Nonsymbolic Estimation of Ratios and Differences with Noisy Feedback

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Abstract

Grace et al. (2018) showed that observers quickly learned to respond based on differences and ratios of stimulus magnitudes in a nonsymbolic task without explicit instruction. But were observers really responding based on differences and ratios, suggesting the perceptual system effectively computes these operations? As a stronger test of relational versus exemplar responding, we studied a version of this task with noisy feedback. In three experiments, observers viewed pairs of stimuli that varied in area, brightness, or length, and responded by clicking on a horizontal bar. For different groups, feedback was based on the ratio or difference of stimulus magnitudes, linearly scaled to the response bar, plus a Gaussian noise component that was constant for repetitions of each pair. Multiple regressions showed that for every observer across experiments, coefficients for the added noise were negative when entered in a model with trained values, and were statistically significant in 76% of individual cases. These results indicate that observers responded based on the differences or ratios of stimulus magnitudes, and effectively ignored or suppressed the added noise. Regressions also showed that differences and ratios described the data well, with the untrained relation accounting for significant variance in approximately half of individual cases, similar to Grace et al.'s (2018) results. Overall, these results suggest that the perceptual system automatically computes two operations, corresponding to differences and ratios, when comparing stimulus magnitudes.

How we compare stimulus magnitudes that vary along a quantitative dimension, such as brightness, length or area, has been an enduring question for psychophysics and experimental psychology. Theoretical accounts have typically assumed that observers are able to judge either differences or ratios between perceived magnitudes (Birnbaum, 1978, 1980; Dzhafarov & Colonius, 1999, 2005, 2006; Luce, 2002; Stevens 1961, 1975), a debate which has persisted since the earliest theorizing about psychophysics (Fechner, 1860; Plateau, 1872; see Murray, 1993, and Scheerer, 1987, for review). However, progress has been hindered because ratios and differences can be transformed into one another by simple mathematical functions (i.e., logarithmic, exponential), so alternative models can be difficult, if not impossible, to distinguish empirically (Torgerson, 1961). A similar mathematical indeterminacy has contributed to the well-known Fechner-Stevens controversy about the psychophysical law (Krueger, 1989; Shepard, 1966).

A potential resolution to this debate is that the perceptual system computes both differences and ratios. This possibility is suggested by results of Grace et al. (2018), who trained observers to estimate differences and ratios of stimulus magnitudes in a novel non-symbolic task. In their study, observers viewed a pair of stimuli that varied in brightness, area, or numbers of dots and clicked along a horizontal response bar below the stimuli. Feedback was provided based on the ratio or difference of the nominal stimulus values, linearly mapped along the response bar. They found that observers responded accurately to both differences and ratios after little training. Moreover, analyses showed that responding depended on both differences and ratios even though only one operation was trained, suggesting the perceptual system automatically computed both operations. They concluded that evidence for two operations suggested that the perceptual system might represent an

algebraic field, a mathematical structure that generalizes the intuitions underlying arithmetic over the real numbers. A possible implication is that such a representation might support ‘computationally complex behavior’, that is, adaptive behavior such as spatial navigation that appears to require the equivalent of algebraic computation by the organism and is found in many nonhuman species, including insects (Grace et al., 2019).

But does the perceptual system really compute differences and ratios, or is some other operation or process responsible for performance in Grace et al.’s (2018) task? An important caveat to consider is the ‘tools-to-theories’ heuristic proposed by Gigerenzer (1991; see also Gigerenzer & Murray, 1987), in which tools or methods that are used to describe cognitive phenomena tend to evolve into explanatory theories. A prime example is how Neyman-Pearson’s method for statistical inference led to signal detection theory, with both accounts based on fundamentally the same mathematical model (Gigerenzer & Murray, 1987). According to this view, we should be cautious to infer that the perceptual system is computing differences and ratios simply because we use differences and ratios to design stimuli for psychophysical experiments. Given the potentially far-reaching implications of understanding how the perceptual system compares magnitudes, it is necessary to have stronger evidence.

An alternative explanation for Grace et al.’s (2018) task results might be exemplar learning – that is, observers performed accurately by associating locations on the response bar with particular stimulus pairs, rather than responding based on the differences and ratios between stimulus magnitudes. Grace et al. tested for exemplar learning in a second experiment with transfer trials, in which new pairs were presented in the final block without feedback. On average, responses on transfer trials were close to predictions based on the trained values, suggesting that observers learned differences and ratios rather than particular exemplars. However, a limitation of this transfer design is that it does not provide a strong

test for individual observers. Thus it is unclear whether observers might have used different response strategies, that is, some learned differences or ratios while others responded based on generalization from exemplars.

The possibility of heterogeneity at the level of individual observers is suggested by research on function learning (DeLosh, Bussemeyer & McDaniel, 1997; McDaniel, Cahill, Robbins & Weiner, 2014). In a typical function learning task, observers learn to make a continuous response (e.g., keypress duration) when given a stimulus magnitude (e.g., a line length; Koh & Meyer, 1991). McDaniel et al. (2014) found that after observers had been classified as exemplar or rule learners based on their responses to stimuli outside the range of training values, that they showed similar strategies in other function learning tasks. Evidence for stable individual differences in category learning tasks was also reported by Little and McDaniel (2015).

Because Grace et al.'s (2018) task is similar to function learning – observers learn to make a continuous response based on a pair of stimulus values – it is reasonable to assume that individuals may have used different response strategies. To provide a stronger test of relational learning, we used a task in which observers were trained on differences or ratios but with random noise added to the feedback. Noise values were sampled once for the exemplar set, so that multiple presentations of the same exemplar have the same feedback value. Following the rationale of classic prototype studies in categorization (Posner, Goldsmith & Welton, 1967; Posner & Keele, 1968), if observers are learning to respond based on differences or ratios they should show a stronger tendency to respond to the noise-free values rather than the noisy feedback, that is, they should effectively ignore or suppress the noise to some degree. By contrast, if observers respond on the basis of exemplars, their learning should include the noise. An advantage of this design is that whether the noise is learned or suppressed can be tested at the level of individual observers.

We report three experiments in which observers were trained to estimate differences or ratios via noisy feedback in a nonsymbolic task. The three experiments used stimuli that, in order of presentation below, varied in area, brightness or length. Although we were not primarily interested in whether the amount of noise affected responding, we increased the variance of the Gaussian distribution used to sample the noise from 3.6% (area) to 10% (brightness, line length) of the variance in the noise-free difference or ratios. We increased the noise in Experiments 2 and 3 because we thought that increased noise amounts might provide a more sensitive test of relational vs. exemplar learning. In each experiment, correlation and regression analyses were used to test whether responding for individual observers was more closely related to the underlying differences or ratios rather than the trained values with noisy feedback.

Experiment 1

On each trial, observers saw a pair of circles that varied in area. There were 8 different circles, and each of the possible 28 pairs was presented 3 times in a randomized order in each of 4 blocks in the session, for a total of 336 trials. The circle pairs were the same as those used by Grace et al. (2018, Experiment 1c). Observers were instructed to compare the sizes of the circles by making a mouse click along a horizontal response bar that contained no markings. Feedback was provided based on either the differences or ratios of the areas of each pair, with added noise. Specifically, feedback training values were defined as $Difference_{train} = Difference + \mathcal{N}(0, \sigma_d^2)$ and $Ratio_{train} = Ratio + \mathcal{N}(0, \sigma_r^2)$, where *Difference* and *Ratio* are calculated based on nominal differences or ratios of stimulus values, linearly scaled along the response bar, and $\mathcal{N}(0, \sigma^2)$ is a random (Gaussian) variate with mean =

0. The variance of the added noise (σ_d^2 , σ_r^2) was determined by calculating the percentage of variance associated with 1 of the 28 stimulus pairs ($1/28 = 3.6\%$), and so was defined as .036 times the variance of the difference or ratio training values across the 28 pairs. Noise values were sampled once for each stimulus pair in both ratio and difference conditions and were constant across trials and observers (see Figure 1, upper panels).

We planned to use multiple regressions to test whether the observers ignored or suppressed the added noise. Responses of individual observers across the 28 pairs were entered into multiple regressions with the trained values (i.e., difference or ratio plus noise) and noise as separate predictors. If observers respond by exemplar learning, then regression coefficients for the added noise should not differ systematically from zero. However, if observers are learning the underlying difference or ratio relation, then the noise should have significantly negative coefficients, suggesting that observers are suppressing the noise. We also planned regressions in which scaled ratios and differences were used to predict responding, both to replicate Grace et al.'s (2018) analyses that assessed whether responding depended on one or two operations, and to characterize whether responding was biased toward ratios or differences across groups.

Method

Participants. Eighteen undergraduate psychology students served as observers (7m, 11f; $M = 19.5$ years). All reported normal or corrected-to-normal vision. None was familiar with the purpose of the research, and they received course credit in exchange for participation.

Materials. The stimuli were pairs of circles with areas of 1.67, 2.93, 4.75, 7.31, 10.81, 15.48, 20.83, and 28.27 cm² (the radii, in pixels, were 148, 170, 192, 214, 235, 257, 278, and

300; Grace et al., 2018). The chosen values were equally spaced according to a square-root transformation, in order to enhance the ordinal discrepancies in perceived area between ratios and differences (Rule, Curtis & Mullin, 1981). The grayscale value for all circles was 205. The stimuli were arranged horizontally, and displayed against a black background, 8 cm from the top of the screen. Located 6.5 cm below the circles was a 16.5 cm (length) x 2 mm (width) grey response bar that contained no markings.

Nominal differences and ratios of stimulus values were linearly scaled along the response bar (0% to 100%, from left to right) using the same linear transformation as Grace et al. (2018), in which *Difference* values ranged from 9.24% to 91.18% ($M = 39.08\%$, $SD = 22.98\%$) and *Ratio* values from 4.83% to 92.54% ($M = 28.17\%$, $SD = 22.60\%$). These transformations were arranged to avoid floor and ceiling effects associated with endpoints of the response bar. Gaussian noise equivalent to 1 stimulus pair's variance was then generated by sampling from $N(0, .036 * \sigma_{d,r}^2)$, where σ_d^2 and σ_r^2 are the variances of the scaled difference or ratio values, and added to the scaled values ($1/28 = .036$). The resulting training values (scaled + noise) were truncated at $[0,1]$, that is, values <0 or >1 were changed to 0 and 1, respectively. The noise distributions were sampled once and the resulting training values were the same for all observers, and for each presentation of a particular stimulus pair. Figure 1 (top panels) shows the training values with added noise used for Experiment 1, plotted as a function of the scaled difference or ratio values.

Procedure. Participants were randomly assigned to ratio ($n = 9$) or difference ($n = 9$) conditions, and were tested individually or in small groups. Each was seated at a desk in a cubicle with an HP Elitedesk core i7 computer with a Samsung 22" LCD monitor. The monitor had a resolution of 1680 x 1050 pixels and a maximum brightness of 300 cd/m² and contrast ratio of 1000:1. They were given the following instructions: "In this experiment

you'll see a series of pairs of circles on the screen. For each pair, you need to compare the areas of the circles by clicking on the horizontal bar. The purpose is to learn how to compare the areas as accurately as you can. You'll receive feedback for each correct response, but if your response is incorrect, you'll have a chance to repeat with the same pair of circles. You should respond at whatever pace feels comfortable and natural for you, and will have several chances to take a break during the session."

Each session consisted of four blocks of 84 trials, separated by brief rest periods. Each of the 28 stimulus pairs was presented on three trials during each block. The smaller stimulus was always presented on the left, and the order of stimulus pairs within each block was randomised. Using a mouse, participants indicated their response by clicking on the horizontal response bar. There was no time limit to respond. After a 100 ms delay, feedback consisting of a green oval (10 mm diameter), centred on the training value (i.e., scaled difference or ratio plus noise) and extending 7% of the response bar in either direction, was presented. Responses that were within the green oval produced audible feedback and were counted as correct. After 500 ms, the stimuli and response bar were removed, and the next trial began after a 2-s interval. Responses that were outside 7% of the designated value were followed by a red oval centred on the correct response for that pair. This was followed by a single correction trial in which the same stimulus pair was presented. Responses on correction trials were omitted for all analyses.

Results

Data were omitted from one participant in the difference group who responded unsystematically (R^2 for trained values = .000).

Figure 2 shows the average absolute deviation of responses from trained values for both difference and ratio groups. Deviations decreased over blocks, indicating that responding became more accurate with training but appeared to stabilize in the second half of sessions. A repeated-measures ANOVA (Greenhouse-Geisser adjusted) with block as a within-subjects factor and group (ratio vs. difference) as a between-subjects factor found a significant effect of block, $F(1.168, 24.270) = 6.867, p < 0.05, \eta_p^2 = .314$, whereas the effects of group and the group $\times$ block interaction were not significant ($ps > .08$). Because average deviations did not differ significantly between Blocks 3 and 4 ($Ms = .105$ and $.096$, respectively), for subsequent analyses responding was pooled across Blocks 3 and 4.

Figure 3 shows the average responses for each group plotted against trained, scaled ratio, and scaled difference values. Analysis of individual data showed that for each observer, responding was more highly correlated with the scaled differences or ratios than the trained values. Average correlations of responses with trained values were $Ms = .92$ and $.94$, for the Difference and Ratio groups, respectively; and were $Ms = .93$ and $.89$ for scaled differences, and $Ms = .93$ and $.95$ for scaled ratios. For the Difference group, responding was most highly correlated with the scaled difference values for 3 out of 8 observers, and most highly correlated with scaled ratio values for 5 of 8 observers. For the ratio group, responding was most highly correlated with scaled ratio values for 9 of 9 observers. Thus, for both groups, individual responses were always more highly correlated with the scaled ratios and/or differences than with the trained values.

Hierarchical regressions were conducted to test whether the added noise accounted for significant variance in individual observers' responding beyond the trained values. Specifically, for each observer we conducted two regressions in which the trained value was entered at the first step, and the deviation between the scaled difference or ratio (depending on group) and the trained value was entered at the second step.

Results are shown in Table 1. The average R^2 based on trained values was similar for difference ($M = .851$) and ratio ($M = .885$) groups, $t(15) = -1.18$, $p > .25$. The coefficients for noise were always negative, and significant ($p < .05$) for 14 of 17 individual cases (6 of 8 in the difference group; 8 of 9 in the ratio group). Averaged across observers, the noise coefficients were significantly negative for both difference, $M = -0.688$ [95% CI: -0.880, -0.496], and ratio groups, $M = -0.728$ [95% CI: -0.907, -0.549]. The average incremental R^2 for noise was 2.6% and 2.9% for the difference and ratio groups, respectively (*ns*). These results are consistent with our prediction that observers would suppress the noise and respond based on the underlying relation.

Next we conducted multiple regressions with the scaled difference and ratio values as predictors. As Table 2 shows, the average R^2 was high and similar for the difference ($M = .914$) and ratio ($M = .916$) groups, $t(15) = -0.07$, $p > .90$, indicating that responding was closely related to the differences and ratios of the circle areas. Difference coefficients were positive and significant for 7 of 8 (difference group) and 2 of 9 observers (ratio group), while ratio coefficients were significant for 6 of 9 (difference) and 9 of 9 observers (ratio). Thus the untrained operation accounted for significant variance in 8 of 17 (47%) of individual observers overall.

To test if responding was biased overall towards differences or ratios across groups, we calculated the difference between the regression coefficients for the trained and untrained operations and compared them between groups (i.e., for the difference group: Difference B – Ratio B ; and for the ratio group, Ratio B – Difference B). Trained B – Untrained B was significantly greater for the ratio ($M = 0.634$) than the difference group ($M = 0.088$), $t(15) = -3.28$, $p = .005$, $d = -1.59$. This shows that responding in the ratio group was characterized by stronger control by the trained operation than responding in the difference group.

In summary, results showed that observers suppressed the added noise, responding more closely to the noise-free ratios or differences of circle areas, as indicated by the significantly negative regression coefficients in Table 1. Multiple regressions confirmed that approximately half of individual observers' data showed evidence of two operations even though only one was trained, consistent with Grace et al.'s (2018) results. Finally, responding in the ratio group was more strongly controlled by the trained operation than responding in the difference group, suggesting that it might have been easier for observers to learn ratios than differences of circle areas in this task.

Experiment 2

Experiment 2 used a similar design to Experiment 1, except that stimuli varied in brightness. Observers ($N = 14$) were randomly divided between ratio and difference groups. There were 8 brightness values (ranging from 50 to 205 in 8-bit grayscale values) and one of the possible 28 pairs was presented on each trial. The brightness values were the same as those used by Grace et al. (2018, Experiment 1a) and similar to Experiment 1, were spaced according to a square-root transformation (Rule et al., 1981). Feedback was based on the scaled ratio or difference of the grayscale values, plus added noise that was determined by sampling from a normal distribution with a mean of 0 and variance equal to 10% of the variance of the scaled ratio or difference values. As in Experiment 1, noise values were computed separately for each stimulus pair but were constant across observers and presentations of each pair (see Figure 1, middle panels). Analyses were planned to determine if responding was more closely related to the scaled difference and ratio values than the training values with noisy feedback.

Method

Participants. Fifteen students recruited from undergraduate Psychology courses at the University of Canterbury served as observers (4m, 11f; $M = 20$ years). All reported normal or corrected-to-normal vision. None were familiar with the purpose of the research, and they received course credit in exchange for participation.

Materials. The stimuli were pairs of circles with grayscale values of 50, 66, 84, 104, 126, 150, 176, and 205, generated using an 8-bit monochrome palette [0 - 255]. Each circle was 6 cm diameter (6.84° visual angle), displayed side by side against a black background, 8 cm from the top of the screen and separated by 3.3 cm. Luminance (cd/m^2) is a nonlinear (gamma corrected) function of the grayscale value, to compensate for differential sensitivity of the visual system so that perceived brightness is approximately linear with grayscale value. The grayscale values were defined by setting the lower and upper values 50 points from the scale limits [$0 = \text{minimum luminance, } 0.33 \text{ cd/m}^2$; $255 = \text{maximum luminance, } 231 \text{ cd/m}^2$], while the six interior values were equally spaced according to a square-root transformation.

As in Experiment 1, a grey horizontal response bar containing no markings was positioned below the stimuli. Nominal differences and ratios of stimulus values were scaled along the response bar using the same linear transformation as Grace et al. (2018). To these scaled values, we added Gaussian noise by sampling from a normal distribution $N(0, .10 * \sigma_{d,r}^2)$, where σ_d^2 and σ_r^2 are the variances of the scaled difference or ratio values. The noise was added to each scaled difference or ratio value; the resulting training values with added noise are plotted in Figure 1 (middle panels) as a function of the scaled values.

Procedure. All other details of the procedure were the same as for Experiment 1. Participants were randomly assigned to ratio ($n = 7$) or difference ($n = 7$) groups and given

the following instructions: "In this experiment you'll see a series of pairs of circles on the screen. For each pair, you need to compare the brightness of the circles by clicking on the horizontal bar. The purpose is to learn how to compare the brightnesses as accurately as you can. You'll receive feedback for each correct response, but if your response is incorrect, you'll have a chance to repeat with the same pair of circles. You should respond at whatever pace feels comfortable and natural for you, and will have several chances to take a break during the session."

Results

Data from one observer in the ratio group was omitted who responded unsystematically (R^2 for trained values = .207).

Figure 4 shows the average absolute deviation of responses from trained values for both difference and ratio groups. Similar to Experiment 1, deviations decreased over blocks, indicating that responding became generally more accurate with training. A repeated-measures ANOVA (Greenhouse-Geisser adjusted) with block as a within-subjects factor and group (ratio vs. difference) as a between-subjects factor found significant effects of block, $F(1.577, 18.929) = 13.723, p < 0.01, \eta_p^2 = .533$, and the group x block interaction, $F(1.577, 18.929) = 3.988, p < 0.05, \eta_p^2 = .249$, but the main effect of block was not significant. Analysis of simple effects showed that average deviations were significantly greater for the ratio than difference group in Block 1, but not Blocks 2-4. As in Experiment 1, we based subsequent analyses on responding pooled across Blocks 3-4.

Figure 5 shows the average responses for each group plotted against trained, scaled ratio, and scaled difference values. Analysis of individual data showed that for all but one observer, responding was more highly correlated with either the scaled differences or ratios

than trained values. Average correlations of responses with trained values were $M_s = .823$ and $.893$, for the difference and ratio groups, respectively; and were $M_s = .846$ and $.903$ for scaled differences, and $M_s = .650$ and $.917$ for scaled ratios. For the difference group, responding was most highly correlated with the scaled difference values for 7 out of 8 observers, and most highly correlated with trained values for 1 of 8. For the ratio group, responding was most highly correlated with scaled ratio values for 8 of 11 observers, and most highly correlated with scaled differences for the other three.

As in Experiment 1, hierarchical regressions were conducted to test whether the added noise accounted for significant variance in individual observers' responding beyond the trained values. Results are shown in Table 3. The average R^2 based on trained values was $M = .814$ and $M = .768$ for difference and ratio groups, respectively, and was not significantly different, $t(12) = 0.901$, $p > .38$. The regression coefficients for noise were always negative, and significant ($p < .05$) for 11 of 14 individual cases (5 of 7 in the difference group; 6 of 7 in the ratio group). On average, the noise coefficients were significantly negative for the difference, $M = -0.518$ [95% CI: $-0.670, -0.367$], and ratio groups, $M = -0.695$ [95% CI: $-0.993, -0.397$]. The average incremental R^2 for noise was 4.3% and 7.7% for the difference and ratio groups, respectively, which were not significantly different ($p > .13$). These results are similar to Experiment 1 and suggest that observers in both groups learned to respond based on differences and ratios, suppressing the added noise.

Multiple regressions with scaled differences and ratios as predictors fit the data well, with average R^2 values of $.857$ and $.877$ for the difference and ratio groups, respectively (Table 4). Difference coefficients were positive and significant for 7 of 7 (difference group) and 6 of 7 observers (ratio group), while ratio coefficients were significant for 1 of 7 (difference) and 6 of 7 observers (ratio). Thus the untrained operation accounted for significant variance in 7 of 14 (50%) of individual observers, but this was largely due to the

ratio group. Indeed, the difference between the trained and untrained regression coefficients was significantly greater for the difference group, $M = 0.695$, than the ratio group, $M = .048$, $t(12) = 4.44$, $p < .001$, $d = 2.37$. This suggests that responding was more strongly controlled by differences than ratios across groups.

Results of Experiment 2 are similar to Experiment 1 in that observers appeared to suppress the added noise, and responded based on the underlying difference or relation. However, unlike Experiment 1, stronger control was evidenced by differences than ratios. This suggests that observers more easily responded based on differences rather than ratios with brightness in this task.

Experiment 3

In Experiment 3 observers compared the lengths of two lines, in a procedure similar to Experiments 1 and 2. Observers ($N = 19$) were randomly divided between ratio ($n = 11$) and difference ($n = 8$) groups. One of the possible 28 pairs of 8 line lengths (ranging from 22 to 137 mm) was presented on each trial. Feedback was based on the scaled ratio or difference of the grayscale values, plus added noise that was determined by sampling from a normal distribution with a mean of 0 and variance equal to 10% of the variance of the scaled ratio or difference values. As in Experiments 1 and 2, noise values were computed separately for each stimulus pair but were constant across observers and presentations of each pair (see Figure 1, lower panels). We tested if responding was more closely related to the scaled difference and ratio values than the training values with noisy feedback.

Method

Participants. Twenty-one students from undergraduate Psychology courses at the University of Canterbury community served as observers (7m, 14f; $M = 21.6$ years. All reported normal or corrected-to-normal vision. None were familiar with the purpose of the research, and they received a \$15 shopping voucher in exchange for participation.

Materials. The stimuli were pairs of yellow (R255, G255, B0) lines 1mm in width, presented side-by-side against a black background. Their lengths were 22, 29, 45, 58, 75, 94, 115 and 137 mm. These values were spaced according to a square-root transformation that enhanced the ordinal discrepancies between differences and ratios, similar to Experiments 1 and 2. To discourage the use of direct measurement strategies, the lines were presented at randomised angles (which differed by at least 30° within pairs). Their midpoints were positioned 18 cm apart and 10 cm from the top of the screen. Table 3 shows the stimulus pairs used and their respective differences and ratios.

As in Experiments 1 and 2, a grey horizontal response bar containing no markings was positioned below the stimuli. Nominal differences and ratios of stimulus values were scaled along the response bar (left-right, 0-100%) using a linear transformation that mapped the minimum difference (or ratio) to 5% and the maximum difference (or ratio) to 95%. To these scaled values, we added Gaussian noise by sampling from a normal distribution $N(0, .10 * \sigma_{d,r}^2)$, where σ_d^2 and σ_r^2 are the variances of the scaled difference or ratio values. The noise was added to each scaled difference or ratio value; the resulting training values with added noise are shown in Figure 1 (lower panels) as a function of the scaled values.

Procedure. All other details of the procedure were the same as Experiments 1 and 2. Participants were randomly assigned to ratio ($n = 4$), or difference ($n = 4$) groups, and given the following instructions: "In this experiment you'll see a series of pairs of lines on the screen. For each pair, you need to compare the length of the lines by clicking on the

horizontal bar. The lines will appear at different angles, but the angles aren't important; we want you to focus on the lines' lengths. The purpose is to learn how to compare the lengths as accurately as you can. You'll receive feedback for each correct response, but if your response is incorrect, you'll have a chance to repeat with the same pair of circles. You should respond at whatever pace feels comfortable and natural for you, and will have several chances to take a break during the session."

Results

Data from two observers (one each from difference and ratio groups) were omitted because they responded unsystematically (R^2 for trained values = .179 and .102).

Figure 6 shows the average absolute deviation of responses from trained values for both difference and ratio groups. Deviations decreased over blocks, indicating that responding became more accurate with training. A repeated-measures ANOVA (with block and group entered as factors, with Greenhouse-Geisser adjustments) showed a significant main effects for block, $F(2.104, 35.765) = 18.322, p < .001, \eta_p^2 = .519$, and group, $F(1, 17) = 8.114, p = .01, \eta_p^2 = .323$, but the interaction was not significant. As Figure 6 shows, deviations were overall greater for the difference group. As with Experiments 1 and 2, subsequent analyses pooled responding across Blocks 3 and 4.

Figure 7 shows the average responses for each group plotted against trained, scaled ratio, and scaled difference values. Analysis of individual data showed that for each observer, responding was more highly correlated with the scaled difference or ratio values than the trained values. Average correlations of responses with trained values were $M_s = .823$ and $.893$, for the difference and ratio groups, respectively; and were $M_s = .846$ and $.903$ for scaled differences, and $M_s = .650$ and $.917$ for scaled ratios. For the difference group,

responding was most highly correlated with the scaled difference values for 7 out of 8 observers, and most highly correlated with trained values for 1 of 8. For the ratio group, responding was most highly correlated with scaled ratio values for 8 of 11 observers, and most highly correlated with scaled differences for the other three.

Results of hierarchical regressions with trained values and added noise are shown in Table 5. The average R^2 based on trained values for the difference group was $M = .602$, which was significantly lower than for the ratio group ($M = .797$), $t(17) = -2.16$, $p < .05$, $d = -1.00$. Several observers in the difference group (e.g., 3, 4 and 8) had R^2 that were $< .50$, unlike the ratio group and Experiments 1 and 2.

The regression coefficients for noise were always negative and significant ($p < .05$) for 13 of 19 individual cases (4 of 8 in the difference group; 8 of 10 in the ratio group). On average, the noise coefficients were significantly negative for the difference, $M = -0.570$ [95% CI: -0.831, -0.309], and ratio groups, $M = -0.650$ [95% CI: -0.768, -0.532]. The average incremental R^2 for noise was 6.2% and 4.6% for the difference and ratio groups, respectively, which did not significantly differ ($p > .16$). These results are similar to Experiments 1 and 2, and suggest that observers in both groups were likely to respond based on the underlying difference and ratio relations, suppressing the added noise.

When scaled differences and ratios were used to predict responding, the average R^2 values increased to $M_s = .728$ and $.922$ for the difference and ratio groups, respectively (Table 6), with the average for the difference group still significantly lower, $t(17) = -2.78$, $p = .01$, $d = -1.29$. Difference coefficients were positive and significant for 8 of 8 (difference group) and 10 of 11 observers (ratio group), while ratio coefficients were significant for 4 of 8 (difference; with two negative coefficients) and 9 of 11 observers (ratio). Overall the untrained operation accounted for significant variance in 13 of 19 (68%) of individual observers. The difference between the trained and untrained regression coefficients was

significantly greater for the difference group, $M = 0.667$, than the ratio group, $M = .170$, $t(17) = 3.07$, $p < .01$, $d = 1.43$. This suggests that similar to Experiment 2, responding was more strongly controlled by differences than ratios across groups.

Results of Experiment 3 are similar to the previous experiments in that responding more closely corresponded to the noise-free difference or ratio relation, suggesting that observers suppressed or ignored the added noise. Across groups, stronger control was shown by differences rather than ratios, similar to brightness (Experiment 2) but unlike circle area (Experiment 1).

General Discussion

Grace et al. (2018) reported that observers learned to respond accurately in a non-symbolic ‘artificial algebra’ task in which correct responses were determined by the difference or ratio of stimulus magnitudes. But did they learn the difference and ratio relations, or could some other process, such as exemplar learning, have explained their performance? To compare these alternatives, we studied a version of the task with noisy feedback. In three experiments, observers saw pairs of circles that varied in area (Experiment 1) or brightness (Experiment 2), or pairs of lines that varied in length (Experiment 3). In each, there were 28 stimulus pairs and each was presented 12 times (in randomized order) so that observers completed a total of 336 training trials. Feedback was based on either the difference or ratio of stimulus values, plus Gaussian noise. The noise was sampled once for the set of stimulus pairs so training values were the same for each presentation of a pair. We ran hierarchical regressions in which the trained values (difference or ratio plus noise) were entered at the first step and the deviation between the difference or ratio values and the trained value were entered at the second step. Across experiments, the noise coefficients

were negative for every observer and statistically significant in 38 of 50 individual cases (76%). This shows that observers learned to respond based on the underlying differences or ratios, and effectively filtered or suppressed the noisy feedback. Although deviations from differences and ratios were embedded in the training feedback and observers had no external cues to separate the ‘true’ difference or ratio relation from the noise, their responding counteracted the added noise.

Our data provide strong evidence of relational learning even though no explicit instruction about differences or ratios was given. The speed with which observers learned to respond was comparable to Grace et al.’s (2018) results, and the degree of accuracy achieved, as measured by R^2 explained by differences and ratios was also similar. The average R^2 for circle areas and brightness in the present study was .915 and .867, respectively, which were close to the average R^2 for Grace et al. (2018) with the same stimuli (.917 and .907; see their Table 2).

These results suggest that differences and ratios are computed automatically by the perceptual system when observers are comparing magnitudes, that is, that these operations are performed at a relatively low level and do not require explicit cognition. An alternative interpretation, based on the ‘tools to theories’ heuristic (Gigerenzer, 1991), would be that mathematical operations provide a convenient descriptive model for how stimulus magnitudes are compared, but it is unwarranted to assume that the perceptual system performs the equivalent of computation. Although our results cannot definitively reject the latter view, we believe that the speed of learning and accuracy of responding despite the noisy feedback is strong evidence of automaticity. If this is correct, magnitude comparison would be ‘computationally complex’ in Grace et al.’s (2019) definition, because it would have an action in the perceptual system that is functionally equivalent to algebraic computation as a proximate cause.

What kind of neural architecture could support algebraic computation? Although it is difficult to speculate, one possibility is that the perceptual system represents an algebraic field, which is a mathematical object that describes algebra over the real numbers in an abstract way. A field consists of two algebraic groups, each with an operation (i.e., addition and multiplication), and connected by a distributive law (i.e., $a*(b+c) = a*b + b*c$). Algebraic fields generalize the intuitions underlying arithmetic over the real numbers, and highlight the structure necessary for arithmetic to work. Groups are associated with *symmetries* – they effectively code the symmetries of an object that can be transformed in various ways while preserving its structure – and it may be helpful to look for evidence of groups when considering the neural processes that might support computation by the perceptual system (see Grace et al., 2019).

There was heterogeneity across experiments in terms whether responding more closely aligned with differences or ratios. Similar to Grace et al. (2018), we used multiple regression to measure the control over responding by differences and ratios. We calculated the difference between the trained and untrained regression coefficients (i.e., Difference B – Ratio B for the difference groups, and Ratio B – Difference B for the ratio groups) as an index of how strongly responding was aligned with the trained operation. We found that this index was greater for the ratio group in Experiment 1 with stimuli that varied in circle area, but greater for the difference groups in Experiment 2 (brightness) and Experiment 3 (line length).

In a recent study, Masin, Brancaccio and Tomassetti (2019) found that observers were unable to estimate ratios of brightness accurately, but did produce accurate estimates of line length pairs. Based on these results as well as prior work (e.g., Masin, 2013, 2014; Masin & Brancaccio, 2017), they proposed that observers may be unable to estimate ratios of intensive stimuli (e.g., brightness, heaviness), instead relying on differences, whereas they can judge

ratios of extensive stimuli. Our finding that responses were more strongly controlled by ratios with circle areas (Experiment 1), but by differences with brightness (Experiment 2) is consistent with Masin et al.'s proposal. However, we also found significantly stronger control by differences with line length (Experiment 3), which is an extensive dimension.

It is important to note that our studies were not devised as a test of Masin et al.'s proposal. There are considerable procedural differences between Masin et al. and the present study. Masin et al. used an explicit task in which observers were asked to judge differences or ratios and made numerical responses, contrary to the implicit task used here in which the instructions did not refer to differences or ratios and responses were non-symbolic (analogue). Also, comparisons across our experiments should be made cautiously because the amount of noise varied. We calculated the added noise by sampling from a Gaussian distribution with $\mu = 0$ and σ^2 defined as a percentage of the variance in the scaled difference or ratio values, which was 3.6% for Experiment 1 and 10% for Experiments 2 and 3. Interesting questions for future research include whether the nature of the task and instructions affect observers' ability to respond based on differences or ratios independently of the stimulus modality, and if responding varies when the amount of noise is varied within a modality.

The logic of our experiments was similar to Posner and Keele's (1968) classic study on prototype learning. In their study, observers learned distortions of four categories (the letters A and M, a triangle, and a random pattern in a 30x30 dot array) and subsequently were tested on both the prototypes (schemas) and new distortions of the categories. They found that observers were faster and more accurate to classify the schemas than the new distortions, and that performance with the schemas was similar to that of the original learned exemplars. They concluded that the observers had formed abstract representations of the prototypes, which facilitated their responding when they saw them for the first time during test. In both

Posner and Keele (1968) and the present experiments, observers were apparently able to filter distortion and respond on the basis of an underlying prototype or relationship between stimulus magnitudes.

Our experiments also provide a replication and extension of Grace et al.'s (2018) tests of Torgerson's (1961) conjecture. According to this conjecture, observers compare stimulus magnitudes with one operation, either a difference or ratio, but which one cannot be determined empirically. Results of multiple regressions with our data showed that both differences and ratios were significant predictors of responding for a substantial proportion of observers (see Tables 2, 4 and 6). Pooled across experiments, the untrained operation (i.e., ratios for the difference group, differences for the ratio group) accounted for significant variance in responding in 28 of 50 individual cases (56%). This result is similar to Grace et al. (2018) and provides additional evidence that Torgerson's conjecture, as applied to our implicit task, is false. The present data also extend this finding to line length as a stimulus modality.

In summary, our results show that observers compare stimulus magnitudes based on differences and ratios even when they are not given explicit instructions to do so, and suggest that the perceptual system automatically computes differences and ratios. When feedback was degraded by adding noise to scaled differences and ratios, observers suppressed or ignored the noise and responded based on the underlying relation. An interesting question for future research is whether feedback is necessary at all for observers to produce difference and ratios.

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Observer	Group	R^2	Train B	Noise B	R^2_{inc}	p
1	Diff	0.758	0.739	-1.074	0.056	0.011
2	Diff	0.863	0.827	-0.679	0.02	0.046
3	Diff	0.886	0.91	-0.668	0.017	0.049
4	Diff	0.872	0.638	-0.254	0.005	0.331
5	Diff	0.817	0.779	-0.642	0.02	0.096
6	Diff	0.929	0.893	-0.66	0.018	0.008
7	Diff	0.796	0.679	-0.854	0.044	0.014
8	Diff	0.883	0.744	-0.674	0.025	0.014
9	Ratio	0.763	0.837	-0.91	0.036	0.046
10	Ratio	0.883	0.769	-0.944	0.053	< .001
11	Ratio	0.911	0.84	-0.68	0.024	0.006
12	Ratio	0.797	0.575	-0.173	0.003	0.56
13	Ratio	0.935	0.957	-0.772	0.024	0.001
14	Ratio	0.937	0.84	-0.637	0.021	0.001
15	Ratio	0.948	0.919	-0.887	0.035	< .001
16	Ratio	0.918	0.884	-0.829	0.032	0.001
17	Ratio	0.875	0.74	-0.719	0.033	0.006
Avg	Diff	0.851	0.776	-0.688	0.026	
	Ratio	0.885	0.818	-0.728	0.029	

Table 1. Results of multiple regressions for Experiment 1 (Area) in which training values and added noise predicted each observer's data. Train B and Noise B are the unstandardized coefficients for training and noise values when entered in a simultaneous regression, with R^2_{inc} the incremental variance accounted for by added noise (with significance level p).

Observer	Group	R^2	Diff B	Ratio B	p (Diff)	p (Ratio)
1	Diff	0.857	0.357	0.467	0.033	0.008
2	Diff	0.882	0.841	0.009	< .001	0.953
3	Diff	0.964	0.382	0.613	< .001	< .001
4	Diff	0.909	0.323	0.360	0.004	0.002
5	Diff	0.929	0.208	0.660	0.078	< .001
6	Diff	0.949	0.768	0.163	0.000	0.134
7	Diff	0.896	0.298	0.458	0.021	0.001
8	Diff	0.927	0.525	0.269	< .001	0.019
9	Ratio	0.798	-0.001	0.879	0.995	< .001
10	Ratio	0.937	0.102	0.716	0.319	< .001
11	Ratio	0.934	-0.075	0.940	0.506	< .001
12	Ratio	0.790	0.130	0.464	0.379	0.004
13	Ratio	0.962	0.174	0.830	0.077	< .001
14	Ratio	0.956	0.059	0.815	0.517	< .001
15	Ratio	0.989	0.181	0.791	0.001	< .001
16	Ratio	0.965	0.280	0.662	0.003	< .001
17	Ratio	0.914	0.161	0.623	0.169	< .001
Avg	Diff	0.914	0.463	0.375		
	Ratio	0.916	0.112	0.747		

Table 2. Results of multiple regressions for Experiment 1 (Area) in which scaled differences and ratios (noise free) predicted each observer's data. Diff B and Noise B are the unstandardized coefficients with corresponding p values when the predictors were entered simultaneously.

<u>Observer</u>	<u>Group</u>	<u>R^2</u>	<u>Train B</u>	<u>Noise B</u>	<u>R^2_{inc}</u>	<u>p</u>
1	Diff	0.908	0.853	-0.566	0.041	< .001
2	Diff	0.843	0.767	-0.785	0.089	< .001
3	Diff	0.701	0.558	-0.358	0.029	0.112
4	Diff	0.797	0.602	-0.348	0.027	0.062
5	Diff	0.886	0.81	-0.443	0.027	0.01
6	Diff	0.718	0.715	-0.672	0.064	0.012
7	Diff	0.843	0.883	-0.457	0.023	0.049
8	Ratio	0.853	0.82	-0.526	0.038	0.007
9	Ratio	0.547	0.24	-0.105	0.011	0.431
10	Ratio	0.76	0.807	-0.873	0.096	< .001
11	Ratio	0.768	0.787	-0.977	0.128	< .001
12	Ratio	0.818	0.91	-0.771	0.063	0.001
13	Ratio	0.752	0.744	-1.034	0.157	< .001
14	Ratio	0.877	0.803	-0.578	0.049	< .001
Avg	Diff	0.814	0.741	-0.518	0.043	
	Ratio	0.768	0.730	-0.695	0.077	

Table 3. Results of multiple regressions for Experiment 2 (Brightness) in which training values and added noise predicted each observer's data. Train B and Noise B are the unstandardized coefficients for training and noise values when entered in a simultaneous regression, with R^2_{inc} the incremental variance accounted for by added noise (with significance level p).

Observer	Group	R^2	Diff B	Ratio B	p (Diff)	p (Ratio)
1	Diff	0.952	0.608	0.218	< .001	0.023
2	Diff	0.932	0.762	-0.021	< .001	0.837
3	Diff	0.732	0.398	0.141	0.016	0.384
4	Diff	0.813	0.577	-0.018	< .001	0.894
5	Diff	0.900	0.820	-0.077	< .001	0.543
6	Diff	0.810	0.975	-0.325	< .001	0.066
7	Diff	0.858	0.992	-0.197	< .001	0.248
8	Ratio	0.939	0.488	0.342	< .001	0.003
9	Ratio	0.563	0.109	0.127	0.271	0.217
10	Ratio	0.910	0.487	0.365	0.001	0.009
11	Ratio	0.930	0.392	0.446	0.001	< .001
12	Ratio	0.931	0.517	0.421	< .001	0.002
13	Ratio	0.935	0.371	0.433	0.001	< .001
14	Ratio	0.931	0.214	0.588	0.045	< .001
Avg	Diff	0.857	0.733	-0.040		
	Ratio	0.877	0.368	0.389		

Table 4. Results of multiple regressions for Experiment 2 (Brightness) in which scaled differences and ratios (noise free) predicted each observer's data. Diff B and Noise B are the unstandardized coefficients with corresponding p values when the predictors were entered simultaneously.

<u>Observer</u>	<u>Group</u>	<u>R^2</u>	<u>Train B</u>	<u>Noise B</u>	<u>R^2_{inc}</u>	<u>p</u>
1	Diff	0.886	0.691	-0.621	0.058	< .001
2	Diff	0.863	0.844	-0.888	0.077	< .001
3	Diff	0.408	0.388	-0.346	0.026	0.29
4	Diff	0.318	0.201	-0.381	0.093	0.059
5	Diff	0.788	0.645	-0.704	0.076	0.001
6	Diff	0.499	0.453	-0.489	0.047	0.12
7	Diff	0.809	0.813	-1.05	0.109	< .001
8	Diff	0.246	0.14	-0.082	0.007	0.64
9	Ratio	0.811	0.627	-0.641	0.051	0.006
10	Ratio	0.508	0.31	-0.406	0.052	0.097
11	Ratio	0.904	0.817	-0.779	0.049	< .001
12	Ratio	0.806	0.668	-0.624	0.042	0.014
13	Ratio	0.812	0.703	-0.568	0.032	0.033
14	Ratio	0.86	0.849	-0.833	0.05	0.001
15	Ratio	0.831	0.674	-0.604	0.04	0.01
16	Ratio	0.916	0.783	-0.627	0.035	< .001
17	Ratio	0.868	0.793	-0.93	0.071	< .001
18	Ratio	0.608	0.408	-0.349	0.027	0.19
19	Ratio	0.839	0.767	-0.79	0.053	0.002
Avg	Diff	0.602	0.522	-0.570	0.062	
	Ratio	0.797	0.673	-0.650	0.046	

Table 5. Results of multiple regressions for Experiment 3 (Line length) in which training values and added noise predicted each observer's data. Train B and Noise B are the unstandardized coefficients for training and noise values when entered in a simultaneous regression, with R^2_{inc} the incremental variance accounted for by added noise (with significance level p).

<u>Observer</u>	<u>Group</u>	<u>R^2</u>	<u>Diff B</u>	<u>Ratio B</u>	<u>p (Diff)</u>	<u>p (Ratio)</u>
1	Diff	0.950	0.610	0.123	< .001	0.065
2	Diff	0.951	0.729	0.184	< .001	0.027
3	Diff	0.449	0.513	-0.144	0.005	0.420
4	Diff	0.459	0.366	-0.179	0.001	0.092
5	Diff	0.904	0.897	-0.282	< .001	0.004
6	Diff	0.567	0.615	-0.180	0.001	0.286
7	Diff	0.936	0.653	0.259	< .001	0.008
8	Diff	0.607	0.404	-0.330	< .001	< .001
9	Ratio	0.887	0.196	0.478	0.027	< .001
10	Ratio	0.836	0.407	-0.035	< .001	0.598
11	Ratio	0.962	0.147	0.716	0.021	< .001
12	Ratio	0.973	0.469	0.274	< .001	< .001
13	Ratio	0.871	0.240	0.508	0.024	< .001
14	Ratio	0.942	0.298	0.616	0.001	< .001
15	Ratio	0.916	0.282	0.445	0.001	< .001
16	Ratio	0.948	0.014	0.792	0.840	< .001
17	Ratio	0.951	0.172	0.679	0.017	< .001
18	Ratio	0.903	0.485	-0.011	< .001	0.853
19	Ratio	0.952	0.362	0.475	< .001	< .001
Avg	Diff	0.728	0.598	-0.069		
	Ratio	0.922	0.279	0.449		

Table 6. Results of multiple regressions for Experiment 3 (Line length) in which scaled differences and ratios (noise free) predicted each observer's data. Diff B and Noise B are the unstandardized coefficients with corresponding p values when the predictors were entered simultaneously.

Figure 1. Scatterplots of training values (with added noise) as a function of scaled (noise-free) differences and ratios for Experiment 1 (Area; upper panels), Experiment 2 (Brightness; middle panels), and Experiment 3 (Line length; lower panels) .

Figure 2. Average absolute deviation of responses from trained values for each trial block in Experiment 1 (area). Results for difference group are shown with unfilled squares and solid line; results for ratio group are shown with filled diamonds and dotted line. Bars indicate one standard error.

Figure 3. Average responses plotted against trained (noisy) values (upper panels), and underlying difference or ratio relation for difference (left panels) and ratio (right panels) groups, respectively. Dotted lines indicate perfect accuracy; solid lines show best-fitting linear regressions (equations and R^2 also shown).

Figure 4. Average absolute deviation of responses from trained values for each trial block in Experiment 2 (brightness). Results for difference group are shown with unfilled squares and solid line; results for ratio group are shown with filled diamonds and dotted line. Bars indicate one standard error.

Figure 5. Average responses plotted against trained (noisy) values (upper panels), and underlying difference or ratio relation for difference (left panels) and ratio (right panels) groups, respectively. Dotted lines indicate perfect accuracy; solid lines show best-fitting linear regressions (equations and R^2 also shown).

Figure 6. Average absolute deviation of responses from trained values for each trial block in Experiment 3 (line length). Results for difference group are shown with unfilled squares and solid line; results for ratio group are shown with filled diamonds and dotted line. Bars indicate one standard error.

Figure 7. Average responses plotted against trained (noisy) values (upper panels), and underlying difference or ratio relation for difference (left panels) and ratio (right panels) groups, respectively. Dotted lines indicate perfect accuracy; solid lines show best-fitting linear regressions (equations and R^2 also shown).