The Reliability Factor: Modeling individual reliability with multiple items from a single assessment

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Reliability is a crucial concept in psychometrics. Although it is typically estimated as a single fixed quantity, previous work suggests that reliability can vary across persons, groups, and covariates. We propose a novel method for estimating and modeling case-specific reliability without repeated measurements or parallel tests. The proposed method employs a "Reliability Factor" that models the error variance of each case across multiple indicators, thereby producing case-specific reliability estimates. Additionally, we use Gaussian process modeling to estimate a non-linear, non-monotonic function between the latent factor itself and the reliability of the measure, providing an analogue to test information functions in item response theory. The reliability factor model is a new tool for examining latent regions with poor conditional reliability, and correlates thereof, in a classical test theory framework.

Keywords: Omega, Bayesian, reliability

Reliability is one of the most important properties of a measurement instrument and is therefore of interest to any researcher using or developing a psychometric tool. However, classical approaches to reliability assessment assume that reliability is a fixed property of the measure. That is, reliability is not permitted to vary across persons, groups, nor the factor space itself. Although item response theory has long allowed reliability to vary across the latent domain, classical approaches often lack this useful feature (de Ayala, 2009).

Due to the customary assumption underlying its computation – namely, that the *error variance* giving rise to the residual distribution is constant across all cases, groups, covariates, and the latent space – reliability is treated as fixed in most applications. However, it is entirely possible, and likely, that a measure will be more reliable or less error-prone for some observations than for others (Lek & Van De Schoot, 2018; Feldt, Steffen, & Gupta, 1985; Harvill, 1991). In fact, as will be discussed briefly, reliability is not constrained to be constant in its basic definition.

In this paper, we propose a method that permits composite score reliability to vary across cases (i.e., individuals), factor scores, and covariates. When multiple indicators are employed, this method permits varying reliability to be estimated even when each case is only assessed on a single occasion. Altogether, the proposed method permits varying reliability without requiring parallel tests, tau-equivalent models (Lek & Van De Schoot, 2018), nor repeated observations (Hu et al., 2016; Geldhof, Preacher, & Zyphur, 2014). Therefore, this approach may be used in the scenario commonly found in psychological research: Cross-sectional study designs with single observations from multiple item instruments. This permits psychometricians to examine the variance in and predictors of reliability in the population.

The method that we present here employs a "reliability factor" in addition to the latent factor(s) typically used. In order to motivate the reliability factor, we briefly describe reliability within a latent modeling framework. The proposed method is illustrated in three examples.

Measurement error variance and reliability

Classical test theory assumes that the estimated score ($\hat{s}_{it}$) is equal to the true (asymptotic) score (s_i) and error (ϵ_{it}). That is, $\hat{s}_{it} = s_i + \epsilon_{it}$, where i is the case and t is the occasion. As Lord and Novick (2008) pointed out, the variance of the observed score can be expressed as: $V_i(\hat{s}_{it}) = V_i(s_i) + V_i(\epsilon_{it})$, where V_i refers to a conditional variance for case i. It is important to note here that the variances are not restricted to be homogeneous. In fact, Lord and Novick (2008, p. 32) noted that "We need not assume that the propensity distribution

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variances... are equal for different persons. Thus we allow the possibilities that some persons' responses are inherently more consistent than those of others, and that we are able to measure some persons' responses more accurately than others'." This means that measurement error variance can depend on cases, covariates, groups, or other random variables. Notably, the unconditional variance term, $V(\epsilon_{it})$, is implicitly an expectation across conditional error variance dictated by the law of total variance :

$$V(\epsilon_{it}) = E(V_i(\epsilon_{it})) + V(E_i(\epsilon_{it}))$$
$$V(\epsilon_{it}) = E(V_i(\epsilon_{it})).$$

Thus, when variance terms are estimated in measurement models as unconditional fixed quantities, they are implicitly marginalized over all possible conditional variances. As a consequence, common reliability coefficients – as functions of these homogenous true score and measurement error variance components (or other unconditional fixed components in Generalizability theory) – are implicitly marginalized over various important random variables (Raju, Price, Oshima, & Nering, 2007; Brennan, 2005; Williams, Martin, & Rast, 2019):

$$\rho = \frac{V(s_i)}{V(s_i) + V(\epsilon_{it})}.$$

This unconditional approach therefore only provides expected measurement error variances and homogenenous reliability, and disposes of useful psychometric information such as how they change across cases, scores, or covariates. This loss of information is unfortunate, but avoidable by retaining and modeling the variance components underlying response consistency. Conditional variance models allow variance components to be conditionalized by random variables; such approaches include (mixed effects) location scale models, dynamic structural equation models, generalized autoregressive conditional heteroskedasticity models, and more (Asparouhov, Hamaker, & Muthén, 2018; Rast, Hofer, & Sparks, 2012; Rast & Ferrer, 2018; Williams, Martin, & Rast, 2019; Rast, Martin, Liu, & Williams, 2020; Kapur, Li, Blood, & Hedeker, 2015).

Given that variance components underlying reliability in classical test theory can be modeled, researchers can relax the strict homogeneous reliability assumption, and regain this useful psychometric information. In these regards, the error variance is of specific interest as it incorporates disturbances that alter reliability. In sum, one can retain the i subscript in the variance components to produce conditionalized reliability:

$$\rho_i = \frac{V_i(s_i)}{V_i(s_i) + V_i(\epsilon_{it})}.$$

Because the true scores, s_i, are never known, population reliability is never computed. Nevertheless, several estimates of reliability ($\hat{\rho}$) exist, and vary depending on the model assumptions. Indeed, previous work has also acknowledged that reliability and measurement error variance can be conditionalized (Hedeker, Mermelstein, Berbaum, & Campbell, 2009; Asparouhov et al., 2018; Li & Hedeker, 2012; Hedeker, Mermelstein, & Demirtas, 2012; Williams, Liu, Martin, & Rast, 2019; Leckie, French, Charlton, & Browne, 2014; Raju et al., 2007; Brunton-Smith, Sturgis, & Leckie, 2017; Hu et al., 2016). Here, we focus on the ω coefficient as an estimator for reliability (Bentler, 2009; Bacon, Sauer, & Young, 1995; Raykov, 1997, 2001; McNeish, 2018; Jöreskog, 1971), in the context of cross-sectional study designs.

Heterogenous Omega Coefficients

The ω coefficient is recommended as an estimate of composite score reliability (or communality), and is appropriate for congeneric latent variable models. The composite score is taken to be the unweighted sum-score of the indicators. The composite score reliability is therefore the estimated proportion of variance in the composite score that is due to the underlying latent variable. The ω coefficient for a unidimensional measure with no residual covariance is derived as follows (Bentler, 2009; Raykov, 2001; Jöreskog, 1971). Let s_i represent the true (asymptotic) sum score of the J indicators for case i. Let η_i represent the latent quantity of interest. The factor loading for each item j is represented by λ_j. We can now define the estimated true score ($\hat{s}$) and derive the associated conditional variance term $V_i(\hat{s}_i)$ decomposed into error components and latent components:

$$\begin{aligned}
\hat{s}_i &= s_i + e_i \\
\hat{s}_i &= \sum_{j=1}^{J} y_{ij} \\
\sum_j y_{ij} &= \sum_j \left(\nu_{ij} + \lambda_{ij}\eta_i + \epsilon_{ij}\right) \\
V_i(\hat{s}_i) &= \underbrace{\left(\sum_j \lambda_{ij}\right)^2 V_i(\eta_i)}_{V_i(s_i)} + \underbrace{\sum_j V_{ij}(\epsilon_{ij})}_{V_i(e_i)}.
\end{aligned}$$

Note that the variance V_i is subscripted indicating that each individual i retains its own variance term for its latent variables (η_i) and residuals (ϵ_{ij}). ν_{ij} represents an intercept term. The individual ω_i coefficient can be computed as

$$\omega_i = \frac{\left(\sum_j \lambda_{ij}\right) V_i(\eta_i)}{\left(\sum_j \lambda_{ij}\right) V_i(\eta_i) + \sum_j V_i(\epsilon_{ij})}. \quad (1)$$

If one assumes homoskedasticity and scalar invariance, Equation (1) simplifies to the more familiar form:

$$\omega = \frac{\left(\sum_j \lambda_j\right)^2 \sigma_\eta^2}{\left(\sum_j \lambda_j\right)^2 \sigma_\eta^2 + \sum_j \sigma_{\epsilon,j}^2}. \quad (2)$$

Coefficient ω therefore represents the sum-score reliability given the latent measurement model. That is, if the measurement model is accurate, then if a researcher were to measure each individual on the J items, and sum those items to produce a score, the coefficient ω reflects how much variability in the sum scores is explained by the latent factor.

ω improves upon other common measures of reliability by acknowledging that items vary in their relationship to the latent score and measurement error (McNeish, 2018; Dunn, Baguley, & Brunsden, 2014). In this form, ω assumes that the loadings and error variances are equivalent across time, persons, regions, or other grouping variables. Moreover, measurement error variance is constant across all possible cases yielding one fixed reliability value.

The omega coefficient comprises three components: The (expected) variance of the latent variable, the loadings, and the (expected) residual measurement error variances. As noted earlier by Raju et al. (2007), the commonly used constant ω simply reflects the expectation across case-specific reliability ($\omega = E(\omega_i)$) . Therefore there are at least three ways by which ω_i can vary: 1) via the loadings, 2) the latent factor variance, or 3) the error variances. We discuss these three in turn.

One approach to permitting case-specific, and conditional reliability is to make loadings conditional. That is, permit $\boldsymbol{\lambda}_i$ to vary between persons or by covariates. Indeed, continuous non-invariance approaches conditionalize loadings, such that the loadings are moderated by covariates, implying conditional reliability coefficients (Bauer, 2017). This approach, however, does not permit random effects of individuals on the reliability. This could be remedied by estimating a second latent variable that acts as an individual-specific random effect on each loading, but such a latent variable would be difficult to interpret, the corresponding model would be difficult to identify, and the scalar invariance assumption would be unmet (Meredith, 1993).

A second approach is to include conditional latent factor variance. This is possible using a scale model on the latent variance term, similar to modeling the between-group variance term in mixed effect location scale models (Hedeker, Mermelstein, & Demirtas, 2008; Rast et al., 2012). However, this again does not permit individual random effects on reliability. Therefore, the reliability estimates would not account for unique individual differences. Additionally, the reliability would vary only because of the variability in latent scores, and not due to variability in measurement error variance. Therefore, any covariates included in such a model would not provide insight into measurement insufficiencies (i.e., error variance; model misfit), and would be limited in psychometric utility.

Given the limitations in the first two approaches, we focus on the third: Allowing error variances to vary across cases and covariates. The idea is that individuals may have more or less measurement error variance, even after accounting for the possibility that it changes across circumstances and covariates. That is, both standard error of measure and, consequently, reliability can vary between persons and across covariate values. Therefore, we define the case-specific, heterogenous ω_i coefficient as follows:

$$\omega_i = \frac{(\sum \lambda_j)^2 \sigma_\eta^2}{(\sum \lambda_j)^2 \sigma_\eta^2 + \sum_j \sigma_{\epsilon,ji}^2}. \quad (3)$$

The loadings λ_j are fixed across all i cases, ensuring scalar invariance. The latent factor variances σ_η^2 also remain fixed for reasons outlined above. As discussed in the third approach, the residual variances, $\sigma_{\epsilon,ji}^2$, are allowed to change over different cases and indicators. It is important to underscore that the *variance* is allowed to vary across j and i – whereas the common assumption is one of a fixed variance across all i. By estimating case-specific residual variances, as indicated by the subscript i in $\sigma_{\epsilon,ji}^2$ (in contrast to Equation 2), ω_i will also vary across cases. Individuals with greater residual error variance terms are poorly represented by the measurement model (their responses are expected to spread further around their latent-projected values), which causes reliability to decrease, and latent score uncertainty to increase.

This approach has at least three benefits. First, it models a quantity of immediate psychometric interest – measurement error variance. That is, rather than modeling the true score variance, or allowing the loadings to be moderated, we model and vary the measurement error variance directly. Simply stated – for understanding individual differences in reliability, there is no clearer target than modeling the error variances directly. Psychometricians therefore obtain a reliability distribution, and the individual differences therein can be explained by modeling the error variance. An ideal measure would perhaps have little variability in the reliability distribution, with small expected error variance terms. This would imply that the expected departure from the measurement model is small, and consistent in the population of individuals; therefore, the uncertainty in latent scores is relatively small, and similar across covariates and the latent value itself.

Second, by keeping the loadings fixed, scalar invariance is assumed and latent score comparisons remain interpretable. The primary effect of allowing conditional error variance is on the *uncertainty* of the latent scores, not on their interpretation. Therefore, structural equation models using these scores remain interpretable, but are also informed more by those with greater reliability (decreased error variance) than

those with lesser reliability (increased error variance, therefore uncertain latent scores). This property is analogous to how reliability affects partial pooling of coefficients in multilevel models (Raudenbush & Bryk, 2002; Gelman, Hill, & Yajima, 2012).

Third, and as shown in the next section, estimating conditional error variance terms has firm foundations in both expectation algebra, and in distributional modeling techniques.

Latent Scale Model for Residual Variances

Lord and Novick (2008) mentioned the possibility of having conditional and unique measurement variance terms, $V_i(\epsilon_{it})$. This corresponds to the $\sum_j \sigma^2_{\epsilon,ji}$ term in equation (3). These variance terms can be a function of both individual random effects and covariates. Borrowing from the residual variance modeling literature (Lee & Nelder, 2006), the case-specific residual variances can be expressed as the exponentiated sum of fixed and random effects:

$$\boldsymbol{\sigma}^2_{\epsilon,i} = \exp(\boldsymbol{\nu}^\sigma + \boldsymbol{u}_{0i}), \tag{4}$$

where $\nu^\sigma_j \in \boldsymbol{\nu}^\sigma$ is the fixed component for the j-th residual variance, and $u_{0ji} \in \boldsymbol{u}_{0i}$ is the random effect of case i for indicator j. This submodel permits individualized reliability coefficients, but requires repeated observations. In fact, Equation (4) is merely a multilevel scale model, wherein loadings and intercepts are assumed fixed, but residual variances have random effects (Martin, Williams, & Rast, 2019; Mehta & Neale, 2005; Raykov & du Toit, 2005; Nestler, 2020). With single-observation data, this is therefore unidentifiable.

However, identifiability can be achieved by reducing the dimensionality of the random variance effects through a latent factor and a set of loadings. This dimension reduction ultimately underpins multivariate latent variable models as a whole. That is, a typical latent measurement model reduces an $N \times J$ matrix of random response effects to an N-length latent vector ($\boldsymbol{\eta}$) and J-length set of loadings ($\boldsymbol{\lambda}$). When combined, they produce an $N \times J$ matrix of projected response locations. The only difference here is that we also add a dimension reduction on the random effects of cases on *residual variances*. Specifically, the $N \times J$ matrix of random variance effects can be reduced to an N-length latent vector, $\boldsymbol{\eta}^\sigma$ and a J-length set of loadings, $\boldsymbol{\lambda}^\sigma$. For example, with J items (and therefore, $N \times J$ random variance effects):

$$\begin{bmatrix} u_{011} & u_{021} & \cdots & u_{0J1} \\ u_{012} & u_{022} & \cdots & u_{0J2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{01N} & u_{02N} & \cdots & u_{0JN} \end{bmatrix} = \begin{bmatrix} \eta^\sigma_1 \\ \eta^\sigma_2 \\ \vdots \\ \eta^\sigma_N \end{bmatrix} \begin{bmatrix} \lambda^\sigma_1 & \lambda^\sigma_2 & \cdots & \lambda^\sigma_J \end{bmatrix}. \tag{5}$$

To reiterate, the otherwise unidentifiable $N \times J$ matrix of random variance effects is instead estimated as an identifiable N-length set of factor scores and J-length set of loadings. Consequently, we obtain a second latent factor, $\boldsymbol{\eta}^\sigma$, that projects to the expected error variances for each case, on each item. We therefore have two types of latent factors, each modeling a different distributional parameter. The first type, $\boldsymbol{\eta}$, is the typical latent factor, which models the response *locations* (i.e., the μ parameter in the normal distribution). The second type, $\boldsymbol{\eta}^\sigma$, is the new latent factor, which models *variances* from the predicted response locations (i.e., extent of model error; the σ parameter in the normal distribution). See Figure 1 for a graphical display of the two factors.

Substituting the reduction in Equation (5) into Equations (4) and (3) yields the case-specific ω_i:

$$\omega_i = \frac{(\sum \lambda_j)^2 \sigma^2_\eta}{(\sum \lambda_j)^2 \sigma^2_\eta + \sum_j \exp(\nu_j + \eta^\sigma_i \lambda^\sigma_j)}. \tag{6}$$

In Equation (6), one can see that $\boldsymbol{\eta}^\sigma$ is effectively a *reliability factor*. If all λ^σ_js are constrained to be positive, then the latent scale factor is scaled such that those higher in $\boldsymbol{\eta}^\sigma$ tend to have greater measurement error variance, and consequently lower reliability. Conversely, if loadings are *negatively* constrained, then those with higher $\boldsymbol{\eta}^\sigma$ will have reduced measurement error variances, and therefore *increased* reliability. The choice of constraint is arbitrary, and only affects the interpretation of the latent scale factor. Because this paper is framed in terms of a *reliability* factor, rather than an *error* factor, we choose the latter constraint. Therefore, loadings are constrained to be negative, measurement error variance decreases with higher $\boldsymbol{\eta}^\sigma$, and reliability *increases*. Simply stated, negative loadings encode that those with a greater reliability factor score should have greater reliability.

In sum, the "reliability factor" is a latent factor which models the extent of model error (residual variability). Whereas the latent factor models the expected response for each item, the corresponding reliability factor models the expected residual variability for each item. Therefore, the reliability factor model is a multivariate location-scale model (Williams, Liu, et al., 2019; Asparouhov et al., 2018; Kapur et al., 2015; Nestler, 2020) with latent variables[1]. The ω_i formulation is composed of individual specific error variances, loadings, and an unconditional (expected) latent score variances. This latter unconditional variance may cause confusion to readers – If one is conditioning on an individual, then their latent value should be fixed, and therefore the variance (and reliability) should be zero. We wish to clarify this matter here. The formulated ω_i in equation (6) (and Eqs 10, 11 below) are conditional on a *reliability* factor score, but *not* on a particular latent score. The case-conditional ω_i therefore represents the expected proportion of composite score variance for a case with a *reliability* factor score, η^σ_i, assuming that latent scores are unknown random variables with variance $V(\eta^\mu_i)$. This approach therefore enables case-specific

[1]Note that the reliability factor models error variance in responses, and *not* intraindividual variability of responses, nor variance in latent factors.

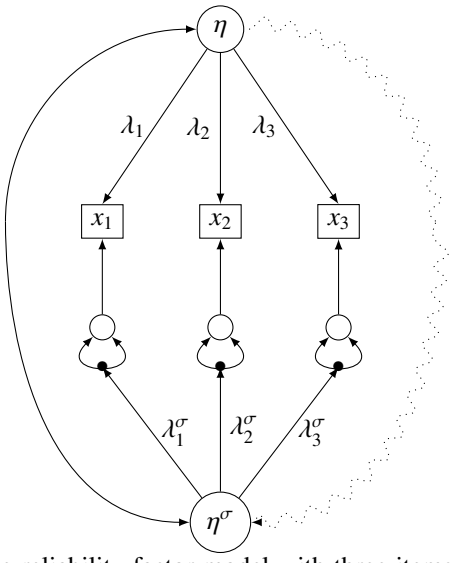

Figure 1. Graphical representation of the reliability factor model with three items. Each indicator has a predicted location and residual variance. The latent factor η underlies the expected responses for each indicator. The second latent factor, η^σ, underlies the residual variances for each indicator. The two factors can covary (black line) or be non-linear functions of another (dotted line). Intercepts for the indicators and residual variances are modeled, but not included in the figure.

reliabilities under the local homogeneity assumption (Ellis & van den Wollenberg, 1993). In the next section, we describe the full model in detail, and then describe how the reliability factor model produces individual ω_i's.

Model Specification and Estimation

It is straightforward to expand the unidimensional case, described above, to the general multidimensional case. That is, multiple latent score factors, and corresponding reliability factors, are permitted, with cross-loadings and correlated residual variances.

Let x_{ij} be the i-th response ($i \in 1 \dots N$) to indicator $j \in 1 \dots J$. As usual, assume each response location is linearly modeled from intercepts, $\boldsymbol{\nu}$, latent variables, $\boldsymbol{\eta_i}$, and respective loadings $\boldsymbol{\Lambda}$,

$$\begin{aligned} \boldsymbol{x}_i &= \boldsymbol{\nu} + \boldsymbol{\eta}_i\boldsymbol{\Lambda} + \boldsymbol{\epsilon}_i \\ \boldsymbol{\epsilon}_i &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\delta}_i). \end{aligned}$$

Setting the measurement error covariance to be equal across all cases, $\boldsymbol{\delta_i} = \boldsymbol{\delta}$, encodes the traditional normal-assumptive latent variable model. To reiterate, the reliability factor model instead assumes that measurement error variance itself varies case by case, such that some cases may have more or less measurement error variance as a whole than others. Therefore, $\boldsymbol{\delta_i}$ is subscripted with an i and is permitted to vary here as a function of an underlying reliability factor (η_i^σ).

Because the reliability factor acts on the residual variability, it is beneficial to separate the residual covariance matrix, $\boldsymbol{\delta}_i$, into a diagonal scale matrix, $\boldsymbol{\sigma}_i$, and a correlation matrix, $\boldsymbol{\Omega}$, such that $\boldsymbol{\delta}_i = \boldsymbol{\sigma}_i\boldsymbol{\Omega}\boldsymbol{\sigma}_i$. The separation strategy is common to scale models (Rast & Ferrer, 2018; Rast et al., 2012; Hedeker et al., 2009), and improves the reliability and efficiency of the estimation algorithm (Liu, Zhang, & Grimm, 2016; Barnard, McCulloch, & Meng, 2000).

As is customary in distributional scale modeling[2], a log-linear model is imposed on the diagonal of the scale matrix, $\boldsymbol{\sigma}_i$, with the latent reliability factor scores as the predictors. The additive log-linear model is primarily used to ensure that, once the linear predictor is exponentiated, the modeled variance remains positive

$$\text{diag}(\boldsymbol{\sigma}_i)' = \exp(\boldsymbol{\nu}^\sigma + \boldsymbol{\eta}_i^\sigma\boldsymbol{\Lambda}^\sigma).$$

Therefore, the latent reliability factors, $\boldsymbol{\eta}_i^\sigma$, are informed by the residual variability through their respective factor loadings. If $\boldsymbol{\Lambda}^\sigma$ is constrained to $\mathbf{0}$, or if an intercept-only scale model is used ($\sigma_{ij} = \exp(\nu_j^\sigma)$), then the reliability factor model is identical to a typical unconditional variance latent model. It is important to clarify that the reliability factor does *not* model the residual *errors*, but instead models the residual error *variances*. It therefore neither implies nor imposes correlation between residual *errors* (as optionally estimated in $\boldsymbol{\Omega}$), and does not conflict with the psychometric goal of having zero correlation between indicators after removing the

[2] A traditional Gaussian model assumes that, e.g., $y \sim \mathcal{N}(\mu = f(X), \sigma^2)$. A location-scale Gaussian model includes a second submodel on the scale parameter, σ^2, to model variance: $y \sim \mathcal{N}(\mu = f(X), \sigma^2 = g(X))$. Because σ^2 must be positive, a log-link function is used in the submodel, $\sigma^2 = \exp(g(X))$. The reliability factor approach uses this exact strategy in its formulation.

substantive latent factor.

The reliability factor model includes two types of latent variables. "Latent factors" are included in the mean-structure model to predict response locations. The latent factors therefore model the $\boldsymbol{\mu}$ parameter of the normal distribution. "Reliability factors" are included in the variance-structure model to predict response error variances. Reliability factors therefore model the diagonal of the $\boldsymbol{\Sigma}$ parameter of the normal distribution. Both sets of latent variables are jointly distributed and permitted to covary. We employ a unit-variance identification, such that

$$\begin{bmatrix} \boldsymbol{\eta}_i \\ \boldsymbol{\eta}_i^\sigma \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}^{ff} & \boldsymbol{\Phi}^{fr} \\ \boldsymbol{\Phi}^{rf} & \boldsymbol{\Phi}^{rr} \end{bmatrix}\right), \tag{7}$$

where $\boldsymbol{\Phi}$ is a correlation matrix (i.e., a covariance matrix with a unit diagonal). This permits all non-zero loadings to be estimated, rather than arbitrarily choosing which to fix to 1. The latent correlation matrix is composed of submatrices containing correlations between the factor scores, $\boldsymbol{\Phi}^{ff}$, between reliability factor scores, $\boldsymbol{\Phi}^{rr}$, and correlations between factor and reliability factor scores, $\boldsymbol{\Phi}^{rf} = \boldsymbol{\Phi}^{fr'}$.

To treat the reliability factors as endogenous, the 0-vector in Equation (7) can be replaced by the desired structural equations, and the stochastic error term can remain fixed to one[3]. Most generally, the reliability factors can be modeled as a function (f) of exogenous variables ($\boldsymbol{X}$) or the factor scores themselves ($\boldsymbol{\eta}$), or both:

$$\begin{bmatrix} \boldsymbol{\eta}_i \\ \boldsymbol{\eta}_i^\sigma \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{0} \\ f(\boldsymbol{\eta}_i, \boldsymbol{X}_i) \end{bmatrix}, \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}^{ff} & \boldsymbol{\Phi}^{fr} \\ \boldsymbol{\Phi}^{rf} & \boldsymbol{\Phi}^{rr} \end{bmatrix}\right).$$

When the reliability factor is an endogenous variable, the covariance matrix, $\boldsymbol{\Phi}$, contains the covariances between the latent factors and the stochastic errors in the reliability factors. If $f(\boldsymbol{\eta}_i, \boldsymbol{X}_i) = f(\boldsymbol{X}_i)$, then $\boldsymbol{\Phi}^{rf}$ contains the standardized residual covariance between factors and reliability factors. If $f(\boldsymbol{\eta}_i, \boldsymbol{X}_i)$ is fully estimated, then $\boldsymbol{\Phi}$ is a block-diagonal covariance matrix, wherein $\boldsymbol{\Phi}^{rf} = \boldsymbol{\Phi}^{fr'} = \mathbf{0}$. Consequently, one correlation matrix, $\boldsymbol{\Phi}^{ff}$, contains correlations between latent factors, and another correlation matrix, $\boldsymbol{\Phi}^{rr}$, contains residual correlations between reliability factors.

From reliability factor to reliability

We provided the formula for a univariate ω in Equation (2), and individual univariate ω in Equation (6). These were specific, simplified cases of the more general ω coefficients. In this section, we provide the formulas for two coefficients, ω_1 and ω_2, for multidimensional measurement models with cross-loadings and residual covariances. The general ω formulas are then expanded to include the reliability factor, and therefore to provide individual coefficients.

Let f index the factor, ψ_f represent the factor variance, $\boldsymbol{\Lambda}_f$ represent the row of loadings for factor f, and $\boldsymbol{\delta}$ represent the residual covariance matrix. Let $\mathbf{1}_f$ represent a dummy coded vector of length J (the number of items), that is 1 when the item loads onto the factor, and 0 when it does not. Then coefficients ω_1 and ω_2 for factor f are defined as:

$$\omega_{1,f} = \frac{(\boldsymbol{\Lambda}_f \mathbf{1}_f)^2 \psi_f}{(\boldsymbol{\Lambda}_f \mathbf{1}_f)^2 \psi_f + \mathbf{1}_f' \boldsymbol{\delta} \mathbf{1}_f} \tag{8}$$

$$\omega_{2,f} = \frac{(\boldsymbol{\Lambda}_f \mathbf{1}_f)^2 \psi_f}{\mathbf{1}_f' \hat{\boldsymbol{\Sigma}} \mathbf{1}_f} \tag{9}$$

$$\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Lambda}' \boldsymbol{\Psi} \boldsymbol{\Lambda} + \boldsymbol{\delta}$$

$$\boldsymbol{\delta} = \boldsymbol{\sigma} \boldsymbol{\Omega} \boldsymbol{\sigma}.$$

The coefficients in Equations (8) and (9) each represent a proportion of variance in sum scores due to a latent factor. The numerators represent the variance in sum scores that is predicted by the factor f. The corresponding denominators represent an estimate of total variance in sum scores. The two coefficients differ only in the denominator. ω_1 indicates the reliability of the measure after removing the predictive variance of other factors. Conversely, ω_2 indicates the reliability of the measure without removing the variance of other factors. Therefore, the two coefficients are equivalent when employing a unidimensional model, or multidimensional models with no cross-loadings.

As defined thus far, the ω coefficients depend only on estimated, fixed model parameters. However, by including a reliability factor for each latent factor, the estimated residual covariance matrix varies across observations. Consequently, the ω coefficients vary across observations. Computing the observation-specific ω coefficients is straightforward.

First, the model-implied residual standard deviations are obtained from the reliability factor model:

$$\text{diag}(\boldsymbol{\sigma}_i)' = \exp(\boldsymbol{\nu}^\sigma + \boldsymbol{\eta}_i^\sigma \boldsymbol{\Lambda}^\sigma).$$

Second, the observation-specific residual covariance is estimated by combining the diagonal SD matrix, $\boldsymbol{\sigma}_i$ with the residual correlation matrix, $\boldsymbol{\Omega}$:

$$\boldsymbol{\delta}_i = \boldsymbol{\sigma}_i \boldsymbol{\Omega} \boldsymbol{\sigma}_i.$$

When estimating $\omega_{2,f,i}$, the model-implied covariance matrix can then be computed for each case:

$$\hat{\boldsymbol{\Sigma}}_i = \boldsymbol{\Lambda}' \boldsymbol{\Psi} \boldsymbol{\Lambda} + \boldsymbol{\delta}_i,$$

where $\boldsymbol{\Psi}$ is the covariance of latent factors (i.e., *not* including reliability factors). When the latent factors are exogenous and standardized, $\boldsymbol{\Psi} = \boldsymbol{\Phi}^{ff}$. Finally, the observation-specific coefficients can be computed by plugging the quantities into

[3]The latent factors ($\boldsymbol{\eta}_i$) can be endogenously modeled as per usual, but this is not discussed here.

Equations (8) and (9), resulting in reliability scores that vary across individuals:

$$\omega_{1,f,i} = \frac{(\boldsymbol{\Lambda}_f \mathbf{1}_f)^2 \psi_f}{(\boldsymbol{\Lambda}_f \mathbf{1}_f)^2 \psi_f + \mathbf{1}'_f \boldsymbol{\delta}_i \mathbf{1}_f} \tag{10}$$

$$\omega_{2,f,i} = \frac{(\boldsymbol{\Lambda}_f \mathbf{1}_f)^2 \psi_f}{\mathbf{1}'_f \hat{\boldsymbol{\Sigma}}_i \mathbf{1}_f}. \tag{11}$$

In order to compare the reliability factor approach to traditional, fixed coefficients, researchers may wish to compute a single "mean" reliability coefficient. The mean reliability can be computed using one of two in-equivalent methods. The first method is to compute $\mathrm{E}(\omega|\boldsymbol{\eta}^\sigma = 0)$, or the value of ω when the reliability factor is at zero. This approach is simple, because it only requires $\boldsymbol{\nu}^\sigma$ when computing $\boldsymbol{\sigma}_i$. However, this does not estimate the expected reliability, per se, because of the non-linear mapping between the reliability factors and the ω coefficients. That is, the coefficient at the mean value of the reliability factor may not be equivalent to the mean coefficient across the reliability factor. Therefore the second approach is to directly estimate the mean ω_i across observations, across the posterior distribution. It has been shown (Raju et al., 2007) that this latter expectation is asymptotically equivalent to the ω value computed using the traditional, fixed approach.

Similarly, researchers may wish to express uncertainty and variability in individual and mean reliability estimates. Because we are employing Bayesian estimation, uncertainty can be expressed through the posterior distribution. Every individual ω_i and both ω mean quantities therefore have posterior variance and credible intervals. Moreover, $\mathrm{Var}(\omega_i)$ can be computed across the posterior, thereby providing an estimate (with uncertainty) about the extent of variability across individual ω_i values. These posterior quantities are all implemented in the "omegad" **R-**package (Martin, Williams, & Rast, 2020) which offers linear and non-linear options for defining the unknown reliability function f.

In sum, the reliability factor model contains two sets of latent variables. In addition to the typical latent factors, which underlie the *location* of the responses, the reliability factors underlie the *residual scale* of the responses. Therefore, the reliability factor encodes predictive inadequacy for each respondent. Constraining the reliability factor loadings to be negative provides an intuitive scaling with respect to reliability, such that those with higher reliability factors will have decreased error variance, and therefore greater reliability estimates. The reliability factor can be linearly or non-linearly predicted from exogenous variables and from the corresponding latent factor itself. This enables researchers to find possible threats to reliability, and examine whether certain regions of the factor space have diminished reliability (Feldt & Quails, 1996; Lek & Van De Schoot, 2018; de Ayala, 2009).

Estimation

The model is estimated in a Bayesian framework. The R package, "omegad" (Martin et al., 2020), provides a user-friendly interface for fitting and post-processing the models presented in this paper. Estimation itself is performed through Stan (Carpenter et al., 2017). Stan implements an efficient Hamiltonian monte carlo sampler that is suitable for high dimensional Bayesian models. It samples from the joint posterior distribution, in turn providing the marginal distributions for each parameter of interest.

It is important to note that the proposed reliability factor model must be fitted to the raw observed data, and not the covariance matrices. Unlike many factor model estimators, covariance or correlation matrices are insufficient. The reliability factor model includes a model on the residual scales for each case. This means that the likelihood for each case includes a case-specific expected covariance matrix. Therefore, a sample-wide summary statistic is insufficient for estimation.

Priors

We recommend standardizing the indicators and continuous exogenous variables. The scaling is arbitrary, but standardization can improve MCMC efficiency and ease prior elicitation. Moreover, ω_i and ω are invariant to scaling. The suggested priors all assume approximately standardized variables. Therefore, all numeric exogenous predictors and all indicators are standardized prior to estimation. However, the posterior distributions can be transformed to their unstandardized equivalents if desired. With standardized data, we use the following priors for the parameters. The non-zero loadings ($\lambda \in \Lambda$) are given independent standard half-normal priors, such that:

$$\boldsymbol{\lambda} \sim \mathcal{N}^+(0, 1)$$
$$\boldsymbol{\lambda}^\sigma \sim \mathcal{N}^-(0, 1).$$

These priors are weakly informative for standardized data. Note that we impose a negativity constraint ($\mathcal{N}^-$) on the reliability factor loadings. The reliability factor loadings are all negative because its relationship to error should be strictly unidirectionally negative. Under this parameterization, larger reliability factor scores should imply less error variability, and therefore greater reliability estimates.

However, the factor loadings are given a positivity constraint ($\mathcal{N}^+$) for identifiability. Strictly speaking, only one factor loading needs a positivity constraint to be identified. In practice, we find that imposing a positivity constraint on all loadings, and reversing the direction of reverse-scaled indicators, leads to more efficient MCMC sampling. This is especially the case when the one sign-constrained indicator is relatively weakly related to the latent factor.

The intercepts are given independent normal priors:

$$\begin{aligned}\boldsymbol{\nu} &\sim \mathcal{N}(0, 1)\\ \boldsymbol{\nu}^{\sigma} &\sim \mathcal{N}(-.5, 1).\end{aligned}$$

Recall that the residual standard deviations are modeled through a log link function. Therefore, the prior for $\boldsymbol{\nu}^{\sigma}$ implies a $\mathcal{LN}(-.5, 1)$ prior on the residual SD when the reliability factor is at its mean of zero. On standardized data, this yields a weakly informative prior over the residual SDs.

Both the latent factor scores and reliability factor scores are assumed normally distributed. Most generally, the prior can be expressed as:

$$\begin{bmatrix}\boldsymbol{\eta}_i\\ \boldsymbol{\eta}_i^{\sigma}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\mathbf{0}\\ f(\boldsymbol{\eta}_i, \boldsymbol{X}_i)\end{bmatrix}, \boldsymbol{\Phi}\right).$$

The prior on $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^{\sigma}$ encodes a unit-variance latent factor identification. The latent and reliability factors are assumed to have a mean of zero. In the case of endogenous reliability factors, the intercepts are set to zero. The use of a correlation matrix identifies the model by fixing both exogenous and endogenous variances to 1.

The correlation matrices are assigned a uniform joint prior:

$$\begin{aligned}\boldsymbol{\Omega} &\sim \text{LKJ}(1)\\ \boldsymbol{\Phi} &\sim \text{LKJ}(1) && (\text{If } \boldsymbol{\Phi}^{rf} \neq \mathbf{0})\\ \boldsymbol{\Phi}^{ff} &\sim \text{LKJ}(1) && (\text{If } \boldsymbol{\Phi}^{rf} = \mathbf{0})\\ \boldsymbol{\Phi}^{rr} &\sim \text{LKJ}(1). && (\text{If } \boldsymbol{\Phi}^{rf} = \mathbf{0})\end{aligned}$$

The LKJ prior (Lewandowski, Kurowicka, & Joe, 2009) is a spherical prior over correlation matrices with a single parameter. The parameter determines how much prior weight is placed on an identity matrix. When it is set to one, the prior is uninformative, and is uniform over all permissible correlation matrices.

Finally, any parameters defined in $f(\boldsymbol{\eta}_i, \boldsymbol{X}_i)$ must be assigned a prior. One such function is a linear model containing exogenous variables ($\boldsymbol{X}$) or the latent factors ($\boldsymbol{\eta}$), or both. Linear predictors of the reliability factors are assigned independent standard normal priors over their coefficients, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, respectively.

$$\begin{aligned}\beta &\sim \mathcal{N}(0, 1)\\ \gamma &\sim \mathcal{N}(0, 1)\end{aligned}$$

Altogether, the full model can be written as follows:

$$\begin{aligned}\boldsymbol{x}_i &\sim \mathcal{N}\left(\boldsymbol{\nu} + \boldsymbol{\eta}_i\boldsymbol{\Lambda}, \boldsymbol{\delta}_i\right)\\ \boldsymbol{\delta}_i &= \boldsymbol{\sigma}_i\boldsymbol{\Omega}\boldsymbol{\sigma}_i\\ \text{diag}(\boldsymbol{\sigma}_i)' &= \exp(\boldsymbol{\nu}^{\sigma} + \boldsymbol{\eta}_i^{\sigma}\boldsymbol{\Lambda}^{\sigma})\\ \begin{bmatrix}\boldsymbol{\eta}_i\\ \boldsymbol{\eta}_i^{\sigma}\end{bmatrix} &\sim \mathcal{N}\left(\begin{bmatrix}\mathbf{0}\\ f(\boldsymbol{\eta}_i, \boldsymbol{X}_i)\end{bmatrix}, \boldsymbol{\Phi}\right)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nu} &\sim \mathcal{N}(0, 1)\\ \boldsymbol{\nu}^{\sigma} &\sim \mathcal{N}(-.5, 1)\\ \boldsymbol{\lambda} &\sim \mathcal{N}^{+}(0, 1)\\ \boldsymbol{\lambda}^{\sigma} &\sim \mathcal{N}^{-}(0, 1)\\ \boldsymbol{\Omega} &\sim \text{LKJ}(1)\\ \boldsymbol{\Phi} &\sim \text{LKJ}(1)\end{aligned}$$

Model Comparison

As mentioned in the estimation section above, the reliability factor model requires a full information approach, therefore covariance matrices are not sufficient statistics for estimation. By construction, the implied covariance matrix for each case is estimated. Consequently, traditional model fit indices (e.g., RMSEA, CFI) cannot be computed – there are multiple model-implied matrices (one per case), but only one observed covariance matrix (computed across the sample) considered by fit indices (Zhang & Savalei, 2019). Similarly, the computation of fit indices under FIML requires saturated models for the discrepancy statistic (Zhang & Savalei, 2019); however, case-specific saturated models are not possible without repeated measures. Therefore computing fit indices is not currently possible. For model robustness checking, criticism, and improvement, we recommend using techniques from the modern Bayesian workflow (Gelman et al., 2020), including posterior predictive checking.

Additionally, likelihood-based metrics and model comparisons can be used. Approximate, pareto-smoothed leave-one-out cross-validation (LOO) is a recent, but popular tool for model comparison in a Bayesian framework. To that end, our "omegad" package implements LOO for model comparison (Vehtari, Gelman, & Gabry, 2017; Martin et al., 2020). LOO can be used to assess whether the reliability factor model improves predictive accuracy over and above a "null model" that excludes the reliability factor. If the LOO performance of the null model is better than, or is not sufficiently different from, the reliability factor model, then the reliability factor model can be excluded, and reliability may be sufficiently described more traditionally as a single coefficient. In all examples below, we provide the difference in the LOO performance (with standard errors) between the null and candidate models.

Examples

In the next several sections, we employ the reliability factor model on simulated data (Examples 1 and 2) and the classic Holzinger-Swineford dataset (Example 3). The examples are increasingly complex, starting with a univariate model and linearly covarying reliability factor. The second example includes a non-linear Gaussian process between the factor and reliability factor. Finally, the third example is multi-

dimensional, with a Gaussian process model between factors and their reliability factors, and with exogenous predictors of reliability factors. In all demonstrations, the reliability factor model is fitted, and individual-specific ω_i coefficients are computed using the above methodology.

Example 1: Covariance between a factor and reliability factor

In this section, we fit the model on a simulated unidimensional dataset with 300 observations and eight indicators. The first model is a covariance-only model with no residual covariances. That is, the relationship between the factor and reliability factor is estimated through their covariance only, and no exogenous variables nor residual correlations are included: $f(\eta_i, \boldsymbol{X_i}) = 0$.

The true and estimated parameters are provided in Table 1. The model converged with all Rhats ≈ 1, and no divergent transitions (Gelman & Rubin, 1992; Betancourt, 2017). The parameters were recovered adequately, with all true values falling within the credible intervals. The estimated expected reliability, $E(\omega_i)$, is approximately .879. As expected, this is close to the traditional ω estimate of .854. However, the traditional estimate is unable to convey the changing and varying reliability. Indeed, the difference in LOO performance between the null model (no reliability factor, constant reliability) and the fitted model is 642.8 (63.9) in favor of the fitted model. This clearly suggests that the reliability factor improves fit, and therefore a single reliability estimate is insufficient.

Indeed, the factor and reliability factor negatively covary. As Figure 2 shows, this negative covariance implies that reliability is monotonically decreasing across the factor space. A greater reliability factor implies smaller residual variances, and therefore increased reliability. Therefore, as factor scores increase, reliability is expected to substantially decrease. Although the average and traditional reliability estimates are both reasonably high, reliability at higher factor scores is predicted to be abysmal.

The posterior intervals are well-calibrated, if slightly conservative. The "true" ω_i values can be computed from their generated reliability factor scores and the model parameters. The true ω_i values fell within the 95% credible intervals 98% of the time. Similarly, the model predicted new ω_i's for newly generated factor scores; the true ω_i values fell within the 95% posterior predictive intervals 96% of the time. Therefore, the uncertainty estimates of ω_i were well-calibrated both in and out of sample.

Example 2: Reliability factor as a flexible function of a latent factor

Although Example 1 adequately estimated the loadings and intercepts, the model assumed that the latent and reliability factors covary linearly. In reality, the reliability factor was truly generated as a non-linear, non-monotonic function of the latent factor. Specifically, $\eta_i^\sigma = f(\boldsymbol{\eta})_i + \epsilon_i = -.6\eta_i - 3\sin(2\eta_i) + \epsilon_i$, $\epsilon_i \sim \mathcal{N}(0, 1)$. This was chosen as a relatively obvious non-linear example to demonstrate the utility of a Gaussian process (GP) approach.

When estimating GP models, the "omegad" package utilizes an approximation technique that greatly reduces the computational burden and improves convergence (Solin & Särkkä, 2019). The technique reduces the otherwise $N \times N$ gram matrix to an $N \times M$ matrix and M-length vector of eigenvalues, where M is the number of basis functions and $M < N$. The number of basis functions must be defined at estimation time. We recommend a minimum of ten, but the approximation accuracy is improved with a greater number of basis functions, at the cost of estimation time. The approximation requires a constant (L) for which the range (-L, L) adequately bounds the predictor values. Assuming standardized predictors and factors, a boundary constant of 7.5 is more than enough to contain the full range of plausible values.

In this example, we permit the reliability factor to be non-monotonically modeled from the factor through an approximate Gaussian process (with $M = 40$ basis functions). That is, we assume the reliability factor is an unknown function of the latent factor, with stochastic error: $\eta_i^\sigma \sim \mathcal{N}(f(\eta)_i, 1)$. Despite GP's being underutilized within the structural equation modeling literature, they permit estimation of and prediction from arbitrarily structured unknown functions between variables, with expressed uncertainty. Because GP's are non-parametric and highly flexible, we require very few assumptions about the functional form of f. Specifically, we assume that $f \sim GP(\boldsymbol{\eta}\gamma, K(\boldsymbol{\eta}, \boldsymbol{\eta'}))$, where K is an approximate exponential quadratic kernel. The function is therefore assumed to have a linear component ($\boldsymbol{\eta}\gamma$), and a flexible component (encoded in $K(\boldsymbol{\eta}, \boldsymbol{\eta'})$). The exponential quadratic kernel assumes that nearby inputs produce nearby outputs. This simple assumption permits the GP to recover a universal set of continuous, non-linear, non-monotonic functions that are otherwise difficult to express, parameterize, or estimate. It is therefore a generic method for estimating any continuous function, including linear functions. The kernel only has two parameters: The marginal standard deviation of the function output, α, and the length scale, l. We do not interpret the parameters, but only use the Gaussian process to model and predict the reliability factor, and therefore the implied ω_i coefficients.

The true and estimated parameters are provided in Table 2. Once again, the true loadings and intercepts were recovered adequately, and are comparable to those obtained from the covariance-only model. However, the Gaussian process model between the reliability factor and the latent factor permits non-monotonic relationships between the two, and therefore between the factor scores and reliability. Indeed,

Table 1
Estimates for example 1 (Covariance between latent and reliability factor).

Item	$\hat{\nu}_j$ (0)	$\hat{\lambda}_j$ (.7)	$\hat{\nu}_j^\sigma$ (-.5)	$\hat{\lambda}_j^\sigma$ (-.6)
1	-0.005 [-0.111, 0.104]	0.753 [0.661, 0.848]	-0.549 [-0.679, -0.422]	-0.650 [-0.784, -0.525]
2	-0.058 [-0.172, 0.056]	0.773 [0.678, 0.877]	-0.444 [-0.581, -0.313]	-0.699 [-0.831, -0.578]
3	-0.049 [-0.160, 0.063]	0.791 [0.694, 0.894]	-0.545 [-0.665, -0.423]	-0.592 [-0.718, -0.481]
4	-0.073 [-0.178, 0.030]	0.724 [0.637, 0.818]	-0.551 [-0.676, -0.425]	-0.663 [-0.786, -0.549]
5	-0.054 [-0.161, 0.054]	0.746 [0.650, 0.846]	-0.490 [-0.604, -0.375]	-0.564 [-0.675, -0.458]
6	-0.085 [-0.188, 0.019]	0.693 [0.598, 0.791]	-0.515 [-0.635, -0.394]	-0.604 [-0.733, -0.490]
7	-0.040 [-0.153, 0.072]	0.767 [0.668, 0.875]	-0.426 [-0.549, -0.306]	-0.568 [-0.698, -0.442]
8	-0.051 [-0.159, 0.056]	0.740 [0.646, 0.842]	-0.497 [-0.614, -0.379]	-0.583 [-0.701, -0.470]
	$\Phi_{1,1'}$ (-.496)	E($\omega|\eta^\sigma = 0$) (.933)	E(ω_i) (.894)	SD(ω_i) (.106)
	-.539 [-.639, -.427]	.923 [.904, .939]	.879 [.860, 897]	.121 [.105, .138]

Estimates are provided [with 95% credible intervals]. Values in parentheses next to parameters are the true values.

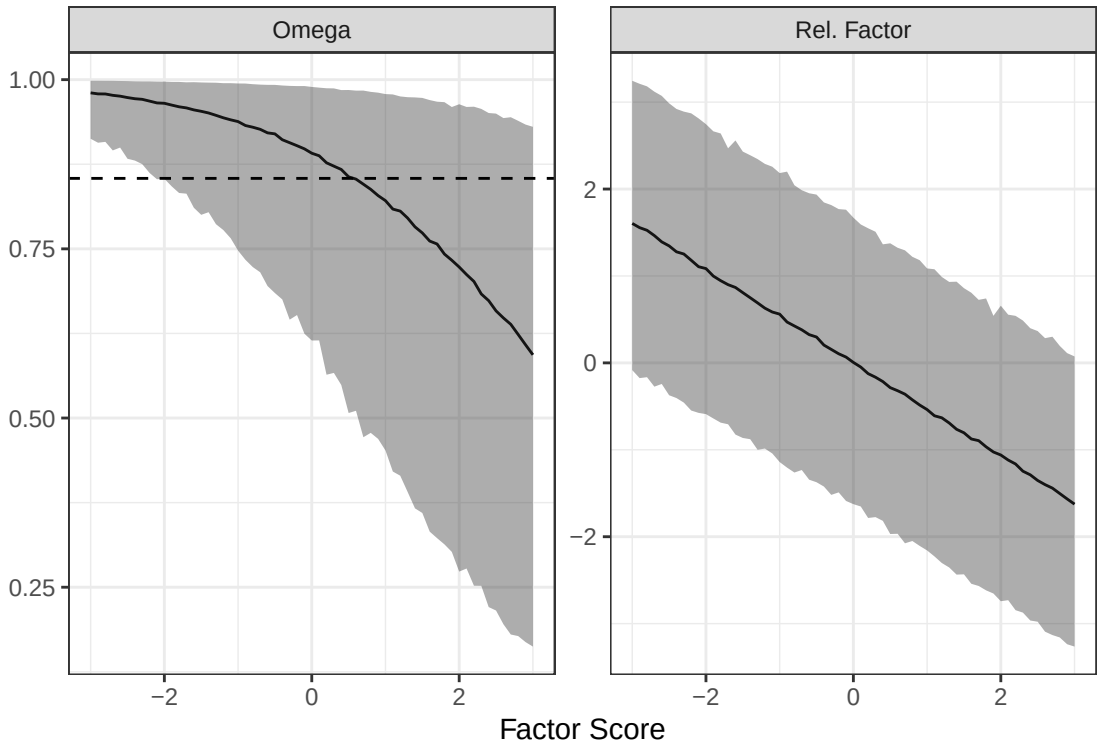

Figure 2. Predicted values of ω_i (Left) and Reliability factor scores, η^σ (right) as a function of the factor score. The factor and reliability factor have notable negative covariance. Higher factor scores imply lower reliability factors. Higher reliability factors imply smaller residual variances, and therefore increased reliability. The negative covariance between the factor and reliability factor therefore implies a monotonically decreasing ω reliability. The shaded regions represent 95% *predictive* intervals. The horizontal dashed line is the traditional ω estimate.

Table 2
Estimates for example 2 (Gaussian process between latent and reliability factor).

Item	$\hat{\nu}_j$ (0)	$\hat{\lambda}_j$ (.7)	$\hat{\nu}_j^\sigma$ (-.5)	$\hat{\lambda}_j^\sigma$ (-.6)
1	-0.015 [-0.080, 0.050]	0.727 [0.654, 0.798]	-0.534 [-0.634, -0.438]	-0.638 [-0.741, -0.536]
2	-0.050 [-0.120, 0.021]	0.757 [0.678, 0.833]	-0.454 [-0.555, -0.352]	-0.712 [-0.819, -0.613]
3	-0.040 [-0.110, 0.029]	0.780 [0.704, 0.854]	-0.553 [-0.652, -0.455]	-0.589 [-0.689, -0.488]
4	-0.068 [-0.134, -0.001]	0.717 [0.647, 0.790]	-0.544 [-0.641, -0.443]	-0.636 [-0.732, -0.543]
5	-0.044 [-0.111, 0.021]	0.735 [0.661, 0.807]	-0.499 [-0.599, -0.402]	-0.586 [-0.680, -0.496]
6	-0.077 [-0.145, -0.010]	0.702 [0.626, 0.773]	-0.524 [-0.622, -0.423]	-0.575 [-0.681, -0.478]
7	-0.045 [-0.116, 0.025]	0.757 [0.681, 0.833]	-0.449 [-0.546, -0.353]	-0.569 [-0.679, -0.462]
8	-0.050 [-0.117, 0.020]	0.736 [0.659, 0.814]	-0.499 [-0.594, -0.406]	-0.589 [-0.685, -0.496]
β	α, l	E($\omega|\eta^\sigma = 0$) (.933)	E(ω_i) (.894)	SD(ω_i) (.106)
-.108	1.159, .820	.913 [.802, .971]	.876 [.857, .893]	.121 [.106, .138]

Estimates are provided [with 95% credible intervals]. Values in parentheses next to parameters are the true values. Because the reliability factor is endogenous (non-monotonically modeled from the factor), the variance is greater than one, and therefore the loadings differ. To facilitate comparison with table 1, the estimates were rescaled assuming the reliability factor variance is one. β is the linear coefficient between η and η^σ. α and l are the gaussian process kernel parameters.

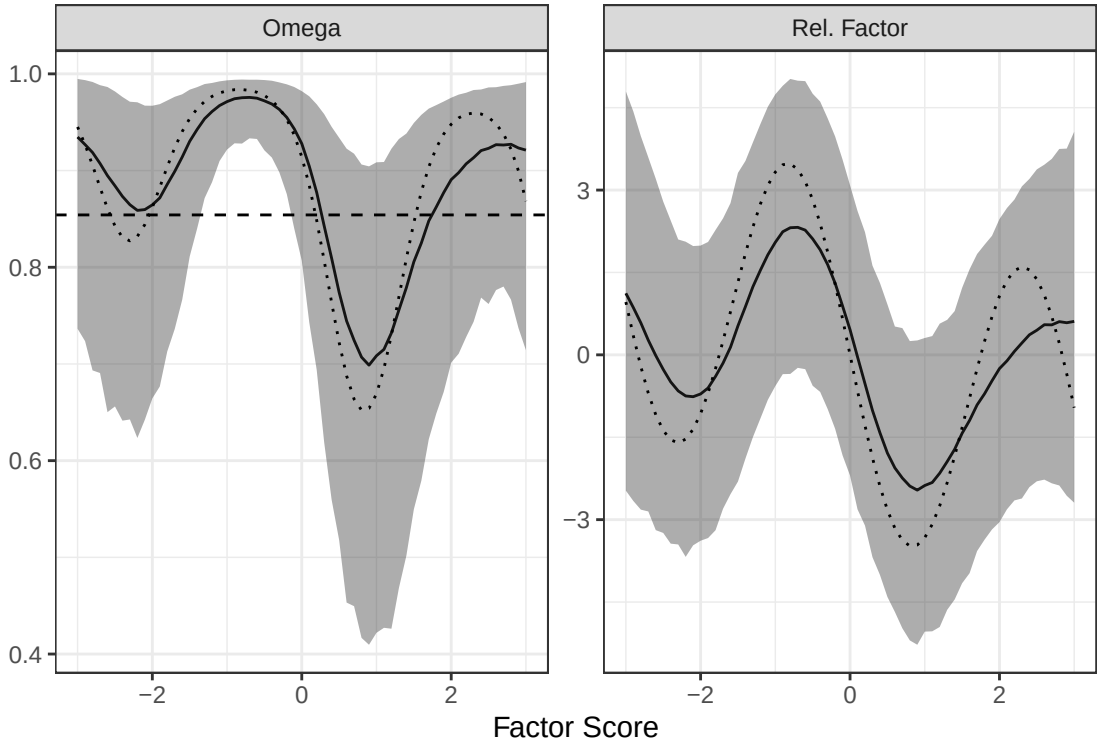

Figure 3. Expected values of ω_i (Left) and reliability factor scores, η^σ (Right), as a function of the factor score. The factor is non-monotonically related to its reliability factor. Consequently, reliability can be higher or lower across the latent space. The dotted line represents the true generative function: $\eta_i^\sigma = -.6\eta_i - 3\sin(2\eta_i) + \epsilon_i$. The shaded regions represent 95% *predictive* intervals. The horizontal dashed line is the traditional ω estimate.

the addition of the GP model greatly improves LOO performance relative to the covariance-only model (69.8 [12.9]) and the null model (712.7 [64.3]).

This non-monotonicity is readily seen in Figure 3. Note that the Gaussian process model was able to approximate the true non-monotonic function reasonably well, without defining the functional structure. The true function was indeed within the 95% credible intervals of the estimated function. The GP permits the non-linear relationship between the latent factor and ω_i to manifest. Although the mean reliability coefficients are nearly equivalent between examples 1 and 2, the conditional reliabilities starkly differ. Unlike in Example 1, the reliability of the instrument is strong for low scores and high scores, but weaker in particular regions therein. Specifically, Figure 3 suggests that reliability for factor scores between .4 and 1.5 is expected to be below .8.

Once again, the posterior intervals are conservatively calibrated. The "true" ω_i values were computed from the generated reliability factor scores and parameters, and they fell within the 95% credible intervals 98% of the time. Similarly, the fitted model was used to predict ω_i from new latent scores; the corresponding true ω_i values fell within the 95% posterior predictive intervals 99% of the time. Therefore, both the in-sample and out-of-sample performance of ω_i estimation were conservatively calibrated (Yang, Bhattacharya, & Pati, 2017).

In sum, the example demonstrates how the reliability factor can be non-linearly and non-monotonically related to its corresponding factor. Consequently, the conditional reliability of the measure non-monotonically changes across the span of the latent factor itself. That is, there is no single estimate of reliability, because the reliability of the measure depends on the respondent's factor score.

Example 3: Exogenous predictors and multiple factors

For this example, we turn to a classic dataset from Holzinger and Swineford (1939). We chose this study on purpose, for one because it is very well known but more so, because we can extract information that has not been reported before. This dataset contains nine continuous indicators for three latent factors: Visual, language, and math abilities. Additionally, it contains subject age and sex. We utilize the 300 complete cases from this dataset.

We fit a three factor model with no cross-loadings or residual covariances. Each of the three factors has a corresponding reliability factor. These reliability factors are each in turn modeled from age and sex. We utilize the Gaussian process approach, such that each reliability factor is modeled as an additive gaussian process (with linear and non-linear components) from each corresponding latent factor, age, and sex. Therefore, omega reliability coefficients can non-linearly vary across factor scores, age, and sex.

The model estimates are summarized in Tables 3 and 4. The LOO performance of the reliability factor model was notably larger (82.1, [12.8]) than the typical constant error variance model. The measurement model suggests that the reliability factors indeed influence the residual variances, albeit weakly. That is, the latent-standardized loadings are relatively small, but notably differ from zero. Nevertheless, Figure 4 reveals how even relatively weak reliability factor loadings imply meaningful variability in ω coefficients.

Figure 4 conveniently shows three different trajectories in ω coefficients. Reliability in the visual factor ($E(\omega_i)$ = .631[.565, .696]) is higher at lower factor scores, then decreases as scores increase, then increases for higher factor scores. That is, reliability is at its lowest between visual scores of 0 and 2, and at its highest between -3 and -1. This suggests the visual battery is more reliable for those with poor visual ability than for those who are of average or high ability.

Conversely, the language factor ($E(\omega_i)$ = .87[.846, .896]) reliability is greatest at low factor scores, and remains relatively stable up until higher factor scores. Reliability declines beyond the average factor score, suggesting the language battery is highly reliable for those below average, and decreases in reliability thereafter.

Finally, the math factor ($E(\omega_i)$ = .704[.646, .755]) is essentially linear across the typical range of latent space. That is, in the typical standardized latent range of -3 to 3, reliability declines in a linear manner. Similar to the other batteries, the math battery is most reliable for those with lower math ability.

Both age and sex were entered as exogenous predictors of the reliability factors. Note that the effects of each covariate (age, sex, latent factor) on each reliability factor control for one another, as per usual. Although the difference is small, females have a higher expected reliability in the visual battery. Otherwise, reliability is effectively equal between sexes. Age predicts small differences in the language and math battery reliabilities. Namely, older children are expected to have lower reliability than younger children. This relationship is most apparent in math reliability, where a difference of .05 (at lower scores) and .15 (at higher scores) is expected between the lowest and highest age groups.

In sum, the reliability factor model elucidates new measurement information even on the thoroughly explored Holzinger-Swineford data. Traditional ω estimates for the visual, language, and math factors are .626, .885, and .689, respectively. The reliability factor model indicates that these are indeed only the *average* reliability coefficients, and that ω can vary considerably across cases, and across the latent factor scores. Whereas the visual and math factors have low average reliability, the reliability factor model suggests that reliability is notably larger for those with lower abilities. Therefore, these batteries may be more efficiently applied when assessing those with lower abilities than those

Table 3
Estimates of measurement model for Holzinger-Swineford data.

Factor	Item	$\hat{\lambda}_j$	$\hat{v}_j^{\sigma}$	$\hat{\lambda}_j^{\sigma}$
Visual	1	0.826 [0.650, 1.023]	-0.208 [-0.571, 0.059]	-0.098 [-0.224, -0.028]
	2	0.506 [0.366, 0.650]	-0.094 [-0.864, 0.598]	-0.289 [-0.439, -0.145]
	3	0.714 [0.547, 0.870]	-0.334 [-1.330, 0.420]	-0.306 [-0.541, -0.127]
Lang.	4	0.950 [0.846, 1.063]	-0.413 [-0.836, 0.087]	-0.200 [-0.329, -0.087]
	5	1.144 [1.021, 1.276]	-0.444 [-0.689, -0.165]	-0.089 [-0.189, -0.028]
	6	0.806 [0.708, 0.915]	-0.454 [-1.188, 0.447]	-0.369 [-0.510, -0.244]
Math	7	0.680 [0.549, 0.809]	-0.210 [-0.685, 0.135]	-0.174 [-0.311, -0.066]
	8	0.761 [0.644, 0.881]	-0.598 [-1.447, 0.052]	-0.338 [-0.535, -0.178]
	9	0.610 [0.485, 0.740]	-0.274 [-0.711, 0.051]	-0.160 [-0.276, -0.063]

Note: For ease of interpretation, the loadings were standardized such that the latent factor and reliability factor have variances of one. The mean structure was estimated, but is not reported here.

Table 4
Estimates of Gaussian Process models for Holzinger-Swineford data.

		Reliability Factor		
Predictor	Parameter	Visual	Lang.	Math
Factor	γ	-.321 [-1.517, 1.158]	-.924 [-1.819, .265]	-.927 [-1.855,-.142]
	α	2.045 [.217, 5.452]	1.028 [.035, 3.500]	.962 [.028, 3.487]
	l	1.056 [.123, 2.154]	1.034 [.030, 2.576]	.945 [.029, 2.395]
Female	β	.359 [-1.189, 1.701]	.096 [-1.355, 1.506]	-.183 [-1.666, 1.473]
	α	1.307 [.039, 4.749]	1.227 [.041, 4.368]	1.304 [.044, 4.471]
	l	.856 [.027, 2.350]	.833 [.020, 2.331]	.847 [.021, 2.365]
Age	β	-.114 [-.805, .531]	-.341 [-1.063, .526]	-.425 [-1.224, .378]
	α	.683 [.025, 2.589]	.886 [.035, 2.979]	.735 [.021, 2.780]
	l	.926 [.012, 2.422]	.922 [.021, 2.400]	.934 [.010, 2.559]

Note: β is the linear coefficient. α is the marginal standard deviation of the non-linear Gaussian process predictions. l is the length scale of the Gaussian process model.

with average or higher abilities. Similarly, the math battery is more reliable for younger subjects. This suggests that the test may need items with less error for older subjects.

Discussion

The reliability factor model expands the typical latent variable model. In addition to the latent factors, it includes a second set of latent factors for modeling residual variance. That is, latent factors model the expected location of responses, and the reliability factors model the expected variance of responses; therefore, it is a location-scale model with underlying latent variables.

The reliability factor permits each observation to have different expected error variances. Because reliability coefficients depend on the extent of error variance, the reliability factor therefore permits observation-specific reliability and, by extension, standard error of the measure. Allowing observation-specific reliability brings with it several benefits. Below, we discuss some properties and benefits of employing the reliability factor model.

Properties and benefits of the reliability factor

Including a reliability factor for each latent factor is a relatively safe default. The primary disadvantages are an increased model estimation time and risk of convergence failure. We discuss the latter in the limitations section. Nevertheless, the reliability factor approach offers new measurement insights and affords beneficial properties when embedded into models.

Importantly, the approach can be readily used within a common design in psychology: cross-sectional single-assessment multivariate data. To date, individual-specific ω requires repeated measurements (Hu et al., 2016; Geldhof et al., 2014; Raykov & du Toit, 2005), which are then modeled with multilevel (Martin et al., 2019; Muthén, 1994) or multigroup latent variable models. This data is comparatively un-

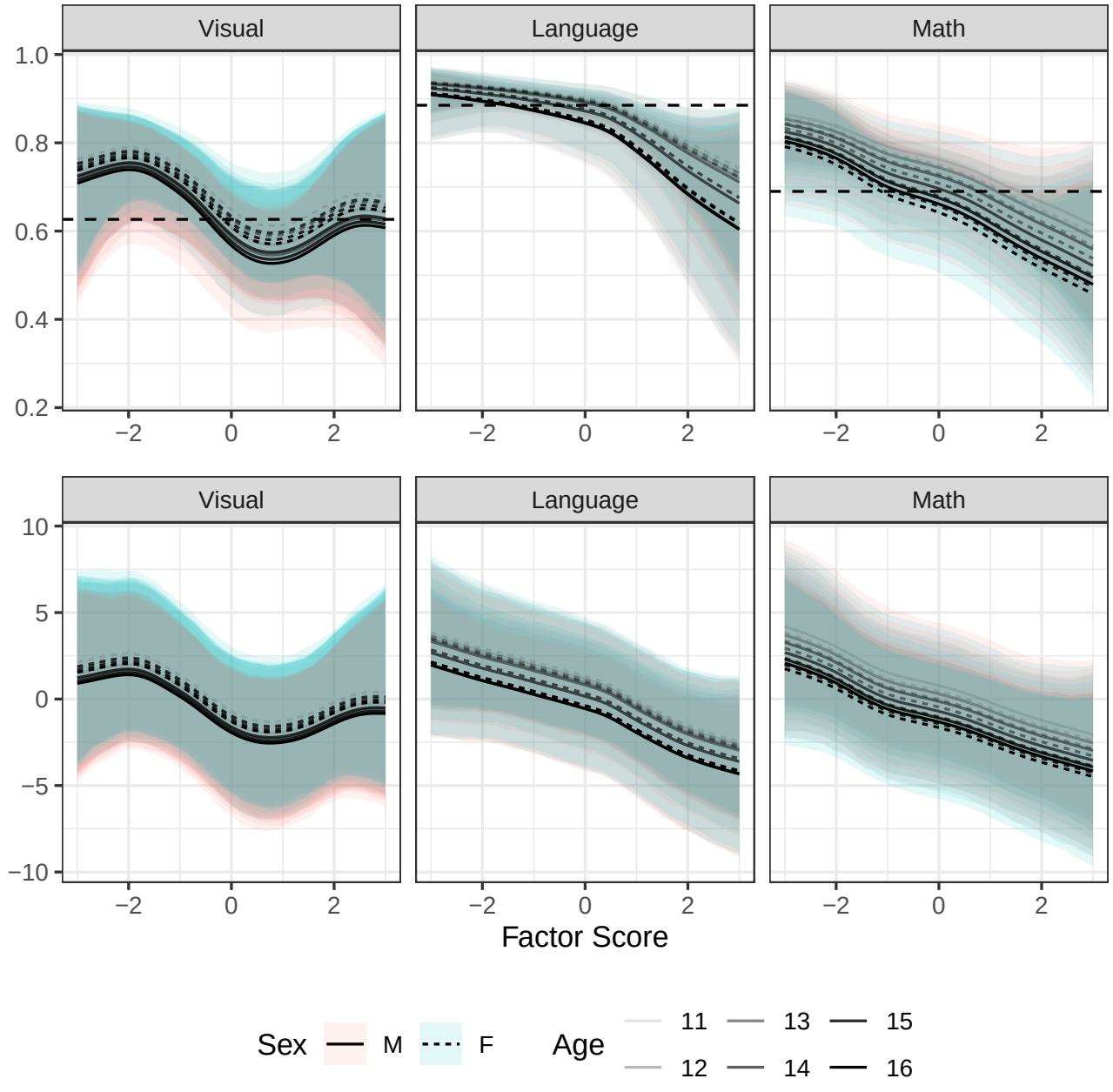

Figure 4. Predicted ω_i (Top) and reliability factor scores (Bottom) across the latent space. The horizontal dashed line is the traditional ω estimate.

common due to the greater cost and participant demands of collecting multiple variables, repeatedly or over time. Conversely, the reliability factor approach enables individual ω_i's to be estimated without the need for repeated assessments.

Again, the traditional, fixed ω coefficient is merely the expectation over these conditional ω_i coefficients, such that $E(\omega_i) = \omega$ (Raju et al., 2007). Therefore, one can interpret the fixed ω coefficient as the unconditional expected reliability, marginalized across both unknown factor scores, η^{μ}, *and* unknown reliability factor scores, η^{σ}. This formulation therefore provides a distribution of conditional measurement error variance terms, and therefore a distribution of ω_i terms with the expected value equal to the traditional fixed value.

The reliability factor approach enables novel insights into the measurement qualities of an instrument. It allows psychometricians to characterize and explore the variance or in-

variance of reliability itself. In the ideal case, homogeneity of ω will be observed. This would imply that the measurement is equally reliable across cases in the sampled population. When this is the case, all $\boldsymbol{\lambda}^{\sigma}$ will be approximately zero. Consequently, all $\boldsymbol{\eta}^{\sigma}$ will be estimated to be approximately zero. The model therefore effectively simplifies to a typical latent variable model with constant residual variance.

Conversely, when $\boldsymbol{\lambda}^{\sigma}$ are not jointly zero, then reliability will vary across cases. The variance in ω_i depends on the magnitude of the loadings. The greater the magnitude, the more variable the residual variances are, and therefore the more variable the ω_is are.

Although it is useful to characterize the distribution of reliability, the model permits more fine-grained analyses of heterogenous reliability. Namely, the reliability factor can be treated as an endogenous factor. By predicting the reliability factor, one can determine correlates of reliability. As observed in example three, respondent demographics may be used as exogenous predictors of the factor to explore how sex or age differ in reliability.

Moreover, the reliability factor may depend on the latent factor score itself (i.e., $\text{Cov}(\eta_i, \eta_i^{\sigma}) \neq 0$, or $\hat{\eta}_i^{\sigma} = f(\eta)_i$). As observed in the math factor of example three, reliability can linearly decrease with factor scores. Alternatively, Examples 2 and 3 demonstrate how reliability can exhibit complex non-linear, non-monotonic relationships with the factor score. Therefore, the reliability factor approach can illuminate measurement insufficiencies within certain ranges of the factor space. This may be important if, for example, a researcher wishes to compare groups with different latent means. Because reliability can vary across factor scores, the uncertainty in the mean comparison of groups must take into account the differing reliability between the group estimates. If a reliability factor model were employed in such an analysis (and Bayesian estimation were employed), the posterior distributions of latent mean differences would automatically account for varying reliability due to marginalizing over factor scores and their respective variations in uncertainty.

When the error variances differ across cases, the standard errors and posterior uncertainty of individual factor score estimates will similarly differ. This has implications for structural equation models that embed the reliability factors into the measurement models. Structural coefficients within a structural equation are estimated by integrating over factor scores: $p(\beta|D) = \int p(\beta|\boldsymbol{\eta}, D)p(\boldsymbol{\eta}|D)d\boldsymbol{\eta}$. Those with greater reliability have greater posterior information about their factor scores, thereby constraining the uncertainty in the linear structural equations. Consequently, structural equation coefficients will be informed more by those with greater reliability than by those with lesser reliability in a manner similar to shrinkage or partial pooling in multilevel models. That is, the reliability factor permits more robust estimates by implicitly weighting those with greater reliability more when estimating structural relationships.

Identification and Model Misspecification

The reliability factor approach is, at its core, a multivariate location scale model with a missing covariate (the latent value). It is also a re-expression of a mixed effects location scale model (Hedeker et al., 2012). Like the multivariate location scale models, the reliability factor approach merely adds a submodel to the scale parameter vector of the multivariate normal likelihood (Kapur et al., 2015; Williams, Liu, et al., 2019). Due to their equivalence, the reliability factor model is identified just as a multivariate location scale model, conditional on an identified direction, location, and scaling of the latent covariates. That is – it is identifiable once the distributional properties of the latent and reliability factors are fixed; this is no different from a typical latent variable model. Because the approximate Hessian and posterior covariance matrices were non-singular, full rank, and invertible, the models estimated in this paper were indeed locally identified (Viallefont, Lebreton, Reboulet, & Gory, 1998; Rothenberg, 1971).

Therefore, the reliability factor model is identified in the same manner that factor models are generally identified (Merkle & Wang, 2018). Although we chose a standardized approach, one could instead fix a reliability factor loading to one and an intercept to zero. We opted for a standardized latent identification because prior elicitation is simpler, and it is unclear which reliability factor indicators should have fixed parameters. We additionally chose a negative-loading identification, such that all reliability factor loadings are negative. One could instead use a positive-loading identification, and interpret the reliability factor as a *measurement error* factor, such that a higher score implies a greater error variance.

Once the latent and reliability factor distributions are identified, the identification requirements for linear components are met. However, the GP parameters can be unidentified *if* the approximated function is truly linear. This occurs because both the linear and the quadratic kernels can approximate a linear function; therefore, the posterior can form a ridge or a bimodal distribution over the GP parameters if the true function is linear. Nevertheless, it is important to note that all other model parameters are unaffected, and will remain identified. In practice, researchers tend to treat GP parameters as nuisances, and use the GP as a predictive model – because the predictive distributions are invariant across equivalent linear GPs, this niche case is not a pragmatic concern.

Finally, in latent variable and other high-dimensional models, there is an incalculable number of ways in which a model may be misspecified, and the reliability factor approach is no different. Like any statistical model, the reliability factor model can be, and probably is, misspecified in practice. For example, we assume that the reliability factor

loading structure mirrors that of the latent factor's, so that each latent factor has a corresponding reliability factor that gauges measurement precision of it. Theory may necessitate replacing this loading structure with others, or to include a higher order factor.

Likewise, we opted to model conditional error variance, instead of conditional loadings, because of its centrality to reliability estimation broadly. Despite the severe interpretation and inferential problems accompanying scalar noninvariance (Meredith, 1993), the generative process may indeed produce variable loadings and homogenous error variances. In this situation, the reliability factor approach would instead estimate expected loadings, with variable error variances. In effect, the further an individual's true loadings are from the expected loading, the lower their estimated reliability factor score would be, *even if* their true loadings are remarkably high. Although this could occur, we believe this concern is neither commonplace nor practical. Measurement models are used as approximations to a theorized process; most researchers would rather posit a single theorized process, from which people's fits may vary, than to posit a different process for every single case.

Although understanding the effects of misspecification on the robustness of inferences can be important, in practice one never knows which misspecification regime one is in. Instead, we advocate for the continuous model expansion, checking, and refinement approach advocated by Gelman and others (Gelman et al., 2020; Schad, Betancourt, & Vasishth, 2019). In sum, the reliability factor can be misspecified in all the same ways a latent factor model can be misspecified. The reliability factor approach replaces the improbable assumption of homogenous error variance with a more flexible assumption of conditional error variance. Future work may wish to examine the impact of reliability factor model misspecification.

Limitations and future directions

Despite the advantages and new insights afforded by the reliability factor, it is not without limitations. The primary limitation is that it requires approximately continuous indicators for model convergence. If too few response patterns are possible (i.e., few indicators with few response options), then the likelihood can be ill-defined and produce invalid estimates. For example, if a combination of latent scores, loadings, and intercepts can produce the exact item response of a respondent at any point during the estimation, then the respondent's reliability factor score will trend toward negative infinity and residual variance will be zero. The likelihood therefore contains an undefined infinite point mass, and estimation will fail. However, in this problematic scenario with few response options and patterns, it is likely that item response theory is more appropriate anyway.

Although the proposed model is constructed to handle heterogenous reliability without repeated measures, it can be extended to include repeated measures. Whereas heterogenous ω is possible with multilevel designs (Hu et al., 2016; Raykov & du Toit, 2005), a multilevel reliability factor model would permit two levels of heterogenous coefficients – within-group (k), case-specific reliability, ω_{ik}, and group-specific mean reliability, ω_k. That is, it could estimate per-observation reliability within level, and include between-level random effects on the model parameters. This would permit multiple interesting metrics. For example, one could estimate, compare, and model the variance in individual ω_i's in order to explore whether certain groups are more homogenously reliable. We hope to formalize this extension in future work.

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