Internal References in Cross-modal Matching: A Global Psychophysical Perspective

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Abstract

Cross-modal matching builds upon our ability to compare intensities across modalities. For example, we can easily match the brightness of a visual stimulus to the loudness of a tone, and vice versa. The regression effect in cross-modal matching relates to the observation that the intensity of whatever stimulus is adjusted exhibits a tendency toward some mean magnitude, which is interpreted as serving as an internal reference. Internal references, however, are not part of classical psychophysical theory in the tradition of Stevens, but form a key ingredient for the theory of global psychophysics initiated by R. Duncan Luce. The present paper extends and generalizes Luce's theory of what he called cross-dimensional magnitude production (Luce et al., 2010, *Psychological Review*, 117, 1247-1258) to integrate potentially role-dependent internal references (pertaining to the standard, or the adjusted stimulus) as suggested by the regression effect. The attribute cross-dimensional is more comprehensive than cross-modal, because it also includes intra-modal tasks where stimuli differ in an additional dimension (e.g., loudness of tones of different frequency). Reevaluating the available empirical evidence for magnitude production and matching experiments on the basis of these theoretical developments leads to a coherent picture. The results support role-dependence of internal references in cross-dimensional tasks (intra- or cross-modal), but clearly reject it in intra-dimensional tasks. Moreover, predictions derived from the generalized theory provide a close fit of classical data on cross-modal matching. Further directions for empirical as well as theoretical research are highlighted.

Keywords: cross-modal matching, magnitude production, global psychophysics, regression effect in psychophysical judgments, internal reference

Introduction

We are able to compare stimulus intensities across modalities without any effort. For instance, we can easily match the brightness of a visual stimulus to the loudness of a tone, and vice versa. This is all the more remarkable because this correspondence cannot be learned from contingencies in the environment: in everyday life bright things do not tend to be loud. Stevens' approach to psychophysics has drawn heavily upon this striking cognitive ability (Stevens, 1975). Cross-modal matches, however, merely served as a means to establish a direct relation between physical stimuli of different modalities, avoiding the widely criticized numerical characterization of sensations requested by Stevens' direct scaling methods (like magnitude estimation, and magnitude production). They have been extensively used as an experimental paradigm by Stevens and his disciples, and a rich body of data has been collected.

Figure 1 provides an adaptation of Figure 1 in Stevens & Marks (1965), and shows average results of ten subjects when matching the loudness of a band of noise to the brightness of a white light, and vice versa. Squares represent average settings when adjusting the sound pressure level (SPL) of the noise to match the brightness of the light, and circles the average settings when adjusting the luminance of the light to match the loudness of the noise. The vertical and horizontal line segments in gray indicate the interquartile ranges of the adjustments.

Figure 1. Data for cross-modal matching of brightness and loudness (adapted from Stevens & Marks, 1965, Figure 1). See text for details.

Stevens & Marks (1965) consider the matching curves to be approximately linear in dB (and thus in logarithmic) coordinates, which they take as evidence for Stevens' assump-

Figure 2. Cross-dimensional matching data of Zwicker (1958). The curves, fitted through independent locally linear regressions with a bandwidth of 10 dB, closely mimic those presented in Stevens & Greenbaum (1966, Figure 2).

tion that the respective psychophysical functions conform to the power law (see Equation (1) below). That the slopes of the regression lines fitted independently to the two matching curves differ as depicted in Figure 1 is a typically observed result. The conclusion drawn from this is that “the observer tended to shorten the range of whichever stimulus he adjusted” (Stevens & Marks, 1965, p. 409). This phenomenon was attributed to a central tendency of judgment (Hollingworth, 1910), and was thus referred to as a *regression effect* (Stevens & Greenbaum, 1966). The underlying idea is that adjustments tend to a respondent-generated internal reference of a mean magnitude, which already Hollingworth (1910) considered to be dependent upon the experimental setting (e.g., the range of stimuli presented as standards in the matching task).

Figure 2 provides an example of what actually constitutes a match which is not cross-modal in the strict sense. The data are due to Zwicker (1958), who let the subjects match the loudness of a pure tone to the loudness of wide-band noise, and vice versa. The match thus is intra-modal, but the stimuli do not only differ with respect to sound pressure level, but also in another attribute, or dimension, which is their frequency spectrum. Following Luce et al. (2010) we call these kind of matches *cross-dimensional*. From a technical perspective the term cross-dimensional match may be conceived to also comprise cross-modal matches, and the notation introduced below will capture all those cases.

In Figure 2 the matching curves are clearly curvilinear in dB coordinates. Apart from this they are interpreted as exhibiting the same regression effect as in Figure 1 (Stevens & Greenbaum, 1966). Fitting independent locally linear regression curves (with 10 dB

bandwidth) to the data seems to back up this view.

The Received View

This section describes how the regression effect in cross-dimensional matching is accounted for in Stevens' psychophysical theory (Stevens, 1975), recasting the approach in a notation following Luce et al. (2010).

Let f and g refer to two dimensions in which the stimuli can be distinguished, like luminance and sound pressure level as in Figure 1, or different spectral forms as in Figure 2. Moreover, let x^f denote a stimulus in dimension f having intensity x , and y^g a stimulus in dimension g having intensity y . Stevens assumes the existence of ratio scale representations ψ_f and ψ_g (Narens, 1996) which, taking the form of a power law, read

$$\psi_f(x^f) = \alpha_f \cdot (x^f)^{\beta_f} \text{ and } \psi_g(y^g) = \alpha_g \cdot (y^g)^{\beta_g} \quad (1)$$

with positive real constants $\alpha_f, \alpha_g, \beta_f, \beta_g$. Cross-modal matching then implicitly assumes that the representations ψ_f and ψ_g actually map to a common ratio scale (this assumption will be made explicit below), and can thus be equated

$$\psi_g(x_1^{f,g}) = \psi_f(x^f) \text{ and } \psi_f(y_1^{g,f}) = \psi_g(y^g). \quad (2)$$

In these equations $x_1^{f,g}$ denotes the match in dimension g provided for x^f , and $y_1^{g,f}$ the match in dimension f provided for y^g . These equations imply the linear log-log relationships

$$\begin{aligned} \log x_1^{f,g} &= \frac{\beta_f}{\beta_g} \log x^f + \frac{\log \alpha_f - \log \alpha_g}{\beta_g} \\ \log y_1^{g,f} &= \frac{\beta_g}{\beta_f} \log y^g + \frac{\log \alpha_g - \log \alpha_f}{\beta_f} \end{aligned}$$

with reciprocal slopes, where only the ratio of power law exponents is determined uniquely, but not their single values. When plotted in log-log coordinates, with dimension f on the abscissa and dimension g on the ordinate, the last equation is to be rewritten as

$$\log y^g = \frac{\beta_f}{\beta_g} \log y_1^{g,f} + \frac{\log \alpha_f - \log \alpha_g}{\beta_g}.$$

This means that Stevens' theory predicts coinciding linear cross-dimensional matching curves. Notice that the same holds to a sufficiently close approximation when predictions are plotted in dB coordinates as in Figures 1 and 2. The differing slopes of the lines independently fitted to the matching data in Figure 1 (but also the somewhat systematic departures from linearity, which are more pronounced in Figure 2) clearly violate this prediction. Because the theory does not include an internal reference as required by an explanation in terms of a regression effect, the discrepancy is attributed to the power law exponents β_f and β_g , and makes them dependent upon the concrete performed matching task $f \rightarrow g$, or $g \rightarrow f$. Assuming varying exponents within the same modality, however, is highly problematic. The psychophysical function is conceived as providing a representation of perceived intensity independent of the procedure used to estimate it. In particular, the

exponent in the power law should not be contaminated by contextual effects due to the experimental setup.

The subsequently outlined theory of global psychophysics (Luce, 2002, 2004) introduces explicitly given, or respondent-generated reference intensities into the representation. The reference intensities may be dependent upon details of the experimental task, and thus may allow for dissociating effects related to the experimental design (such as the role of the stimulus in the matching task) from the psychophysical function.

Global Psychophysics

Narens (1996) provided a measurement theoretic foundation of Stevens' direct scaling methods (Stevens, 1975). He examined Stevens' claim that a ratio scale representation of the corresponding sensation may be established by experimental methods, like magnitude estimation and magnitude production. Narens (1996) identified the qualitative conditions warranting the kind of representation intended by Stevens. Luce (2002, 2004) further generalized Narens' approach. His theory of global psychophysics draws upon results obtained in the context of utility theory. It therefore represents an extraordinary piece of psychology which succeeds in unifying theories across quite different areas of cognitive psychology.

Luce's theory is introduced by referring to the method of magnitude production, first within a single dimension and then for two dimensions. Given a stimulus with intensity x , and a positive number p , let x_p denote the intensity of the stimulus which is perceived to be p -times as intense as stimulus x . Given this notation, Luce (2004) derived the representation

$$W(p) = \frac{\psi(x_p) - \psi(\rho_{i_p})}{\psi(x) - \psi(\rho_{i_p})}, \quad i_p = \begin{cases} + & \text{if } p \geq 1, \\ - & \text{if } p < 1. \end{cases} \quad (3)$$

Here, ψ is a ratio scale, and the function W is interpreted as a "cognitive distortion" (Luce et al., 2010, p. 3) of the number p . This means that $W(p)$ in the representation (3) may be distinct from the number p appearing in the instruction, so that we might even have $W(1) \neq 1$. The symbol ρ_{i_p} denotes a reference intensity which might either be explicitly presented by the experimenter, or is a respondent-generated internal reference. Its value may depend on whether p is greater than or equal to 1, or less than 1, as indicated by the index i_p .

The theory describes how judgments are formed, based on perceived intensities $\psi(x)$ and $\psi(x_p)$ of the involved stimuli. It makes explicit that these judgments depend on a reference intensity ρ_{i_p} . In fact, it considers the ratio of intervals $\psi(x) - \psi(\rho_{i_p})$ and $\psi(x_p) - \psi(\rho_{i_p})$ of perceived intensities. Here, the reference intensity ρ_{i_p} need not be interpreted as a "mean magnitude", or "indifference point" as assumed by Hollingworth (1910), but may rather constitute a "zero point" for magnitude production (or, estimation). Notice that, beyond supposing ψ to conform to a power law, Stevens' assumptions captured in (2) amount to W being the identity as well as setting $\rho_+ = 0$.

In the more general situation, with arbitrary strictly increasing function W , the following commutative property must hold — in fact, together with other conditions it is also sufficient for deriving Equation (3). For all stimuli x and all numbers p, q

$$x_{p,q} = x_{q,p}.$$

This means that the result of performing two consecutive magnitude production tasks is not affected by their order. The commutative property assumes that first producing a stimulus x_p “perceived p -times x ” and then a stimulus $x_{p,q}$ “perceived q -times x_p ” leads to the same result as first producing a stimulus x_q “perceived q -times x ” and then a stimulus $x_{q,p}$ “perceived p -times x_q ”. It seems a priori plausible that the reference intensity in (3) will depend on whether p is larger, or smaller than 1. The index i_p takes care of this. There is converging evidence supporting the empirical validity of the commutative property in various modalities (e.g., brightness, loudness, perceived contrast, size, pitch interval, and short-time duration) whenever $p, q \geq 1$ or $p, q < 1$, i.e., if $i_p = i_q$. A detailed review of experimental results will be given below.

Luce et al. (2010) generalized the representation still further to cover intra-modal judgments of stimuli varying in more than one attribute, or dimension. The examples that they had in mind include loudness of pure tones with varying frequency, or brightness of visual stimuli of varying hue. Actually, the question they addressed was whether pure tones of varying frequency are represented on the same loudness scale. But what does it mean to be on a common scale? Consider two different dimensions (e.g., pure tones of different frequency), which will be denoted by f and g in the sequel. The experimental paradigm to which Luce et al. (2010) refer is a cross-dimensional magnitude production task. In the experiment the subject is presented with a stimulus x^f of dimension f and a positive number p , and is instructed to produce a stimulus $x_p^{f,g}$ in dimension g such that its perceived intensity is p -times that of x^f . Then dimension-specific ratio scale representations ψ_f and ψ_g form a *common ratio scale* if there is a ratio scale $\psi_{f,g}$ defined on all pairs (x, k) with x a physical intensity and k an index for the dimension (denoted by x^k), satisfying

$$\psi_{f,g}(x^k) = \begin{cases} \psi_f(x^f) & \text{if } k = f \\ \psi_g(x^g) & \text{if } k = g \end{cases} . \quad (4)$$

In this cross-dimensional context, the representation ψ in (3) is to be replaced by the representation $\psi_{f,g}$, of (4) and all stimuli (including the reference intensities) have to be indexed by the respective dimension. Notice that intra- or cross-dimensional matches correspond to the case $p = 1$. The commutative property was generalized accordingly to a property called *cross-dimensional commutativity* (Luce et al., 2010), the formulation of which has to account for different cases of cross-dimensional magnitude productions tasks. For $f \rightarrow g \rightarrow f$ (a cross-dimensional production from f to g , followed by one from g to f), and $f \rightarrow g \rightarrow g$ (a cross-dimensional production from f to g , followed by an intra-dimensional production within g) Luce et al. (2010) showed that, given the considered representation by a common scale $\psi_{f,g}$, it follows that

$$x_{p,q}^{f,g,f} = x_{q,p}^{f,g,f} . \quad (5)$$

and

$$x_{p,q}^{f,g,g} = x_{q,p}^{f,g,g} \quad (6)$$

On the contrary, the condition

$$x_{p,q}^{f,g,f} = x_{q,p}^{f,f,f} \quad (7)$$

was shown to hold under additional constraints on the involved reference intensities only. This may point to differences between intra-dimensional (righthand side of the equation)

and cross-dimensional (lefthand side of the equation) magnitude productions. The notation of Luce et al. (2010), however, does not allow for fully exploring dependencies like this. The subsequent section extends the notation to allow for a systematic characterization of the reference intensities with respect to their role in intra- and cross-dimensional magnitude productions.

Role-specific Reference Stimuli

In the sequel, Luce’s theory (Luce, 2002, 2004; Luce et al., 2010) for single as well as two dimensions is reconsidered to account for a more sophisticated treatment of the reference intensities, which is motivated by the above outlined regression effect in psychophysical judgments. Luce et al. (2010) already felt that such a generalization is necessary. Their implementation, however, remains rudimentary and incomplete (limited to the case treated in Equation (7) above), and leads to inconsistencies in the proofs of their Propositions 3 and 4.

Consider the generalized representation

$$W(p) = \frac{\psi_{f,g}(x_p^{f,g}) - \psi_{f,g}(\rho_{i_p}^{g \leftarrow f})}{\psi_{f,g}(x^f) - \psi_{f,g}(\rho_{i_p}^{f \rightarrow g})} \quad (8)$$

of a magnitude production task, with i_p as defined in (3). The notation employed to characterize the respondent-generated internal reference intensities not only indicates whether the judgment is intra- or cross-dimensional, but also informs on the role of the stimulus in magnitude production to which it refers. To be precise, consider a cross-dimensional magnitude production from f to g as sketched above. We use the notation $\rho_{i_p}^{g \leftarrow f}$ to denote the reference intensity in dimension g pertaining to the stimulus $x_p^{f,g}$ adjusted by the subject to obtain a perceived intensity p -times that of the standard x^f . Analogously, $\rho_{i_p}^{f \rightarrow g}$ denotes the reference intensity in dimension f pertaining to the presented standard stimulus x^f . This also allows to account for dependencies on the particular pair of dimensions considered. Reference intensities in dimension f may not be the same if f is paired with dimension h instead of dimension g . Within the introduced notation we refer to $\rho_{i_p}^{f \rightarrow g}$, $\rho_{i_p}^{f \leftarrow g}$ and $\rho_{i_p}^{f \rightarrow h}$, $\rho_{i_p}^{f \leftarrow h}$, respectively. For intra-dimensional magnitude productions we distinguish $f \rightarrow f$ from $f \leftarrow f$. Notice that for cross-dimensional magnitude production Luce et al. (2010) only distinguish between reference intensities that they denote by $\rho_{i_p}^{f,g}$ and $\rho_{i_p}^{g,f}$, which in our notation correspond to $\rho_{i_p}^{g \leftarrow f}$ and $\rho_{i_p}^{f \rightarrow g}$. They lump together all the other cases that are kept apart in the sequel.

For any dimensions f, g the reference intensities in f will be called *role-independent* whenever $\rho_{i_p}^{f \rightarrow g} = \rho_{i_p}^{f \leftarrow g}$ for all positive p . We will speak of *intra-dimensional role-independence* if $f = g$, and of *cross-dimensional role-independence* if $f \neq g$. Moreover, in case of $f \neq g$ reference intensities in f are said to satisfy *cross-dimensional invariance of type 1* whenever $\rho_{i_p}^{f \rightarrow g} = \rho_{i_p}^{f \rightarrow f}$, and to satisfy *cross-dimensional invariance of type 2* whenever $\rho_{i_p}^{f \leftarrow g} = \rho_{i_p}^{f \leftarrow f}$ for all positive p .

Single Dimension

In this case the fixed dimension index is dropped for convenience, and reference intensities are denoted by $\rho_{i_p}^{\leftarrow}$ and $\rho_{i_p}^{\rightarrow}$. Representation (8) then reads

$$W(p) = \frac{\psi(x_p) - \psi(\rho_{i_p}^{\leftarrow})}{\psi(x) - \psi(\rho_{i_p}^{\rightarrow})}, \quad i_p = \begin{cases} + & \text{if } p \geq 1, \\ - & \text{if } p < 1. \end{cases} \quad (9)$$

The following result generalizes Propositions 1 of Luce et al. (2010). Its proof can be found in the Appendix.

Proposition 1. *Assume representation (9). Then intra-dimensional commutativity*

$$x_{p,q} = x_{q,p} \quad (10)$$

holds for all stimuli x and all positive p, q if and only if

$$\rho_{i_p}^{\leftarrow} = \rho_{i_p}^{\rightarrow} = \rho_{i_q}^{\leftarrow} = \rho_{i_q}^{\rightarrow}.$$

Notice that Luce et al. (2010) do not distinguish the different roles in magnitude production when it comes to characterizing the internal reference intensities in the single dimension case. In view of Proposition 1 this amounts to assuming intra-dimensional role-independence, which imposes the implicit constraints $\rho_{i_p}^{\leftarrow} = \rho_{i_p}^{\rightarrow}$ and $\rho_{i_q}^{\leftarrow} = \rho_{i_q}^{\rightarrow}$. Thus, for the cases $p, q \geq 1$ and $p, q < 1$ (i.e., $i_p = i_q$) intra-dimensional commutativity is derived as a seemingly unconditional consequence of representation (9) in Luce et al. (2010). More precisely, their Proposition 1 can be conceived as a corollary to the above stated Proposition. For this notice also that the constraints $\rho_{i_p}^{\rightarrow} = \rho_{i_q}^{\rightarrow}$ and $\rho_{i_p}^{\leftarrow} = \rho_{i_q}^{\leftarrow}$ are trivially satisfied if $i_p = i_q$.

Corollary 1. *Assume representation (9) with role-independent reference intensities $\rho_j = \rho_j^{\leftarrow} = \rho_j^{\rightarrow}$, with $j = +, -$. Then intra-dimensional commutativity (10) holds for all stimuli x and all positive p, q if and only if $\rho_{i_p} = \rho_{i_q}$.*

Two Dimensions

Let us first consider commutativity for magnitude productions $f \rightarrow g \rightarrow f$ (illustrated in Figure 3) and $f \rightarrow g \rightarrow g$ as captured by (5) and (6) above. The following result covers both cases. See the Appendix for a proof.

Proposition 2. *Assume representation (8). Then for all x , and for all positive p, q such that $i = i_p = i_q$*

$$i. \quad x_{p,q}^{f,g,f} = x_{q,p}^{f,g,f} \quad \text{if and only if} \quad \rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}.$$

$$ii. \quad x_{p,q}^{f,g,g} = x_{q,p}^{f,g,g} \quad \text{if and only if} \quad \rho_i^{g \leftarrow g} = \rho_i^{g \rightarrow g}.$$

The result for case $f \rightarrow g \rightarrow f$ is in line with Luce et al. (2010), who claimed cross-dimensional commutativity (5) to hold under the tacit assumption of role-independence (on dimension g), which is exactly the condition in Proposition 2 i. Notice, however, that they

Figure 3. Illustration of cross-dimensional commutativity in case of $f \rightarrow g \rightarrow f$ for $p, q \geq 1$, which holds if and only if $\rho_+^{g \leftarrow f} = \rho_+^{g \rightarrow f}$ (Proposition 2). Notice that no constraints are imposed on the reference intensities $\rho_+^{f \leftarrow g}$ and $\rho_+^{f \rightarrow g}$.

do not provide a proof of this claim. Notice also that the constraint in Proposition 2 ii is at odds with Proposition 3 in Luce et al. (2010), who derived cross-dimensional commutativity for case $f \rightarrow g \rightarrow g$ also assuming role-independence (on dimension g). This discrepancy is due to an error in the proof of Proposition 3 in Luce et al. (2010). The last expression on the bottom right of Luce et al. (2010, p. 9), corresponds to the right fraction in (A.3) in the proof of Proposition 2 in the Appendix. Adopting the present notation, it reads

$$W(q) = \frac{\psi_g(x_{p,q}^{f,g,g}) - \psi_g(\rho_i^{g \leftarrow g})}{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}.$$

This expression equals that of (A.3) whenever $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow g}$, which is exactly the condition stated in Proposition 2 ii.

Let us now consider a situation where the sequence $f \rightarrow g \rightarrow f$ of magnitude productions is to be compared to $f \rightarrow f \rightarrow f$. The following proposition collects the sufficient conditions that result from spelling out cross-dimensional commutativity for this case. It remains unclear whether this implication can be strengthened to an equivalence as in Propositions 1 and 2. Anyway, the result as it stands will be useful in the subsequent section. Although the assertion is immediate, an explicit proof is given in the Appendix.

Proposition 3. *Assume representation (8). Then for all x and for all positive p, q such that $i = i_p = i_q$*

$$(\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f}, \rho_i^{f \leftarrow g} = \rho_i^{f \leftarrow f}, \rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}, \rho_i^{f \leftarrow f} = \rho_i^{f \rightarrow f}) \text{ implies } x_{p,q}^{f,g,f} = x_{p,q}^{f,f,f}.$$

In the proof of their Proposition 4 Luce et al. (2010) lump together all the intra- and cross-dimensional reference intensities that may emerge in dimension f by a symbol denoted $\rho_{i_p}^f$. This amounts to the implicit assumption

$$\rho_i^f = \rho_i^{f \rightarrow f} = \rho_i^{f \leftarrow f} = \rho_i^{f \rightarrow g} = \rho_i^{f \leftarrow g},$$

and thus presupposes all constraints in Proposition 3 except for $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}$. It is only under this assumption that they can then identify the latter as a necessary and sufficient condition for cross-dimensional commutativity in the considered case.

Figure 4. Illustration of cross-dimensional commutativity capturing the compatibility of intra-dimensional magnitude production with cross-dimensional matching (Proposition 4 with $q = 1$).

For providing a foundation of cross-dimensional matching one has to examine still another condition, which seems to be quite natural in this context, but was not considered by Luce et al. (2010). The condition captures the compatibility of intra-dimensional magnitude production with cross-dimensional matching in the sense that both operations can be interchanged. The equation

$$x_{p,1}^{f,f,g} = x_{1,p}^{f,g,g}$$

states that magnitude production within dimension f followed by providing a match in dimension g comes out the same as first providing a match in dimension g and then performing a magnitude production within dimension g . The condition is illustrated in Figure 4. The following result considers the slightly more general case with cross-dimensional magnitude production by factor q replacing cross-dimensional matching, that is

$$x_{p,q}^{f,f,g} = x_{q,p}^{f,g,g} \quad (11)$$

for all x and all positive p, q . It collects the constraints that are sufficient for obtaining (11). Again, this result is immediate (see proof in the Appendix).

Proposition 4. *Assume representation (8). Then for all x and for all positive p, q such that $i = i_p = i_q$*

$$(\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f} = \rho_i^{f \leftarrow f} \text{ and } \rho_i^{g \leftarrow f} = \rho_i^{g \leftarrow g} = \rho_i^{g \rightarrow g}) \text{ implies } x_{p,q}^{f,f,g} = x_{q,p}^{f,g,g}.$$

Reevaluating Empirical Evidence

The following aims at reinterpreting available evidence collected in a quite large number of experiments in global psychophysics as well as classical data on cross-dimensional matching in the light of the theoretical results derived above. This links the axiomatic approach of Luce (2002, 2004) to the curve-fitting approach prevalent in traditional psychophysics. It will be shown that a coherent picture emerges within this integrative perspective.

Commutativity Revisited

Intra-dimensional commutativity was investigated through magnitude production experiments for a variety of modalities and domains, among them loudness (Ellermeier & Faulhammer, 2000; Luce et al., 2010; Steingrímsson & Luce, 2005; Zimmer, 2005), brightness (Steingrímsson, 2009; Steingrímsson et al., 2012), perceived contrast (Steingrímsson, 2016a), size of circles (Augustin & Maier, 2008), pitch intervals (Kattner & Ellermeier, 2014), and short-time duration (Birkenbusch et al., 2015). Without any exception, these studies provide converging evidence on the empirical validity of intra-dimensional commutativity (10) whenever $p, q \geq 1$, or $p, q < 1$. This finding implies that ratio scale representations can be established in all of these domains, and Proposition 1 shows that it is in line with intra-dimensional role-independence

$$\rho_i^{f \rightarrow f} = \rho_i^{f \leftarrow f} \quad (12)$$

for all dimensions f in case of $i = i_p = i_q$. Here, the reference intensities can be assumed to be the same, no matter whether they pertain to the standard or the adjusted stimulus. Because (10) always turned out to be violated in the mixed case $p \geq 1, q < 1$ and $p < 1, q \geq 1$ (i.e., $i_p \neq i_q$), consideration is confined to the case $i_p = i_q$ in the sequel.

Cross-dimensional commutativity (i.e., $f \neq g$) was investigated for the loudness of pure tones of different frequency (Luce et al., 2010), and for brightness of different hues (Steingrímsson et al., 2012). As both conditions (5) and (6) treated in Proposition 2 were assumed to hold without any restrictions on the reference intensities by Luce et al. (2010), only the second condition was subjected to an empirical test in both studies. The condition $x_{p,q}^{f,g,g} = x_{q,p}^{f,g,g}$ was found to hold in both modalities for $i = i_p = i_q$, which allows for concluding

$$\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow g} \quad (13)$$

for all dimensions f, g . Together with (12) this implies

$$\rho_i^{g \leftarrow f} = \rho_i^{g \leftarrow g}, \quad (14)$$

for all dimensions f, g , which amounts to cross-dimensional invariance of type 2 (pertaining to the adjusted stimulus).

On the contrary, the equality $x_{p,q}^{f,g,f} = x_{q,p}^{f,g,f}$ stated in (7) was found to be violated in both modalities for $i = i_p = i_q$. Thus, by Proposition 3 we cannot have the conjunction

$$\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f}, \rho_i^{f \leftarrow g} = \rho_i^{f \leftarrow f}, \rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}, \rho_i^{f \leftarrow f} = \rho_i^{f \rightarrow f}.$$

Due to intra-dimensional role-independence (12) and cross-dimensional invariance of type 2 (14) it follows that

$$\rho_i^{f \rightarrow g} \neq \rho_i^{f \rightarrow f} \text{ or } \rho_i^{g \leftarrow f} \neq \rho_i^{g \rightarrow f}$$

for all dimensions f, g . Claiming that conditions, like (13) or (14), hold for all dimensions f, g includes the symmetric versions resulting from interchanging f and g . In case of the considered intra-model cross-dimensional magnitude productions this simply means that the conditions are independent of the arbitrary assignment of labels f and g to frequencies and hues. Drawing upon this symmetry across dimensions, $\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f}$ together with (13)

implies $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}$, and conversely $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}$ together with (13) implies $\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f}$. Thus, neither of these equations can hold, and we must indeed have

$$\rho_i^{f \rightarrow g} \neq \rho_i^{f \rightarrow f} \text{ and } \rho_i^{g \leftarrow f} \neq \rho_i^{g \rightarrow f} \quad (15)$$

for all dimensions f, g . This means that the data contradict both cross-dimensional invariance of type 1 (pertaining to the standard stimulus) as well as cross-dimensional role-independence in dimension g .

The available evidence thus leads us to expect that an empirical test of commutativity (5) in case of $f \rightarrow g \rightarrow f$ will fail, because (15) contradicts its necessary and sufficient condition according to Proposition 2 i. Moreover, as for $i = i_p = i_q$ the data unequivocally support intra-dimensional role-independence (12) for all dimensions, and cross-dimensional invariance of type 2 (14), Corollary 2 (see Appendix for a proof) reconsiders Proposition 4. It shows that (11) is equivalent to cross-dimensional invariance of type 1 in dimension f . Thus, in the light of the evidence in favor of (12) and (14), intra-dimensional magnitude production is not compatible to cross-dimensional matching due to (15), and an empirical test of commutativity (11) is expected to fail.

Corollary 2. *Assume representation (8), intra-dimensional role-independence (12) for dimensions f, g , and cross-dimensional invariance of type 2 (14) in dimension g . Then for all x and for all positive p, q such that $i = i_p = i_q$*

$$\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f} \text{ if and only if } x_{p,q}^{f,f,g} = x_{q,p}^{f,g,g}.$$

If the experimental results are indeed in line with these predictions then a coherent picture arises. It points to an affirmative answer of the question whether the dimension-specific representations of perceived intensity are on a common scale as expressed by (4). This would mean that there is a common scale of the loudness of pure tones irrespective of their frequency, and a common scale of the brightness of visual stimuli irrespective of their hue.

Cross-dimensional Matching Revisited

Although we have to bear in mind that the classical matching data are almost always reported at the group level only, and artifacts from aggregating data over subjects cannot be ruled out, reevaluating these data will nevertheless cast light on the explanatory power of the derived theoretical results.

The cross-dimensional matching curves as plotted in Figures 1 and 2 correspond to functions $m_{f,g}$ and $m_{g,f}$ defined by

$$\begin{aligned} m_{f,g}(x^f) &= x_1^{f,g} = \psi_g^{-1} \left[W(1) \cdot [\psi_f(x^f) - \psi_f(\rho_+^{f \rightarrow g})] + \psi_g(\rho_+^{g \leftarrow f}) \right] \\ m_{g,f}(y^g) &= y_1^{g,f} = \psi_f^{-1} \left[W(1) \cdot [\psi_g(y^g) - \psi_g(\rho_+^{g \rightarrow f})] + \psi_f(\rho_+^{f \leftarrow g}) \right] \end{aligned} \quad (16)$$

for all x^f in dimension f and y^g in dimension g . The curves coincide whenever $m_{f,g}$ and $m_{g,f}$ are inverse functions of each other, or equivalently $m_{f,g} \circ m_{g,f}$ and $m_{g,f} \circ m_{f,g}$ (with

◦ denoting function composition) are identity functions on appropriately chosen domains. Put differently, the cross-dimensional matching curves coincide, whenever for all x^f and y^g

$$x^f = y_1^{g,f} \text{ implies } y^g = x_1^{f,g}.$$

In order to relate the observed properties of cross-dimensional matching curves to the developed theory we assume that the psychophysical functions ψ_f and ψ_g are strictly increasing on the positive reals, and occasionally that they are unbounded above. The latter, for example, holds if they conform to the power law (1) with positive parameters. The mentioned general properties of ψ_f and ψ_g , but not their specific form (such as the power law), will be essential for characterizing the asymptotic behavior of the cross-modal matching curves in Proposition 5. Notice that this asymptotic behavior may not show in plots within the limited range of physical intensities used in the experiments.

Proposition 5. *Applying representation (8) to cross-dimensional matching (i.e., setting $p = 1$, and thus $i_p = +$) with strictly increasing psychophysical functions ψ_f and ψ_g , we have following assertions.*

- i. *If the reference intensities in f and g are role-independent (i.e., $\rho_+^{f \leftarrow g} = \rho_+^{f \rightarrow g}$ and $\rho_+^{g \leftarrow f} = \rho_+^{g \rightarrow f}$), then the cross-dimensional matching curves $m_{f,g}$ and $m_{g,f}$ coincide if and only if $W(1) = 1$.*
- ii. *If ψ_f and ψ_g are unbounded above on the positive reals then the cross-dimensional matching curves are asymptotically identical for increasing intensity if and only if $W(1) = 1$.*
- iii. *If $W(1) = 1$ then the following statements are equivalent.*
 - a. *The cross-dimensional matching curves $m_{f,g}$ and $m_{g,f}$ coincide.*
 - b. *The reference intensities satisfy the equation*

$$\psi_g(\rho_+^{g \rightarrow f}) - \psi_g(\rho_+^{g \leftarrow f}) = \psi_f(\rho_+^{f \leftarrow g}) - \psi_f(\rho_+^{f \rightarrow g}) \quad (17)$$

Under the additional assumption that the ordering of role-dependent reference intensities is *invariant across dimensions*, i.e., that

$$\rho_+^{g \rightarrow f} \begin{matrix} \geq \\ \leq \end{matrix} \rho_+^{g \leftarrow f} \text{ if and only if } \rho_+^{f \rightarrow g} \begin{matrix} \geq \\ \leq \end{matrix} \rho_+^{f \leftarrow g} \quad (18)$$

holds, we have the following corollary to Proposition 5. This is because by (18) both sides of (17) have equal sign only if the two differences vanish.

Corollary 3. *Applying representation (8) to cross-dimensional matching (i.e., setting $p = 1$, $i_p = +$) with strictly increasing psychophysical functions ψ_f and ψ_g , assume that $W(1) = 1$, and that the ordering of role-dependent reference intensities is invariant across dimensions according to (18).*

Then the cross-dimensional matching curves $m_{f,g}$ and $m_{g,f}$ coincide if and only if $\rho_+^{g \rightarrow f} = \rho_+^{g \leftarrow f}$ and $\rho_+^{f \rightarrow g} = \rho_+^{f \leftarrow g}$.

Figure 5. Data of Stevens & Marks (1965) with different theoretical predictions. See text for details.

Let us consider the data on cross-modal matching of brightness (dimension f) and loudness (dimension g) provided by Stevens & Marks (1965) and illustrated in Figure 1. These data are replotted in Figure 5 together with predictions of different theoretical accounts. In all of them it is assumed that the psychophysical functions ψ_f and ψ_g conform to the power law (1). Moreover, the plots in the top row represent predictions under the assumption of cross-dimensional (here, actually cross-modal) role-independence of the reference intensities in both dimensions f and g , while the predictions in the left column assume the constraint $W(1) = 1$. Parameters were estimated numerically with the R software for statistical computing (R Core Team, 2018) via the function `optim` using the method of Nelder & Mead (1965), and implementing an ordinary least squares criterion.

Figure 5 gives rise to a number of remarks. First, all of the plots show that introducing reference intensities into the representation according to (8) allows for matching curves that

exhibit clear departures from linearity as requested by the data. Second, the top row plots demonstrate that predictions based on role-independent reference intensities are not able to capture essential features of the data. In particular, they fail to predict matching curves that drift apart for low intensities. Third, the left column plots illustrate the theoretical results of Proposition 5 i and ii, which refer to the total and asymptotic coincidence of the matching curves under the assumption $W(1) = 1$. Deviations between predictions and data in the latter case are most pronounced for large intensities, and speak against this assumption. Forth, it is evident that the bottom right plot, representing the most complex model, provides a really close fit to the data. It shows intersecting matching curves similar to Figure 1. Here, however, only a single psychophysical function is assumed for each modality (or, dimension), and not different power laws per modality for the two roles in the matching task as in Figure 1.

Figure 6. Dependence between reference intensities for data of Stevens & Marks (1965). See text for details.

The interpretation of the parameter estimates predicting the matching curves in the bottom right plot, however, has to be taken with a grain of salt. From intra-modal judgments a power law exponent around 0.3 was calculated for brightness within the received view, and an exponent of 0.66 for loudness with respect to sound pressure (Stevens & Marks, 1965). The estimates 0.089 and 0.263 for the power law exponents β_f and β_g in the bottom right plot of Figure 5 are considerably smaller in absolute value. There may be several reasons for this. First, it is obvious that introducing reference intensities into the representation, which absorb some task-specific aspects (or, context dependencies), will not leave the psychophysical functions unchanged. Second, the result may also be affected by the fact that the data do not represent individual matches. Averaging over nonlinear matching functions may seriously bias data analysis. This effect was demonstrated by Steingrímsson & Luce (2012) for an intra-dimensional matching task, where averaging data over subjects produced misleading results. The value 0.885 estimated for $W(1)$ is clearly smaller than

one, and the bottom row plots in Figure 5 show that this contributes to the prediction of intersecting matching curves. Notice that the estimate is well within the range from 0.75 to 1.17 obtained for individual subjects in Steingrímsson & Luce (2012). The estimates of the role-dependent reference intensities, however, are not determined uniquely by (16), but satisfy the equations

$$\begin{aligned} W(1) \cdot \psi_f(\rho_+^{f \rightarrow g}) - \psi_g(\rho_+^{g \leftarrow f}) &= c_{f,g} \\ W(1) \cdot \psi_g(\rho_+^{g \rightarrow f}) - \psi_f(\rho_+^{f \leftarrow g}) &= c_{g,f} \end{aligned} \tag{19}$$

with real constants $c_{f,g}, c_{g,f}$. This means that the models implementing role-dependent reference intensities are not identifiable given the cross-dimensional matching curves. This is somewhat analogous to the situation in Stevens' psychophysics, where only the ratio of power law exponents can be determined uniquely, but not their single values. The functional dependencies $\rho_+^{g \leftarrow f} = H_{f,g}(\rho_+^{f \rightarrow g})$ and $\rho_+^{f \leftarrow g} = H_{g,f}(\rho_+^{g \rightarrow f})$, following from (19) and illustrated in Figure 6, do not give any indication of the absolute values of the reference intensities. From this indeterminacy it follows that their interpretation as a mean magnitude to which judgments gravitate is not necessary for predicting the observed matching curves. Because the respective curves intersect, there is a pair of intensities that is in line with role-independence. The luminance of 78.3 dB corresponding to a sound pressure level of 78.6 dB, however, falls above the range of reference intensities obtained in previous work. Steingrímsson & Luce (2012) report estimates for individual subjects in an intra-dimensional matching task concerning the loudness of 1000 Hz tones that span a range of 30-64 dB SPL. Except for the intersection point, however, the assumption $\rho_+^{f \rightarrow g} = \rho_+^{f \leftarrow g}$ implies that $\rho_+^{g \rightarrow f} \neq \rho_+^{g \leftarrow f}$, and, vice versa, assuming $\rho_+^{g \rightarrow f} = \rho_+^{g \leftarrow f}$ implies that $\rho_+^{f \rightarrow g} \neq \rho_+^{f \leftarrow g}$.

The prediction of the intra-modal cross-dimensional matching data of Zwicker (1958) from Figure 2 are replotted in Figure 7 together with different theoretical predictions based on the power law representations (1). As in Figure 5 the plots in the top row represent predictions under the assumption of cross-dimensional role-independence of the reference intensities in both dimensions f and g , while the predictions in the left column assume the constraint $W(1) = 1$. The systematic deviations between predicted matching curves and data in the top row plots, and the close fit evident in the bottom row plots lets us again conclude that role-independence does not hold. Moreover, because the bottom row plots are highly similar, we may conclude that $W(1)$ is not different from one. Given $W(1) = 1$, the fact that the matching curves do not coincide also points to a violation of role-independence according to Corollary 3. This conclusion is also corroborated by the non-intersecting curves in Figure 8. They indicate that over the whole range assuming $\rho_+^{f \rightarrow g} = \rho_+^{f \leftarrow g}$ implies that $\rho_+^{g \rightarrow f} < \rho_+^{g \leftarrow f}$, and, vice versa, assuming $\rho_+^{g \rightarrow f} = \rho_+^{g \leftarrow f}$ implies that $\rho_+^{f \rightarrow g} < \rho_+^{f \leftarrow g}$. Again, the estimates 0.289 and 0.200 for the power law exponents β_f and β_g in the bottom right plot of Figure 7 turn out to be considerably smaller than the previously mentioned value of 0.66 for loudness with respect to sound pressure. The estimate 1.048 for $W(1)$ cannot be distinguished from one.

Conclusions

The present paper addresses an often replicated finding in cross-modal matching, which is known as the regression effect (Stevens & Greenbaum, 1966). The typical ob-

Figure 7. Data of Zwicker (1958) with different theoretical predictions. See text for details

servation, that the matching curves do not coincide but show a pattern as illustrated in Figures 1 and 2, is attributed to a systematic bias. The adjustments in the matching task are assumed to exhibit a tendency toward a respondent-generated internal reference of a mean magnitude. However, no such internal references occur within the traditional formal account of matching data within Stevens' psychophysics (Stevens, 1975). The regression effect thus directly affects the power law exponents in the psychophysical functions, which may be considered problematic. The psychophysical function, as a representation of perceived intensity within a particular modality, should not be contaminated by contextual effects due to the experimental setup.

Luce's theory of global psychophysics (Luce, 2002, 2004) integrates explicitly given, or respondent-generated internal references into the representation. The presented development generalizes the theory of cross-dimensional magnitude production (Luce et al., 2010) by introducing reference intensities that are sensitive to the role of the respective stimuli

Figure 8. Dependence between reference intensities for the data of Zwicker (1958). See text for details.

in the experimental task (i.e., whether they pertain to the standard vs. the adjusted stimulus). This theoretical framework allows for dissociating task-dependent effects from the psychophysical function. Within Luce’s axiomatic approach the commutativity (or, interchangeability) of two magnitude productions is a key property for establishing a ratio scale representation of perceived intensity. The present paper generalizes and extends previous results on commutativity (Luce, 2002; Luce et al., 2010). For the ratio scale representation (8) it identifies explicit constraints on the internal reference intensities that are sufficient, or even sufficient and necessary for intra- and cross-dimensional commutativity in various situations.

The evidence from extensive experimental tests of the commutativity axioms (for an overview see Steingrímsson, 2016b) unequivocally speaks to the empirical validity of both intra-dimensional role-independence and cross-dimensional invariance of type 2 of the reference intensities. From the derived theoretical results it then follows that the pattern of available data contradict cross-dimensional invariance of type 1 as well as cross-dimensional role-independence (see Table 1). It is notable that cross-dimensional invariance is to be rejected with respect to the standard (type 1), while it can be retained with respect to the adjusted stimulus (type 2). This leads to strong predictions for the (not yet tested) commutativity conditions in Proposition 2 i and Corollary 2 that are expected to fail. Notice that the results summarized in Table 1 do not contradict the assumption that the dimension-specific representations of perceived intensity are on a common scale according to (4).

The reevaluation of classical data from cross-dimensional (including cross-modal) matching (Stevens & Marks, 1965; Zwicker, 1958) confirms the conclusions drawn from checking commutativity axioms. Under the assumption that the psychophysical functions

Table 1

Overview of properties of the reference intensities for magnitude productions according to (8), with positive factor p and $i_p = +, -$ whenever $p \geq 1$ or $p < 1$, respectively.

Property	Equation	Empirical status
Intra-dim. role-independence	$\rho_{i_p}^{f \rightarrow f} = \rho_{i_p}^{f \leftarrow f}$ for all f	Supported
Cross-dim. role-independence	$\rho_{i_p}^{f \rightarrow g} = \rho_{i_p}^{f \leftarrow g}$ for all $f \neq g$	Rejected
Cross-dim. invariance of type 1	$\rho_{i_p}^{f \rightarrow g} = \rho_{i_p}^{f \rightarrow f}$ for all $f \neq g$	Rejected
Cross-dim. invariance of type 2	$\rho_{i_p}^{f \leftarrow g} = \rho_{i_p}^{f \leftarrow f}$ for all $f \neq g$	Supported

conform to the power law, a close fit of the matching data is obtained with role-dependent reference intensities only. It has to be emphasized that, in contrast to Stevens' approach, the presented extension of Luce's theory can produce matching functions that are curvilinear in dB coordinates. In fact, Figures 5 and 7 show that the predicted matching curves are able to capture the essential features of the data. Notice that this is achieved by assuming a single psychophysical function per dimension/modality. The interpretation of the obtained power law exponents, however, is severely compromised by the fact that they are not estimated from individual data, but from values averaged across subjects (Steingrímsson & Luce, 2012). The same may in principle be true for the estimate of $W(1)$, which is interpreted as the numerical representation of the matching task instruction. But Proposition 5 makes clear that its value corresponds to clear-cut properties of the matching curves. A value of 1 indicates that they asymptotically coincide (as in Figure 2), which will not be the case if they intersect at a point that falls well within the range of presented stimuli (as in Figure 1). This, however, means that the function W seems to be sensitive to additional aspects of the matching task (e.g., whether the matching is cross-modal, or cross-dimensional but intra-modal). Let us finally point out that the derived theoretical results together with the empirical evidence on the various commutativity conditions show that all these conclusions do not depend upon the power law assumption.

As an aside we mention that there is an alternative explanation of the regression effect that refers to a sequential adaptation of responses to the previously presented standard stimuli through a trial-by-trial updating of what has also been called an internal reference (Dyjas et al., 2012; Lapid et al., 2008; Nachmias, 2006). In the experimental procedures underlying the tests of commutativity as well as cross-dimensional matching, however, the standard and variable stimulus cycled on and off until the final setting was made (see Luce et al., 2010; Steingrímsson et al., 2012; Stevens & Marks, 1965). This essentially eliminates the effects of previous standards. It stabilizes the assumed adaptive internal reference and makes the judgments depend on the current standard only. Notice that, although similar words are used, the adaptive internal reference is to be distinguished from the reference intensities appearing in (8). The adaptive internal reference updates the standard x^f and thus exerts its effect through $\psi_{f,g}(x^f)$. This, however, is precluded by the used experimental procedure.

The present paper shows that a coherent picture on the internal references in intra- and cross-dimensional magnitude production and matching emerges from the empirical ev-

idence reviewed above. It successfully links classical studies on cross-dimensional matching to the axiomatic approach of global psychophysics initiated by Luce (2002, 2004), and documents the necessity and usefulness of the proposed extension of the theory. It therefore offers a promising route for further empirical as well as theoretical research which may proceed in various directions. First, the classical experiments on cross-dimensional matching need to be replicated to get access to individual data. This will not only cast light on the inter-individual variation, but will also allow for obtaining proper estimates of the parameters that do not suffer from the problems caused by aggregating data over subjects. However, it is the testing of the commutativity conditions treated here that can provide full information on internal references. It remains to investigate whether cross-dimensional but still intra-modal magnitude productions violate the commutativity conditions stated in Proposition 2 i and Corollary 2 as predicted. Moreover, we also have to go through all this checking for the cross-modal case in order to answer the important question whether there is a common scale of perceived intensity across different modalities. Finally, in a posthumous publication, Luce (2013) investigated the potential role of the respondent-generated reference intensities in explaining the discrepancies between data collected by magnitude estimation vs. magnitude production, which is also conceived as a regression effect (Stevens & Greenbaum, 1966). The generalized framework developed here provides the means for further pursuing this unfinished work.

References

- Augustin, T. & Maier, K. (2008). Empirical evaluation of the axioms of multiplicativity, commutativity, and monotonicity in ratio production of area. *Acta Psychologica*, 129, 208–216.
- Birkenbusch, J., Ellermeier, W., & Kattner, F. (2015). Octuplicate this interval! axiomatic examination of the ratio properties of duration perception. *Attention, Perception, & Psychophysics*, 77, 1767–1780.
- Dyjas, O., Bausenhardt, K. M., & Ulrich, R. (2012). Trial-by-trial updating of an internal reference in discrimination tasks: Evidence from effects of stimulus order and trial sequence. *Attention, Perception, & Psychophysics*, 74, 1819–1841.
- Ellermeier, W. & Faulhammer, G. (2000). Empirical evaluation of axioms fundamental to Stevens’s ratio-scaling approach: I. loudness production. *Perception & Psychophysics*, 62, 1505–1511.
- Hollingworth, H. L. (1910). The central tendency of judgment. *The Journal of Philosophy, Psychology and Scientific Methods*, 7, 461–469.
- Kattner, F. & Ellermeier, W. (2014). Fractionation of pitch intervals: an axiomatic study testing monotonicity, commutativity, and multiplicativity in musicians and non-musicians. *Attention, Perception, & Psychophysics*, 76, 2508–2521.
- Lapid, E., Ulrich, R., & Rammsayer, T. (2008). On estimating the difference limen in duration discrimination tasks: A comparison of the 2AFC and the reminder task. *Perception & Psychophysics*, 70, 291–305.

- Luce, R. D. (2002). A psychophysical theory of intensity proportions, joint presentations, and matches. *Psychological Review*, 109, 520–532.
- Luce, R. D. (2004). Symmetric and asymmetric matching of joint presentations. *Psychological Review*, 111, 446–454.
- Luce, R. D. (2013). Analogs in Luce’s global psychophysical theory of Stevens’s psychophysical regression effect. *American Journal of Psychology*, 126, 47–52.
- Luce, R. D., Steingrímsson, R., & Narens, L. (2010). Are psychophysical scales of intensities the same or different when stimuli vary on other dimensions? theory with experiments varying loudness and pitch. *Psychological Review*, 117, 1247–1258.
- Nachmias, J. (2006). The role of virtual standards in visual discrimination. *Vision Research*, 46, 2456–2464.
- Narens, L. (1996). A theory of ratio magnitude estimation. *Journal of Mathematical Psychology*, 40, 109–129.
- Nelder, J. A. & Mead, R. (1965). A simplex algorithm for function minimization. *Computer Journal*, 7, 308–313.
- R Core Team (2018). *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing.
- Steingrímsson, R. (2009). Evaluating a model of global psychophysical judgments for brightness: I. behavioral properties of summations and productions. *Attention, Perception, & Psychophysics*, 71, 1916–1930.
- Steingrímsson, R. (2016a). Empirical evaluation of a model of global psychophysical judgments for perceived contrast: I. behavioral properties of summations and productions. Unpublished manuscript. Ohio State University, Columbus, Ohio, USA.
- Steingrímsson, R. (2016b). Subjective intensity: Behavioral laws, numerical representations, and behavioral predictions in Luce’s model of global psychophysics. *Journal of Mathematical Psychology*, 75, 205–217.
- Steingrímsson, R. & Luce, R. D. (2005). Evaluating a model of global psychophysical judgments: I. behavioral properties of summations and productions. *Journal of Mathematical Psychology*, 49, 290–307.
- Steingrímsson, R. & Luce, R. D. (2012). Predictions from a model of global psychophysics about differences between perceptual and physical matches. *Attention, Perception, & Psychophysics*, 74, 1668–1680.
- Steingrímsson, R., Luce, R. D., & Narens, L. (2012). Brightness of different hues is a single psychophysical ratio scale of intensity. *American Journal of Psychology*, 125, 321–333.
- Stevens, J. C. & Marks, L. E. (1965). Cross-modality matching of brightness and loudness. *Proceedings of the National Academy of Sciences*, 54, 407–411.

- Stevens, S. S. (1975). *Psychophysics: introduction to its perceptual, neural, and social prospect*. New York, NY: Wiley.
- Stevens, S. S. & Greenbaum, H. B. (1966). Regression effect in psychophysical judgment. *Perception & Psychophysics*, 1, 439–446.
- Zimmer, K. (2005). Examining the validity of numerical ratios in loudness fractionation. *Perception & Psychophysics*, 67, 569–579.
- Zwicker, E. (1958). Über psychologische und methodische Grundlagen der Lautheit (On the psychological and methodological basis of loudness). *Acta Acustica united with Acustica*, 8(Supplement 1), 237–258.

Appendix Proofs

Proposition 1

From representation (9) it follows that for all positive p, q ,

$$W(p) \cdot W(q) = \frac{\psi(x_p) - \psi(\rho_{i_p}^{\leftarrow})}{\psi(x) - \psi(\rho_{i_p}^{\rightarrow})} \cdot \frac{\psi(x_{p,q}) - \psi(\rho_{i_q}^{\leftarrow})}{\psi(x_p) - \psi(\rho_{i_q}^{\rightarrow})} \quad (\text{A.1})$$

$$W(q) \cdot W(p) = \frac{\psi(x_q) - \psi(\rho_{i_q}^{\leftarrow})}{\psi(x) - \psi(\rho_{i_q}^{\rightarrow})} \cdot \frac{\psi(x_{q,p}) - \psi(\rho_{i_p}^{\leftarrow})}{\psi(x_q) - \psi(\rho_{i_p}^{\rightarrow})}. \quad (\text{A.2})$$

First, if the equations $\rho_{i_p}^{\leftarrow} = \rho_{i_p}^{\rightarrow} = \rho_{i_q}^{\leftarrow} = \rho_{i_q}^{\rightarrow}$ hold then it is easily seen that intradimensional commutativity follows. Conversely, assume intra-dimensional commutativity and select p such that $x_p = x$ (notice that p may, but need not equal 1, depending on $W(1)$ and/or distinct reference intensities). Replacing x_p by x and $x_{p,q} = x_{q,p}$ by x_q in both (A.1) and (A.2) leads to

$$\frac{\psi(x) - \psi(\rho_{i_p}^{\leftarrow})}{\psi(x) - \psi(\rho_{i_p}^{\rightarrow})} = \frac{\psi(x_q) - \psi(\rho_{i_p}^{\leftarrow})}{\psi(x_q) - \psi(\rho_{i_p}^{\rightarrow})},$$

which in turn provides

$$\psi(x) \cdot [\psi(\rho_{i_p}^{\leftarrow}) - \psi(\rho_{i_p}^{\rightarrow})] = \psi(x_q) \cdot [\psi(\rho_{i_p}^{\leftarrow}) - \psi(\rho_{i_p}^{\rightarrow})].$$

This equation can hold for all positive q and all x only if $\rho_{i_p}^{\leftarrow} = \rho_{i_p}^{\rightarrow}$. By an analogous argument, selecting q such that $x_q = x$, and equating (A.1) and (A.2) implies

$$\psi(x) \cdot [\psi(\rho_{i_q}^{\rightarrow}) - \psi(\rho_{i_q}^{\leftarrow})] = \psi(x_p) \cdot [\psi(\rho_{i_q}^{\rightarrow}) - \psi(\rho_{i_q}^{\leftarrow})].$$

Again, this equation can hold for all positive p and all x only if $\rho_{i_q}^{\leftarrow} = \rho_{i_q}^{\rightarrow}$.

Equating (A.1) and (A.2), plugging in the already derived constraints by setting $\rho_{i_p} = \rho_{i_p}^{\leftarrow} = \rho_{i_p}^{\rightarrow}$ and $\rho_{i_q} = \rho_{i_q}^{\leftarrow} = \rho_{i_q}^{\rightarrow}$, and rearranging terms provides

$$\frac{\psi(x_p) - \psi(\rho_{i_p})}{\psi(x) - \psi(\rho_{i_p})} \cdot \frac{\psi(x_q) - \psi(\rho_{i_p})}{\psi(x_{q,p}) - \psi(\rho_{i_p})} = \frac{\psi(x_p) - \psi(\rho_{i_q})}{\psi(x) - \psi(\rho_{i_q})} \cdot \frac{\psi(x_q) - \psi(\rho_{i_q})}{\psi(x_{p,q}) - \psi(\rho_{i_q})}.$$

Assuming intra-dimensional commutativity (10), both sides of the equation are equal, except for their dependence on the reference intensities ρ_{i_p} and ρ_{i_q} , respectively. It follows that this equation can hold for all intensities x and all positive p, q only if $\rho_{i_p} = \rho_{i_q}$, which finally establishes the asserted equality. $\square$

Proposition 2

i. From representation (8) and (4) it follows that in case of $f \rightarrow g \rightarrow f$ and for $i = i_p = i_q$

$$\begin{aligned} W(p) \cdot W(q) &= \frac{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_f(x_{p,q}^{f,g,f}) - \psi_f(\rho_i^{f \leftarrow g})}{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \rightarrow f})} \\ W(q) \cdot W(p) &= \frac{\psi_g(x_q^{f,g}) - \psi_{f,g}(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_f(x_{q,p}^{f,g,f}) - \psi_f(\rho_i^{f \leftarrow g})}{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \rightarrow f})}. \end{aligned}$$

If equation $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}$ holds then it is easily seen that the stated commutativity follows. Conversely, assuming $x_{p,q}^{f,g,f} = x_{q,p}^{f,g,f}$ implies

$$\frac{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \rightarrow f})} = \frac{\psi_g(x_q^{f,g}) - \psi_{f,g}(\rho_i^{g \leftarrow f})}{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \rightarrow f})}.$$

This equation can hold for all x and for all positive p, q only if $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}$.

ii. Again from (8) and (4) it follows that in case of $f \rightarrow g \rightarrow g$ and for $i = i_p = i_q$

$$\begin{aligned} W(p) \cdot W(q) &= \frac{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_g(x_{p,q}^{f,g,g}) - \psi_g(\rho_i^{g \leftarrow g})}{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \rightarrow g})} \\ W(q) \cdot W(p) &= \frac{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_g(x_{q,p}^{f,g,g}) - \psi_g(\rho_i^{g \leftarrow g})}{\psi_g(x_q^{f,g}) - \psi_{f,g}(\rho_i^{g \rightarrow g})} \end{aligned} \quad (\text{A.3})$$

As in i the equation $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow g}$ implies $x_{p,q}^{f,g,g} = x_{q,p}^{f,g,g}$. Conversely, assuming commutativity implies

$$\frac{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \rightarrow g})} = \frac{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \rightarrow g})}.$$

Again this equation can hold for all x and for all positive p, q only if $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow g}$. $\square$

Proposition 3

Given (8) and (4), confine consideration to the case $i = i_p = i_q$. Then, plugging the constraints $\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f}$, $\rho_i^{f \leftarrow g} = \rho_i^{f \leftarrow f}$, $\rho_i^{g \leftarrow f} = \rho_i^{g \rightarrow f}$, and $\rho_i^{f \leftarrow f} = \rho_i^{f \rightarrow f}$ into

$$\begin{aligned} W(p) \cdot W(q) &= \frac{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_f(x_{p,q}^{f,g,f}) - \psi_f(\rho_i^{f \leftarrow g})}{\psi_g(x_p^{f,g}) - \psi_g(\rho_i^{g \rightarrow f})}, \\ W(p) \cdot W(q) &= \frac{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow f})} \cdot \frac{\psi_f(x_{p,q}^{f,f,f}) - \psi_f(\rho_i^{f \leftarrow f})}{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \rightarrow f})}, \end{aligned}$$

immediately provides $\psi_f(x_{p,q}^{f,g,f}) = \psi_f(x_{p,q}^{f,f,f})$, and the assertion follows by monotonicity. $\square$

Proposition 4

Again, treating the case $i = i_p = i_q$, given (8) and (4), yields

$$\begin{aligned} W(p) \cdot W(q) &= \frac{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow f})} \cdot \frac{\psi_g(x_{p,q}^{f,f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \rightarrow g})} \\ W(q) \cdot W(p) &= \frac{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_g(x_{q,p}^{f,g,g}) - \psi_g(\rho_i^{g \leftarrow g})}{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \rightarrow g})} \end{aligned}$$

Plugging in the constraints $\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f} = \rho_i^{f \leftarrow f}$ and $\rho_i^{g \leftarrow f} = \rho_i^{g \leftarrow g} = \rho_i^{g \rightarrow g}$ immediately provides $\psi_g(x_{p,q}^{f,f,g}) = \psi_g(x_{q,p}^{f,g,g})$, and the assertion follows by monotonicity. $\square$

Corollary 2

Given (8) and (4) and treating the case $i = i_p = i_q$, assume $\rho_i^{f \rightarrow f} = \rho_i^{f \leftarrow f}$ and $\rho_i^{g \leftarrow f} = \rho_i^{g \leftarrow g} = \rho_i^{g \rightarrow g}$. Sufficiency of the constraint is obvious from Proposition 4, so let us consider its necessity. Plugging the assumed equations as well as commutativity $x_{p,q}^{f,f,g} = x_{q,p}^{f,g,g}$ into

$$\begin{aligned} W(p) \cdot W(q) &= \frac{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow f})} \cdot \frac{\psi_g(x_{p,q}^{f,f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \rightarrow g})} \\ W(q) \cdot W(p) &= \frac{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})} \cdot \frac{\psi_g(x_{q,p}^{f,g,g}) - \psi_g(\rho_i^{g \leftarrow g})}{\psi_g(x_q^{f,g}) - \psi_g(\rho_i^{g \rightarrow g})} \end{aligned}$$

provides

$$\frac{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \leftarrow f})}{\psi_f(x_p^{f,f}) - \psi_f(\rho_i^{f \rightarrow g})} = \frac{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow f})}{\psi_f(x^f) - \psi_f(\rho_i^{f \rightarrow g})}.$$

For fixed x^f the righthand side is constant, and so is the lefthand side for all p , which forces $\rho_i^{f \rightarrow g} = \rho_i^{f \rightarrow f}$. $\square$

Proposition 5

According to (8) and (4) cross-modal matching is represented by

$$\begin{aligned} W(1) &= \frac{\psi_g(x_1^{f,g}) - \psi_g(\rho_+^{g \leftarrow f})}{\psi_f(x^f) - \psi_f(\rho_+^{f \rightarrow g})} \\ W(1) &= \frac{\psi_f(y_1^{g,f}) - \psi_f(\rho_+^{f \leftarrow g})}{\psi_g(y^g) - \psi_g(\rho_+^{g \rightarrow f})}. \end{aligned}$$

Rearranging terms provides

$$\psi_g(y^g) = \frac{1}{W(1)} \cdot [\psi_f(y_1^{g,f}) - \psi_f(\rho_+^{f \leftarrow g})] + \psi_g(\rho_+^{g \rightarrow f}) \quad (\text{A.4})$$

$$\psi_g(x_1^{f,g}) = W(1) \cdot [\psi_f(x^f) - \psi_f(\rho_+^{f \rightarrow g})] + \psi_g(\rho_+^{g \leftarrow f}). \quad (\text{A.5})$$

- i. Under the constraints $\rho_+^{f \leftarrow g} = \rho_+^{f \rightarrow g}$ and $\rho_+^{g \leftarrow f} = \rho_+^{g \rightarrow f}$ the conclusion that $x^f = y_1^{g,f}$ implies $y^g = x_1^{f,g}$ if and only if $W(1) = 1$ holds for all intensities due to strict monotonicity of ψ_f and ψ_g .
- ii. If intensity approaches infinity then the terms for the reference intensities in (A.4) and (A.5) can be neglected, because ψ_f and ψ_g are assumed to be unbounded above. Given the strict monotonicity of the psychophysical functions, then $x^f = y_1^{g,f}$ implies $y^g = x_1^{f,g}$ if and only if $W(1) = 1$.
- iii. Given $W(1) = 1$, setting $x^f = y_1^{g,f}$ and $y^g = x_1^{f,g}$ in (A.4) and (A.5) immediately provides the equation stated in (17). Conversely, rearranging terms in (17), and plugging the result into (A.4) and (A.5) provides that $x^f = y_1^{g,f}$ implies $y^g = x_1^{f,g}$. $\square$