PURPOSE = “Waste”, 0.6188; ACCURACY = “Not True”,
526 0.9574; COOPERATION = “Impatient, Hostile”, 0.1009). Latent class K_2
527 is the type of “Believers”, who positively respond toward surveys in the
528 interview (PURPOSE = “Good”, 0.9157; ACCURACY = “Mostly True”,
529 0.6528), without having really understood the interview questions
530 (UNDERSTANDING = “Fair, Poor”, 0.6757).

531 The conditional item probabilities $r(I_l = i_l \mid X = K_t)$ can be
532 represented graphically using a profile plot. This is shown in Figure 10.

[Figure 10 about here]

533 In Figure 10, highlight “Prf-Plot” in the Outline Pane to produce the
534 profile plot in the Contents Pane. To invoke the profile plot Control Panel
535 “Prf-Plot Control” right click on the plot or select “View > Plot Control”

in the menu. The Control Panel allows to customize the profile plot. For example, select the variable ACCURACY in the “Variables” box to see the checked categories of this variable. By default, for dichotomous variables only the last category is displayed (see the ACCURACY variable with its category “Not True”). This can be changed in the Control Panel (see the dichotomous variable UNDERSTANDING). Each latent class is represented by a zigzag line (we have three lines). The conditional item probabilities are placed on the y -axis. The variables with their categories are vertically put on the x -axis. Any of the clusters (e.g., “Cluster3”) can be selected by clicking on the symbol next to the cluster name at the bottom of the plot, and the corresponding profile in the plot is then highlighted in red.

In Figure 11, under “ProbMeans”, we have the reversed conditional probabilities $\mathbb{P}(X = K_t | I_l = i_l)$ for landing in cluster K_t given the indicator category $I_l = i_l$. Click the expand icon “+” next to “ProbMeans” to see the available plot options.

[Figure 11 about here]

For example, given the category “Good” of the PURPOSE indicator, the probabilities for being in the clusters “Ideal”, “Believers”, and “Skeptic” are 0.7185, 0.2443, and 0.0371, respectively. That is, there is a high probability of landing in the cluster “Ideal”, if “Good” is scored on the

PURPOSE indicator. Or, given $\text{PURPOSE} = \text{"waste"}$, the probabilities for the three clusters are 0.2359, 0.0185, and 0.7455, respectively, with the highest probability obtained for the cluster "Skeptic".

These reversed conditional class probabilities $\mathbb{P}(X = K_t | I_t = i_t)$ can be visualized using a tri plot, a barycentric coordinate display. The tri plot for the 3-class LC model is shown in Figure 12.

[Figure 12 about here]

Select "Tri-Plot" in the Outline Pane to obtain the tri plot in the Contents Pane. To view "Tri-Plot Control", the Control Panel for customizing the tri plot, right click on the Contents Pane or tri plot, or select "View > Plot Control" from the menu. For example, we have disabled the variable ACCURACY in the Control Panel, with the effect that this variable is not shown anymore in the tri plot. The edges of the triangle stand for the three clusters. The filled blue triangle delineates the cluster sizes, $p(K_1) = 0.6168$, $p(K_2) = 0.2039$, and $p(K_3) = 0.1793$. We have highlighted the category "Good" of the variable PURPOSE. In the status bar at the bottom of the window, we see that the highest conditional probability $\mathbb{P}(X = K_t | \text{PURPOSE} = \text{"Good"})$ is with "Cluster1" (0.7185), followed by "Cluster2" (0.2443), and the smallest probability is obtained for "Cluster3" (0.0371). These numbers can also be seen in barycentric

574 coordinates of the tri plot.

575 We can use Latent GOLD[®] to classify cases into the clusters by modal
576 assignment. For this, we have to re-estimate our 3-class LC model.
577 However this time, we do request the classification output in the software
578 settings, see Figure 13.

[Figure 13 about here]

579 Call the analysis dialog box by double clicking “Model3” in the Outline
580 Pane. Then click on the sixth tab “Output”. To request the classification
581 by modal assignment results, mark the entry “Classification - Posterior” of
582 the box “Output Sections” and click on “Estimate”.

583 The same 3-class LC model is fitted and appears as the new “Model5”
584 in the Outline Pane. Under “Model5”, click on “Classification” on the
585 left-hand side to obtain the classification output on the right-hand side.
586 The Latent GOLD[®] screen should now look like in Figure 14.

[Figure 14 about here]

587 Two examples are emphasized, the first and last rows of the output.
588 The first row shows that the 419 interviewees with the response pattern
589 (PURPOSE = “Good”, ACCURACY = “Mostly True”,
590 UNDERSTANDING = “Good”, COOPERATION = “Interested”) are
591 classified into the first cluster “Cluster1” / “Ideal”, because the posterior

592 probability for this cluster is the largest among the three (0.9196). Under
593 “Modal”, there is the value 1 to denote this result. In the last row, we have
594 8 interviewees who have the response pattern (PURPOSE = “Waste”,
595 ACCURACY = “Not True”, UNDERSTANDING = “Fair, Poor”,
596 COOPERATION = “Impatient, Hostile”) and are assigned to the third
597 cluster “Cluster3” / “Skeptic”. This cluster has the highest probability
598 (0.9868). Thus we see the value 3 under “Modal”.

599 The classification output can be appended to the data file, see Figure
600 15.

[Figure 15 about here]

601 Double click on “Model3” to open up the analysis dialog box. Select
602 “Classification - Posterior” on the fifth tab “ClassPred”, specify under what
603 name and where to store your file, and click on “Estimate”. The new file is
604 saved and can be found at the specified location.

605 Via the menu “File > Save Results...” we can save the output or
606 results for a highlighted model. The settings of a selected model can also be
607 saved and then opened for later use. For this, select the model that you
608 want to save in the Outline Pane, and go to “File > Save Definition...”.
609 The save dialog box opens. Name the file (“*.lgf”, for Latent GOLD® file)
610 and specify its location, and click on “Save”. Now you can re-open your

611 saved “*.lgf” definition file, without having to re-open the data file
612 separately. Simply go to “File > Open...”. The open dialog box appears.
613 Select the desired file and click on “Open”.

614 **6. Conclusion**

615 In this paper, we have discussed the technique of exploratory latent
616 class cluster analysis. This technique pursues two major goals, clustering
617 followed by classification, both in model-based variants.

618 As the end point of the paper, we recap the basic idea underlying
619 exploratory latent class cluster analysis and we give a schematic overview of
620 the complete procedure.

621 The basic idea can be summarized as follows. The conditional
622 probabilities of a postulated LC model differ across unobserved subgroups
623 (latent classes, clusters, or types). These subgroups form the categories of a
624 categorical latent variable. Having estimated these parameters, assessed the
625 fit of the LC model, and selected the proper number of latent classes, those
626 parameters can be compared between the subgroups to show how the
627 clusters differ from each other. This enables characterizing the unobserved
628 subgroups and hence the nature of the categorical latent variable
629 (model-based clustering). Eventually, cases of the sample can be classified
630 into the latent types based upon posterior membership probabilities

631 estimated directly from the LC model (model-based classification).

632 To sum up the entire presentation, an overall flow diagram of the
633 analysis process of exploratory latent class cluster analysis is depicted in
634 Figure 16.

[Figure 16 about here]

References

- Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle. In Petrov, B. N. and Csaki, F., editors, *Second International Symposium on Information Theory*, pages 267–281, Budapest. Akademiai Kiado.
- Andersen, E. B. (1982). Latent structure analysis: A survey. *Scandinavian Journal of Statistics*, 9:1–12.
- Birch, M. W. (1964). A new proof of the Pearson-Fisher theorem. *Annals of Mathematical Statistics*, 35:818–824. doi:10.1214/aoms/1177703581.
- Bishop, Y. M., Fienberg, S. E., and Holland, P. W. (2007). *Discrete Multivariate Analysis*. Springer, New York. doi:10.1007/978-0-387-72806-3.
- Burnham, K. P. and Anderson, D. R. (2002). *Model Selection and Multimodel Inference: A Practical Information-Theoretic Approach*. Springer, New York. doi:10.1007/b97636.
- Clogg, C. C. (1995). Latent class models. In Arminger, G., Clogg, C. C., and Sobel, M. E., editors, *Handbook of Statistical Modeling for the Social and Behavioral Sciences*, pages 311–359. Plenum Press, New York. doi:10.1007/978-1-4899-1292-3_6.

- 654 Cramér, H. (1999). *Mathematical Methods of Statistics*. Princeton
655 University Press, Princeton, NJ.
- 656 Dayton, C. M. (1998). *Latent Class Scaling Analysis*. Sage Publications,
657 Thousand Oaks. doi:10.4135/9781412984720.
- 658 Enøe, C., Georgiadis, M. P., and Johnson, W. O. (2000). Estimation of
659 sensitivity and specificity of diagnostic tests and disease prevalence when
660 the true disease state is unknown. *Preventive Veterinary Medicine*,
661 45:61–81. doi:10.1016/S0167-5877(00)00117-3.
- 662 Formann, A. K. (1984). *Die Latent-Class-Analyse [Latent Class Analysis]*.
663 Beltz, Weinheim.
- 664 Goodman, L. A. (1974a). The analysis of systems of qualitative variables
665 when some of the variables are unobservable. Part I: A modified latent
666 structure approach. *American Journal of Sociology*, 79:1179–1259.
667 doi:10.1086/225676.
- 668 Goodman, L. A. (1974b). Exploratory latent structure analysis using both
669 identifiable and unidentifiable models. *Biometrika*, 61:215–231.
670 doi:10.1093/biomet/61.2.215.
- 671 Goodman, L. A. (1978). *Analysing Qualitative/Categorical Variables:*

- 672 *Loglinear Models and Latent Structure Analysis*. Cambridge University
673 Press, Cambridge.
- 674 Hagenaars, J. A. and McCutcheon, A. L., editors (2002). *Applied Latent*
675 *Class Analysis*. Cambridge University Press, Cambridge.
676 doi:10.1017/CBO9780511499531.
- 677 Hui, S. L. and Zhou, X. H. (1998). Evaluation of diagnostic tests without
678 gold standards. *Statistical Methods in Medical Research*, 7:354–370.
679 doi:10.1177/096228029800700404.
- 680 Kaufman, L. and Rousseeuw, P. J. (2005). *Finding Groups in Data: An*
681 *Introduction to Cluster Analysis*. Wiley, Hoboken, NJ.
682 doi:10.1002/9780470316801.
- 683 Langeheine, R. and Rost, J., editors (1988). *Latent Trait and Latent Class*
684 *Models*. Springer, Boston. doi:10.1007/978-1-4757-5644-9.
- 685 Lazarsfeld, P. F. and Henry, N. W. (1968). *Latent Structure Analysis*.
686 Houghton Mifflin, Boston.
- 687 Lehmann, E. L. and Romano, J. P. (2005). *Testing Statistical Hypotheses*.
688 Springer, New York. doi:10.1007/0-387-27605-X.
- 689 Lin, T. H. (2006). A comparison of model selection indices for nested latent

- 690 class models. *Monte Carlo Methods and Applications*, 12:239–259.
691 doi:10.1515/156939606778705164.
- 692 Lin, T. H. (2015). Model selection information criteria in latent class
693 models with missing data and contingency question. *Communications in*
694 *Statistics - Simulation and Computation*, 44:319–331.
695 doi:10.1080/03610918.2013.777454.
- 696 Lin, T. H. and Dayton, C. M. (1997). Model selection information criteria
697 for non-nested latent class models. *Journal of Educational and*
698 *Behavioral Statistics*, 22:249–264. doi:10.2307/1165284.
- 699 McCutcheon, A. L. (1987). *Latent Class Analysis*. Sage Publications,
700 Newbury Park. doi:10.4135/9781412984713.
- 701 McLachlan, G. J. and Krishnan, T. (2008). *The EM Algorithm and*
702 *Extensions*. Wiley, New York. doi:10.1002/9780470191613.
- 703 Read, T. R. C. and Cressie, N. A. C. (1988). *Goodness-of-Fit Statistics for*
704 *Discrete Multivariate Data*. Springer, New York.
705 doi:10.1007/978-1-4612-4578-0.
- 706 Rost, J. and Langeheine, R., editors (1997). *Applications of Latent Trait*
707 *and Latent Class Models in the Social Sciences*. Waxmann, New York.

- 708 Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of*
709 *Statistics*, 6:461–464. doi:10.1214/aos/1176344136.
- 710 Vermunt, J. K. and Magidson, J. (2002). Latent class cluster analysis. In
711 Hagnaars, J. A. and McCutcheon, A. L., editors, *Applied Latent Class*
712 *Analysis*, pages 89–106. Cambridge University Press, Cambridge.
713 doi:10.1017/CBO9780511499531.004.
- 714 Vermunt, J. K. and Magidson, J. (2004). Latent class analysis. In
715 Lewis-Beck, M. S., Bryman, A., and Liao, T. F., editors, *The Sage*
716 *Encyclopedia of Social Science Research Methods*, pages 549–553. Sage
717 Publications, Thousand Oaks. doi:10.4135/9781412950589.n472.
- 718 Vermunt, J. K. and Magidson, J. (2016). *Technical Guide for Latent GOLD*
719 *5.1: Basic, Advanced, and Syntax*. Statistical Innovations Inc., Belmont,
720 MA. <https://www.statisticalinnovations.com/user-guides/>.
- 721 Walter, S. D. and Irwig, M. (1988). Estimation of test error rates, disease
722 prevalence and relative risk from misclassified data: A review. *Journal of*
723 *Clinical Epidemiology*, 41:923–937. doi:10.1016/0895-4356(88)90110-2.

Tables

Table 1

The four 1982 GSS questions (McCutcheon, 1987, p. 29) considered in the example application. The PURPOSE and ACCURACY items are directly responded to by the interviewees. The items COOPERATION and UNDERSTANDING are responded to by the interviewers and cover assessments of the interviewers about the interviewees.

PURPOSE: In general, do you feel that surveys usually serve a good purpose, or do you feel that they are usually a waste of time and money? (Good purpose/ Depends/ Waste of time and money)

ACCURACY: How often do you think that you can trust the results of surveys? Do you think they are almost always right, right most of the time, only some of the time, or hardly ever right? (Almost always, Most of the time/ Some of the time, Hardly ever)

COOPERATION: In general, what was the respondent's attitude toward the interview? (Friendly and interested/ Cooperative but not particularly interested/ Impatient and restless/ Hostile)

UNDERSTANDING: Was the respondent's understanding of the questions good, fair, or poor? (Good/ Fair/ Poor)

Note. In the example application, the first and second response categories of the item ACCURACY correspond to “Mostly True” and “Not

734 True”, respectively. For the COOPERATION item, the categories
735 “Impatient and restless” and “Hostile” are merged into one category in the
736 example. For the UNDERSTANDING item, in the application the
737 categories “Fair” and “Poor” are joined.

738 Table 2

739 The 1982 GSS data in contingency table representation (McCutcheon,
740 1987, p. 30). The sample size is $N = 1,202$.

741

			COOPERATION		
			Interested	Cooperative	Impatient, Hostile
PURPOSE	ACCURACY	UNDERSTANDING			
Good	Mostly True	Good	419	35	2
		Fair, Poor	71	25	5
	Not True	Good	270	25	4
Depends		Fair, Poor	42	16	5
	Mostly True	Good	23	4	1
		Fair, Poor	6	2	0
Waste	Not True	Good	43	9	2
		Fair, Poor	9	3	2
	Mostly True	Good	26	3	0
		Fair, Poor	1	2	0
	Not True	Good	85	23	6
		Fair, Poor	13	12	8

742 Table 3

743 Frequency table of the coded 1982 GSS data.

744

	purpose	accuracy	understa	cooperat	frq
1	1	1	1	1	419
2	1	1	1	2	35
3	1	1	1	3	2
4	1	1	2	1	71
5	1	1	2	2	25
6	1	1	2	3	5
7	1	2	1	1	270
8	1	2	1	2	25
9	1	2	1	3	4
10	1	2	2	1	42
11	1	2	2	2	16
12	1	2	2	3	5
13	2	1	1	1	23
14	2	1	1	2	4
15	2	1	1	3	1
16	2	1	2	1	6
17	2	1	2	2	2
18	2	2	1	1	43
19	2	2	1	2	9
20	2	2	1	3	2
21	2	2	2	1	9
22	2	2	2	2	3
23	2	2	2	3	2
24	3	1	1	1	26
25	3	1	1	2	3
26	3	1	2	1	1
27	3	1	2	2	2
28	3	2	1	1	85
29	3	2	1	2	23
30	3	2	1	3	6
31	3	2	2	1	13
32	3	2	2	2	12
33	3	2	2	3	8

Figure Captions

745

746 *Fig. 1.* Schematic representation of the simplified modeling process in
747 the quantitative behavioral or psychological sciences. The contents of the
748 paper are structured according to this scheme.

749 *Fig. 2.* Population partitioned into four subpopulations representing
750 the latent types, clusters, or classes K_1 , K_2 , K_3 , and K_4 . Each class is not
751 empty, any two classes have no element in common (mutual exclusiveness),
752 and the union of all classes is the entire population (exhaustiveness). In
753 other words, every element of the population belongs to one, and exactly
754 one, of the four classes.

755 *Fig. 3.* “File > Open...” dialog box. The red arrows show the File
756 Open shortcut, the selected data file, and the Open button, respectively.

757 *Fig. 4.* “Model > Cluster” dialog box for LC cluster analysis.

758 *Fig. 5.* Select the indicators and automatically set the case weight for
759 LC cluster analysis, then scan the data, and explore information about a
760 variable (e.g., the PURPOSE variable).

761 *Fig. 6.* Pop-up menu for variable scale type. Right click a variable in
762 the indicator box to change its scale type. For the general LC cluster
763 analysis model, we set the scale type of all indicators to “Nominal”.

764 *Fig. 7.* LC cluster analysis dialog box, with specified numbers of

765 “Clusters” $T = 1, 2, 3, 4$. To estimate these four LC models, the “Estimate”
 766 button must be clicked.

767 *Fig. 8.* Model summary output for the fitted 1-class, 2-class, 3-class,
 768 and 4-class LC models. The “Model Summary Display” is also shown,
 769 where additional statistics can be selected. “LL” stands for the
 770 log-likelihood $\ln(\mathcal{L}(\hat{\theta}; \mathbf{x}))$; “BIC(LL)” and “AIC(LL)” are the Schwarz and
 771 Akaike information criteria SIC and AIC, respectively; “Npar” stands for
 772 the number of independent model parameters; “L²” is the deviance
 773 $L^2(\hat{\theta}; \mathbf{x})$; and “p-value” is the significance probability $p = \mathbb{P}(\chi_{df}^2 > L^2(\hat{\theta}; \mathbf{x}))$.
 774 “Model3” is the best model according to the heuristic model selection rule
 775 (cf. Figure 16).

776 *Fig. 9.* Profile output for the 3-class LC model showing the cluster
 777 sizes $p(K_t)$, for $t = 1, 2, 3$, and conditional item probabilities
 778 $r(I_l = i_l \mid X = K_t)$, for $t = 1, 2, 3$, $l = 1, 2, 3, 4$, $i_1, i_4 \in \{1, 2, 3\}$, and
 779 $i_2, i_3 \in \{1, 2\}$. According to the latter parameters, we can interpret
 780 “Cluster1” as the “Ideal” respondent type, “Cluster2” are the “Believers”,
 781 and “Cluster3” is the “Skeptic” type.

782 *Fig. 10.* Profile plot for the 3-class LC model. On the x -axis are the
 783 indicators with their response categories, $I_l = i_l$. On the y -axis the
 784 conditional item probabilities $r(I_l = i_l \mid X = K_t)$ are put, with each of the
 785 three latent classes being represented by a zigzag line. “Cluster3”, the

786 “Skeptic” respondent type, is selected and highlighted in red. The profile
 787 plot Control Panel is displayed, for customizing the legend, clusters,
 788 variables, and categories of the plot.

789 *Fig. 11.* Output of indicator-score specific posterior membership
 790 probabilities $\mathbb{P}(X = K_t \mid I_l = i_l)$. The rows sum up to one. The categories
 791 “Good” and “Waste” of the PURPOSE variable are good indicators for the
 792 “Ideal” (0.7185) and “Skeptic” (0.7455) respondent types, respectively.

793 *Fig. 12.* Tri plot for the 3-class LC model. The axes represent the
 794 three clusters. The base axis is “Cluster1”, the right axis is “Cluster2”, and
 795 the left axis is “Cluster3”. Each indicator category $I_l = i_l$ is shown as a
 796 point in the triangle, with the corresponding coordinates
 797 $P = (\mathbb{P}(X = K_1 \mid I_l = i_l), \mathbb{P}(X = K_2 \mid I_l = i_l), \mathbb{P}(X = K_3 \mid I_l = i_l))$. The
 798 indicator-score PURPOSE = “Good” is selected and the corresponding
 799 coordinates are delineated. The filled blue triangle has the cluster sizes as
 800 the coordinates. In the Control Panel “Tri-Plot Control”, the variable
 801 ACCURACY has been disabled and is not shown in the tri plot.

802 *Fig. 13.* Output tab of the analysis dialog box. The analysis dialog
 803 box appears after double clicking “Model3”. Choose the tab “Output”,
 804 activate “Classification - Posterior”, and click on “Estimate”.

805 *Fig. 14.* Classification by modal assignment output for the 3-class LC
 806 model. The cases in the first and last rows of the output, with completely

807 positive and negative categories, are put into the first (“Modal” = 1) and
808 third (“Modal” = 3) clusters, respectively. For, “Cluster1” and “Cluster3”
809 are most likely, with highest probabilities 0.9196 and 0.9868, respectively.

810 *Fig. 15.* Append the classification output to the data file by choosing
811 “Classification - Posterior” on the “ClassPred” tab of the analysis dialog
812 box, which is retrieved by double clicking “Model3” in the Outline Pane.
813 You can specify the name for the file and where to store it. The file is
814 generated when the “Estimate” button is pressed.

815 *Fig. 16.* Flow chart representation of the analysis process of
816 exploratory latent class cluster analysis. Abbreviations: LC(s) = latent
817 class(es), LCM = latent class model, ML = maximum likelihood, df =
818 degrees of freedom, AIC = Akaike information criterion, and SIC (BIC) =
819 Schwarz (Bayesian) information criterion.

820

Figures

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

LatentGOLD

File Edit View Model Window Help

gss82white.sav

- Model1 - $L^2 = 257,2604$
- Model2 - $L^2 = 79,5079$
- Model3 - $L^2 = 22,0872$
- Model4 - $L^2 = 6,6142$
- Model5 - $L^2 = 22,0872$
- Parameters
- Profile
- ProbMeans
- Bivariate Residuals
- Classification
- EstimatedValues-Model
- Model6

	purpose	accuracy	understa	cooperat	ObsFreq	Modal	Cluster1	Cluster2	Cluster3
	good	mostly true	good	interested	419,0000	1	0,9196	0,0787	0,0017
	good	mostly true	good	cooperative	35,0000	1	0,6378	0,3540	0,0081
	good	mostly true	good	impatient/hostile	2,0000	2	0,0150	0,9439	0,0411
	good	mostly true	fair/poor	interested	71,0000	2	0,0235	0,9732	0,0033
	good	mostly true	fair/poor	cooperative	25,0000	2	0,0037	0,9927	0,0036
	good	mostly true	fair/poor	impatient/hostile	5,0000	2	0,0000	0,9932	0,0068
	good	not true	good	interested	270,0000	1	0,8778	0,0638	0,0584
	good	not true	good	cooperative	25,0000	1	0,5185	0,2444	0,2371
	good	not true	good	impatient/hostile	4,0000	3	0,0066	0,3503	0,6431
	good	not true	fair/poor	interested	42,0000	2	0,0243	0,8529	0,1228
	good	not true	fair/poor	cooperative	16,0000	2	0,0038	0,8643	0,1319
	good	not true	fair/poor	impatient/hostile	5,0000	2	0,0000	0,7759	0,2240
	depends	mostly true	good	interested	23,0000	1	0,8651	0,0969	0,0380
	depends	mostly true	good	cooperative	4,0000	1	0,4929	0,3581	0,1490
	depends	mostly true	good	impatient/hostile	1,0000	2	0,0068	0,5558	0,4374
	depends	mostly true	fair/poor	interested	6,0000	2	0,0171	0,9258	0,0571
	depends	mostly true	fair/poor	cooperative	2,0000	2	0,0027	0,9361	0,0612
	depends	not true	good	interested	43,0000	3	0,3746	0,0356	0,5898
	depends	not true	good	cooperative	9,0000	3	0,0804	0,0496	0,8701
	depends	not true	good	impatient/hostile	2,0000	3	0,0004	0,0292	0,9704
	depends	not true	fair/poor	interested	9,0000	3	0,0060	0,2758	0,7183
	depends	not true	fair/poor	cooperative	3,0000	3	0,0009	0,2656	0,7335
	depends	not true	fair/poor	impatient/hostile	2,0000	3	0,0000	0,1607	0,8393
	waste	mostly true	good	interested	26,0000	1	0,8829	0,0177	0,0994
	waste	mostly true	good	cooperative	3,0000	1	0,5252	0,0682	0,4067
	waste	mostly true	fair/poor	interested	1,0000	2	0,0519	0,5029	0,4452
	waste	mostly true	fair/poor	cooperative	2,0000	2	0,0081	0,5117	0,4801
	waste	not true	good	interested	85,0000	3	0,1979	0,0034	0,7987
	waste	not true	good	cooperative	23,0000	3	0,0347	0,0038	0,9615
	waste	not true	good	impatient/hostile	6,0000	3	0,0002	0,0021	0,9977
	waste	not true	fair/poor	interested	13,0000	3	0,0032	0,0260	0,9708
	waste	not true	fair/poor	cooperative	12,0000	3	0,0005	0,0246	0,9749
	waste	not true	fair/poor	impatient/hostile	8,0000	3	0,0000	0,0132	0,9868

Fig. 15

Fig. 16