The Psychological Scaffolding of Arithmetic

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Authors' note

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This paper is dedicated to the memory of Georgina Carvell, whose intelligence, kindness, and good humor has been an inspiration to us.

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Abstract

Where does arithmetic come from, and why are addition and multiplication its fundamental operations? Although we *know* that arithmetic is true, no explanation that meets standards of scientific rigor is available from philosophy, mathematical logic, or the cognitive sciences. We propose a new approach based on the assumption that arithmetic has a biological origin: Many examples of adaptive behavior such as spatial navigation suggest that organisms can perform arithmetic-like operations on represented magnitudes. If so, these operations – non-symbolic precursors of addition and multiplication – might be optimal due to evolution and thus identifiable according to an appropriate criterion. We frame this as a meta-mathematical question, and using an order-theoretic criterion, prove that four qualitative conditions – monotonicity, convexity, continuity, and isomorphism – are sufficient to identify addition and multiplication over the real numbers uniquely from the uncountably infinite class of possible operations. Our results show that numbers and algebraic structure emerge from purely qualitative conditions, and as a construction of arithmetic, provide a rigorous explanation for why addition and multiplication are its fundamental operations. We argue that these conditions are pre-verbal psychological intuitions or principles of perceptual organization that are biologically based and shape how humans and nonhumans alike perceive the world. This is a Kantian view, and suggests that arithmetic need not be regarded as an immutable truth of the universe, but rather as a natural consequence of our perception. Algebraic structure may be inherent in the representations of the world formed by our perceptual system.

Keywords: arithmetic, number, numerical cognition, comparative cognition, mathematical psychology

“We used to think that if we knew one, we knew two, because one and one are two. We are finding that we must learn a great deal more about 'and'.”

Sir Arthur Eddington

1. Introduction

Where does arithmetic come from? Is it based on our experience of the physical world, psychological intuition, or a Platonic realm of mathematical ideas? This question has been debated for centuries but is especially consequential now because mathematics has become so important for modern life. Not only are math skills critical for success in education and employment (Duncan et al., 2007; Siegler et al., 2012), but we are in the midst of a ‘fourth industrial revolution’ in which smart technologies such as mobile phones, social media and the internet of things – all enabled by binary arithmetic – are transforming our daily lives (Schwab, 2017), and not always for the better (Best, Manktelow, & Taylor, 2014; Lau, 2017; Perloff, 2014). Our approach is to ask why addition and multiplication are the fundamental operations of arithmetic. We show that the answer reveals a key insight into arithmetic’s origins – that addition and multiplication logically emerge from a handful of basic, well-researched psychological concepts.

The origins of number and number-related concepts have been investigated in many empirical studies. Evidence shows that humans have a ‘number sense’, shared with nonhumans, which supports basic abilities such as comparisons of relative numerosity, approximate judgments of quantity (‘approximate number system’ or ANS), and precise enumeration of small numbers of objects (parallel individuation or subitizing; Burr, Anobile, & Arrighi, 2018; Dehaene, 2011; Feigenson, Dehaene, & Spelke, 2004). Performance on numerical tasks is associated with increased activation in the intraparietal sulcus (Piazza &

Izard, 2009; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004) and neurons have been found that respond selectively to number have been found in primates (Nieder, 2016; Nieder, Freedman, & Miller, 2002) avians (Kobylykov et al., 2022) and fish (Messina et al., 2021). Number sense may be part of a general system for processing magnitudes (e.g., space, time, number; Gallistel & Gelman, 2000; Leibovich, Katzin, Harel, & Henik, 2017; Newcombe, Levine, & Mix, 2015; Walsh, 2003) and research has identified neural pathways associated with magnitude processing in nonhuman species (including invertebrates; see Lorenzi, Perrino & Vallortigara, 2021 and Vallortigara et al., 2022, for review). Overall, there is considerable evidence that number sense is linked to specific brain areas and provides an evolutionary basis for numerical cognition, but its role in learning number-related concepts in childhood has been controversial (Carey & Barner, 2019; Le Corre & Carey, 2007; Odic, Le Corre, & Halberda, 2015; Szkudlarek & Brannon, 2017).

What about arithmetic? It seems reasonable to assume that numbers and arithmetic have a common origin, because arithmetic consists of operations performed with numbers. Consistent with this, there is some evidence that number sense provides a scaffolding for learning of symbolic arithmetic and related skills. Longitudinal studies have shown that number sense in infancy predicts childhood numeracy (Starr, Libertus, & Brannon, 2013), and numerosity discrimination in adolescence predicts mathematics achievement (Chen & Li, 2014; Halberda, Mazzocco, & Feigenson, 2008). Number sense has also had significant impact on the teaching of mathematics and interventions for developmental dyscalculia (Berch, 2005; Wilson & Dehaene, 2007; but see Szűcs & Myers, 2017 for a critical view).

Yet arithmetic is remarkable in ways that evolutionary precursors to number and number-related concepts would seem unable to explain. First, it is unique to humans. Arithmetic requires numeric symbols to be used precisely according to well-defined rules, but there is no evidence that nonhumans can use numerical information beyond the subitizing

range except in an approximate way (e.g., Bryer et al., 2022; Cantlon, Merritt, & Brannon, 2016; Ferrigno, Jara-Ettinger, Piantadosi, & Cantlon, 2017; Tomonaga, 2008). In a review of research on the numeracy skills of non-human primates, Matsuzawa (2009, p. 92) concluded: “Chimpanzees do not possess human-like capabilities for representation at an abstract level.”

Second, arithmetic is universal; results agree across all human cultures that have developed it independently. This is true for both elementary (e.g., $2 + 2 = 4$) and more advanced results, such as the Pythagorean theorem (Hodgkin, 2005). The universality of arithmetic contrasts strongly with the widespread variation in numeric symbols used across different cultures (Chrisomalis, 2010; O'Shaughnessy, Gibson, & Piantadosi, 2021). Third, arithmetic (and mathematics more generally) is astonishingly successful at describing the physical world, with reasons for its ‘unreasonable effectiveness’ (Wigner, 1960) a longstanding conundrum for both physicists and philosophers of science (McDonnell, 2017; Steiner, 1998; Weinberg, 2011).

Because arithmetic is both uniquely human and universal across cultures it suggests, perhaps even more strongly than language, a discontinuity between human and non-human minds (Penn, Holyoak, & Povinelli, 2008). Elucidating the provenance of arithmetic is thus vital for understanding the basis of human cognition and our capacity for abstract relational thought.

Arithmetic consists of numeric symbols and rules for their manipulation, which define both a set of numbers and operations that combine pairs of numbers to give another number. We seek clues to arithmetic’s origin by asking why, of all possible ways that numbers might be combined, these operations are addition and multiplication. It might seem pointless to ask because they are obviously the right choice, but our contention is that we do not have a satisfactory answer: Although we *know* that arithmetic is true (in some sense), we do not know *why* it is so. The difficulty of reconciling accounts of knowledge and truth is a long-

standing problem in the philosophy of mathematics (i.e., epistemology and semantics; Benacerraf, 1973; Maddy, 1991; Nutting, 2022).

Here we provide a formal, mathematical explanation for why addition and multiplication are the fundamental operations of arithmetic. To anticipate our results, we prove that if several qualitative conditions are assumed – monotonicity, convexity, continuity and isomorphism – addition and multiplication can be uniquely identified as operations over the real numbers. Thus, the algebraic structure on which numbers and arithmetic are based can be derived from purely qualitative conditions. We argue that these conditions have, prior to their mathematical definition, a psychological existence as basic intuitions about the world or principles of perceptual organization, encompassing notions of magnitude, space and time, objects and their motion, and relations between objects. Evidence for them is found with humans (including infants) and nonhumans, suggesting they are part of our common evolutionary heritage. We conclude that arithmetic's origin is not found in our experience of the world, but in how our mind structures perception of that world. This is fundamentally a Kantian argument that precisely specifies the qualitative conditions, inherent in our perception, which are sufficient for arithmetic and algebraic structure.

The organization of this paper is as follows. First in §2 we consider perspectives on the origin of arithmetic from philosophy, mathematical logic, and psychology/cognitive science. We argue that prior approaches do not give a rigorous explanation for why addition and multiplication are fundamental to arithmetic. Philosophical and psychological accounts have either been based on experience with the physical world or appeals to intuition and thus lack rigor, whereas the foundations of mathematics (e.g., the Dedekind-Peano axioms) give a logical reconstruction of arithmetic, not an explanation.

In §3 we propose a new approach based on the observation that many nonhuman species are capable of adaptive behavior that seems to require the equivalent of arithmetic computation (e.g., spatial navigation by path integration; Gallistel, 1990, 2017; Grace et al., 2020). This suggests that arithmetic may have a biological basis in that organisms have the ability to represent magnitudes and perform operations on them. If evolution has shaped this ability, these operations – the precursors of addition and multiplication – are expected to be optimal and thus identifiable according to an appropriate criterion. We translate this reasoning into a formal mathematical question: Starting with minimal structure – a set and all possible operations mapping two of its elements to another element – we ask what qualitative conditions or assumptions are sufficient for addition and multiplication to be uniquely identified from the class of all possible operations. We prove that given a countably infinite set equipped with an order relation (an $\mathbb{N}$ – chain; see Appendix A and B for details), that conditions of monotonicity, convexity, continuity and isomorphism are sufficient to identify addition and multiplication over the real numbers uniquely with an order-theoretic criterion, that is, as the minimum element in the uncountably infinite class of possible operations. This result is a formal explanation for why addition and multiplication are the fundamental arithmetic operations.

In §4, we summarize evidence that monotonicity, convexity, continuity and isomorphism (MCCI) characterize perception and cognition in humans and nonhumans. We argue that prior to their mathematical definition, these conditions have a psychological existence as basic intuitions about the world regarding time, space, objects and motion, suggesting that algebraic structure may be inherent in perception. In §5 we consider some broader implications of these findings. Our results show that for any symbol system equipped with an order, MCCI are sufficient to yield algebraic structure. If perceptual systems encode environmental magnitudes at the molecular level (e.g., distance travelled,

angle with respect to the sun), as Gallistel (2017) has proposed, MCCI could explain why computations can apparently be performed with them (see also Akhlaghpour, 2022). The universality of addition and multiplication across all cultures that have developed numeral notation systems, together with our results, suggest that the concepts which support numbers and arithmetic are innate and share a common evolutionary basis. Finally, we discuss why symbolic arithmetic is a uniquely human activity and offer some thoughts on the ‘unreasonable effectiveness’ of mathematics in the physical sciences.

2. Perspectives on the Origin of Arithmetic

Our approach to investigating the origin of arithmetic is to ask, why are addition and multiplication its basic operations? Of all the possible ways that two numbers can be combined to give another number, why are addition and multiplication fundamental? An answer could constitute a formal explanation for arithmetic, and provide clues as to its origin. This might seem unnecessary because it is self-evidently true, but because arithmetic is essential to so much of our mathematical and scientific knowledge the question is worth investigating.

In this section, we give a brief overview of various perspectives on the origin of arithmetic. We review accounts from philosophy of mathematics, mathematical logic (i.e., logicism) and psychology/cognitive science, and ask whether they successfully provide a rigorous answer for the question above. Our assessment is based on principles of scientific explanation described by Hempel’s (1965) deductive-nomological (DN) model (see also Hempel & Oppenheim, 1948), where a phenomenon (‘explanandum’) is considered to be scientifically explained when it can be logically deduced from initial conditions and general laws (‘explanans’). Other qualities of good explanations (and theories) are exactness,

fruitfulness and simplicity (Carnap, 1962). Although Hempel's model has been philosophically criticized it is still generally regarded as a useful framework for the physical sciences (Salmon, 2006; Woodward & Zalta, 2017), and given the indispensability of mathematics for the physical sciences (Colyvan, 2001a) it seems appropriate to apply these criteria here.

Philosophy of Mathematics

Mathematics has been a central concern of philosophy from the ancient Greeks to the present day. Here we briefly sketch some of the main themes to illustrate how philosophers have attempted to explain its origins. For general overviews of the philosophy of mathematics, see Avigad (2007), Benacerraf and Putnam (1984), Horsten (2007), Shapiro (2000), Shapiro and Wainwright (2005), and Hacking (2014); and for mathematicians' views of their enterprise see Davis and Hersh (1981), Hadamard (1954), and Mac Lane (2012).

In broad terms, philosophers have endeavored to account for two features of mathematics that appear paradoxical. First, mathematics is both *necessary* and *a priori* – necessary because it is impossible to imagine a world where triangles didn't have three sides or that 18 was a prime number, and *a priori* because mathematical knowledge seems to be arrived at independent of previous sensory experience. But mathematics also accords so well with the physical world. The tension between these features was reflected in the polarity between rationalism and empiricism which dominated the philosophy of mathematics until the 18th century. Rationalist views included Platonism, which holds that mathematical objects existed in an ideal realm that is accessible to thought but independent of any particular observer, and the contributions of Descartes and Leibniz, who maintained that reason was the source of all certain knowledge. Empiricist approaches were developed most

notably by English and Scottish philosophers such as Locke, Hume and Reid, and held that knowledge was wholly derivable from sense experience.

Kant (1781/1908) famously attempted to reconcile rationalist and empiricist accounts by developing a system that was consistent with the exact sciences of his time, Euclidean geometry and Newtonian physics (Friedman, 1992). Central to his system was the distinction between analytic and synthetic statements. According to Kant, analytic statements (e.g., ‘All bachelors are unmarried’) are true by definition, whereas synthetic statements are ‘ampliative’, in the sense that their predicate provides information not contained within their subject (e.g., ‘all crows are black’). While most synthetic statements thus depend for their verification on sense experience, crucially, Kant held that arithmetic knowledge was both synthetic and prior to sense experience (*‘synthetic a priori’*), since statements such as “ $7 + 5 = 12$ ” are ampliative (the concept of ‘12’ is not contained in the concepts of ‘7’, ‘5’ and ‘+’) yet knowable without recourse to the external world. For Kant, arithmetic judgments (such as that above), and geometric judgements (such as Euclid’s axioms), are grounded in the ‘pure intuitions’ of time and space, respectively, which structure and define the limits of human experience.

J. S. Mill (1843) proposed an empiricist account of arithmetic as part of an attempt to describe a ‘science of reasoning’ which provided the basis for inductive and deductive logic. According to Mill, numbers corresponded to properties associated with collections of physical objects. Combining sets of objects defined ‘addition’ in terms of our sense experience; for example, two pebbles arranged next to three pebbles gives a row of five pebbles. Mill’s account has been described as an example of ‘psychologism’, because it attempted to explain mathematical laws via psychological principles (Kusch, 2005).

Although it seems reasonable for small quantities, it was dismissed by Frege, who pointed out that no sense experience could reasonably prove that $135,664 + 37,863 = 173,527$ was true

(Frege, 1980/1884, p. 6). Addition seems to require more than just our physical experience with collections of objects.

While offering different perspectives on the source of mathematical knowledge, Plato's, Kant's and Mill's accounts do not provide a rigorous explanation for why addition and multiplication are the fundamental arithmetic operations. Neither do philosophical views that emerged later in the 20th century, including intuitionism (Dummett, 2000; Heyting, 1966), quasi-empiricism (Lakatos, 1976), naturalism (Kitcher, 1988; Maddy, 1997; Putnam, 2014; Quine, 1969), structuralism (Resnik, 1997), or nominalism (Field, 2016). Likely this is because philosophers have generally been concerned with how mathematics differs from other sciences in terms of its objects of study, methods and knowledge – that is, its ontology and epistemology (Horsten, 2007). Regardless of its origin and ontological status, simple arithmetic is a cornerstone of mathematical knowledge, so *why* arithmetic is as it is doesn't seem to require a philosophical explanation of its own.

Logicism

Developments in mathematics and logic from the 1870s onwards by Dedekind, Frege, Peano and others provided a logical foundation for arithmetic in terms of a set of axioms and stimulated a new analytic approach to philosophy in which arguments were made in formal and rigorous terms. Gillies (2013) provides a comparative analysis of Dedekind, Frege and Peano's work, and for overviews of analytic philosophy see Beaney (2013) and Schwartz (2012). The philosophy of mathematics associated with this approach is known as *logicism*. Here we describe a modern version of the Dedekind-Peano axioms which provide a logical basis for the natural numbers and show how addition and multiplication can be derived from them – in this way, the axioms provide a formal construction of arithmetic.

The Dedekind-Peano system includes two non-logical symbols, a constant symbol 0 , and a unary function symbol S . The axioms are:

1. 0 is a natural number.
2. For each natural number a , $S(a)$ is a natural number.
3. For each natural number a , $S(a) \neq 0$.
4. For all natural numbers a and b , if $S(a) = S(b)$ then $a = b$.
5. If X is a set such that:
 - 0 is in X , and
 - For each natural number a in X , $S(a)$ is in X

then X contains every natural number.

The unary function S is often described as the successor function, and intuitively defines the notion of ‘next’; thus, $S(0)$ corresponds to the natural number 1, $S(S(0))$ to 2, $S(S(S(0)))$ to 3, and so forth. Informally stated, axiom 1 provides a starting point to apply the successor function; axiom 2 requires that the naturals are closed under the successor function; axiom 3 that 0 is not the successor of any natural number; axiom 4 that distinct natural numbers have distinct successors (i.e., S is injective); and axiom 5 that any set containing 0 and closed under the successor function contains all the natural numbers.¹ These axioms define the natural numbers, and can also be used to define the binary operations of addition and multiplication recursively as follows:

Addition (+) is the unique binary operation on the natural numbers satisfying the following recursive constraints:

$$1. \quad a + 0 = a$$

$$2. \quad a + S(b) = S(a + b)$$

With addition uniquely identified, multiplication ($\cdot$) can be defined as the unique binary operation on the natural numbers satisfying the following recursive constraints:

$$1. \quad a \cdot 0 = 0$$

$$2. \quad a \cdot S(b) = a + (a \cdot b)$$

The resulting system is known as ‘Peano arithmetic’.

The story of logicism is well known. Just as Frege was finishing the second volume of his *Grungesetze der Arithmetik* in 1903, he received a letter from the young Bertrand Russell noting a contradiction in one of the logical principles used to describe mathematical functions. Frege acknowledged Russell’s contribution in an appendix but was never able to correct the problem. Whitehead and Russell published *Principia Mathematica* in 1913, which by resolving the contradiction in Frege’s system and other problems in set theory, attempted to provide a logical basis for mathematics (Whitehead & Russell, 1997). Hilbert developed a foundations program in the 1920s in which all branches of mathematics were to be defined in terms of a set of axioms that were both complete (i.e., able to prove all true theorems) and logically consistent (Franks, 2009; Grattan-Guinness, 2011). Interestingly, in justifying his program Hilbert (1926; in Van Heijenoort, 1967) seemed to suggest that arithmetic (which he called ‘elementary number theory’) was infallible:

It is necessary to make inferences everywhere as reliable as they are in ordinary elementary number theory, which no one questions and in which contradictions and paradoxes arise only through our carelessness (p. 376).

However, Hilbert's program ultimately failed when Gödel (1931) famously showed that no logical system rich enough to support arithmetic could be both complete and without contradiction. Quine (1965) commented:

To say that mathematics in general has been reduced to logic hints at some new firming up of mathematics at its foundations. This is misleading. Set theory is less settled and more conjectural than the classical mathematical superstructure that can be founded upon it (p. 125).

Logicism may have failed to provide a comprehensive foundation for mathematics, but we are interested in a more modest goal: Do the Dedekind-Peano axioms explain why addition and multiplication are the fundamental operations of arithmetic? Addition and multiplication are defined in terms of logical concepts, so the criterion of exactness is satisfied. If the axioms are regarded as prior conditions, then addition and multiplication might be considered as deductions from those conditions, consistent with Hempel's (1965) model of scientific explanation. The research program in mathematical logic that grew out of the Dedekind-Peano axioms has also been very fruitful, even if Gödel's work rendered the original goal unattainable – there is a direct lineage from mathematical logic to theoretical computer science (Church, 1936; Turing, 1936).

However, the Dedekind-Peano axioms are properly characterized as a logical reconstruction of addition and multiplication, not an explanation. As such, the axioms fail on the simplicity criterion – they are more complex than the phenomena they seek to explain. After all, no one has ever suggested that children should learn arithmetic by studying them! A similar point was made by Wang (1957) in an article tracing their historical emergence:

The application of the axiomatic method in the development of numbers is not natural. Its rather late appearance is evidence. If, however, our problem is to have a systematic understanding of the domain of number theory, the axiomatic method suggests itself. Indeed, we now realize that the method can be applied to numbers, and that for certain purposes the application is suitable. It remains a rare thing to teach arithmetic from an axiomatic approach. (p. 146).

Finally, we note that recent work in the philosophy of mathematics has argued strongly against the possibility that logicist (and neo-logicist) accounts that rely on axiomatic definitions can explain the origins of our arithmetical knowledge (Nutting, 2013, 2016, 2018):

While... use of explicit definitions in arithmetical proofs helps us develop secure knowledge of arithmetical truths, our ability to understand those proofs presupposes an understanding of the structure of arithmetic. Our most basic arithmetical knowledge is not grounded in these explicit definitions, but is cognitively prior to those definitions (Nutting, 2013, p. 134).

Psychology/Cognitive Science

The most comprehensive attempt from psychology and cognitive science to explain the origin of mathematics is Lakoff and Núñez' (2000) book, "*Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being*". In the preface, they stated their purpose was to create a new discipline of 'mathematical idea analysis', a 'cognitive science of mathematics' (p. xi). This discipline was necessary because the intellectual content of mathematics, Lakoff and Núñez claimed, lies not in symbols but the ideas behind them, so a systematic analysis of human mathematical ideas might finally explain where mathematics comes from. Lakoff and Núñez strongly argued against Platonism and the 'romance of mathematics' – that mathematics has any kind of abstract or objective existence: "the more we have applied what we know about cognitive science to understand the cognitive

structure of mathematics, the more it has become clear that this romance cannot be true” (p. xvi).

Lakoff and Núñez assume that humans have an innate capacity for number sense, which provides abilities to quantify small numbers of objects (subitizing) as well as approximate judgements of quantity. They propose that the ideas underlying mathematics are ‘conceptual metaphors’ which are based on both number sense and experience with objects in the world. There are four ‘grounding metaphors’ which provide the basis for arithmetic, and then ‘linking metaphors’ which are later used to extend arithmetic concepts to other areas. Linking metaphors are associated with abstract concepts – for example, geometrical figures as algebraic equations.

The first grounding metaphor is Arithmetic as Object Collection, which is explained as follows:

If a child is given a group of three blocks, she will naturally subitize them automatically and unconsciously as being three in number. If one is taken away, she will subitize the resulting group as two in number. Such everyday experiences of subitizing, addition, and subtraction with small collections of objects involve correlations between addition and adding objects to a collection and between subtraction and taking objects away from a collection. Such regular correlations, we hypothesize, result in neural connections between sensory-motor physical operations like taking away objects from a collection and arithmetic operations like the subtraction of one number from another. Such neural connections, we believe constitute a conceptual metaphor at the neural level – in this case, the metaphor that Arithmetic is Object Collection. This metaphor, we hypothesize, is learned at an early age, prior to any formal arithmetic training. (p. 55).

Multiplication and division are explained as a ‘pooling/splitting’ extension of the Arithmetic as Object Collection metaphor. Lakoff and Núñez give a list of examples of how the formal properties of arithmetic (e.g., commutativity, associativity, additive and multiplicative identity) are learned through physical experience with object collections. They also describe three other grounding metaphors – Arithmetic as Object Construction, The Measuring Stick

Metaphor, and Arithmetic As Motion Along a Path – that are important for learning arithmetic. In the remaining sections of the book, Lakoff and Núñez show how these principles can be extended by linking metaphors to account for more abstract areas of mathematics, including algebra, logic and set theory, the concept of infinity, the construction of the real numbers, and Euler’s equation. They also include a chapter which summarizes the implications of their embodied cognition perspective for the philosophy of mathematics.

Lakoff and Núñez’ theory has been strongly criticized by mathematicians for its lack of rigor and inability to explain areas of mathematics, such as complex analysis, which are not easily understood in terms of physical metaphors (e.g., Auslander, 2001; Davis, 2005; Gold, 2001). Their Arithmetic as Object Collection metaphor also strongly resembles Mill’s (1843) empiricism. In effect, Lakoff and Núñez have hypothesized a mediating cause (conceptual metaphors) for the inductive inference posited by Mill, but otherwise their theory makes a similar claim that arithmetic works because it corresponds to operations performed with physical objects. In that case, criticisms that have been lodged against empiricist accounts by Frege and others (e.g., what sense experience could show that $135,664 + 37,863 = 173,527$ is true?) would also apply to Lakoff and Núñez’ ‘philosophy of embodied mathematics’.

Finally, their account is also an excellent example of psychologism, in that Lakoff and Núñez assert the primacy of ‘human mathematical ideas’ and metaphors over mathematical concepts defined by symbols. Dummett (1991) was prescient in commenting that “where (Frege and Husserl) failed was in demarcating logical notions too strictly from psychological ones... These failings have left philosophy open to a renewed incursion from psychology, under the banner of ‘cognitive science’” (p. 277-278).

3. Where Does Arithmetic Come From?

None of the accounts reviewed above provide a rigorous explanation for why addition and multiplication are the fundamental operations of arithmetic – one that meets standards for scientific explanation such as those proposed by Hempel (1965). This is a remarkable gap in our knowledge, considering that arithmetic is indispensable for mathematics, which in turn is indispensable for science (Colyvan, 2001a).

The limitation of philosophical and psychologicistic accounts and the possibility of an alternative approach become clear if we approach the question more formally in terms of arithmetic over the set $\mathbb{N}$ ($= \{0, 1, 2, \dots\}$) of natural numbers. Letting $\mathcal{F}_{\mathbb{N}}$ denote the set of all binary operations on $\mathbb{N}$ – that is, $\mathcal{F}_{\mathbb{N}}$ contains those functions that take in ordered pairs of natural numbers and return a natural number – and noting that addition is one operation among uncountably many in $\mathcal{F}_{\mathbb{N}}$, we can ask why addition has a privileged status within $\mathcal{F}_{\mathbb{N}}$. This is a meta-mathematical question, that is, a question about mathematics itself that can be investigated by mathematical methods (Kleene, 1952). We can also pose the question in more general terms. Given a countably infinite set X , and the set $\mathcal{F}_X$ of all binary operations on X , we can ask what kind of structure, when imposed on X and $\mathcal{F}_X$, allows for the emergence of number from X and arithmetic from $\mathcal{F}_X$? Formally answering this question might help to 1) show why addition is ‘special’ and 2) provide a rigorous explanation for arithmetic’s origins.

But what does it mean for arithmetic to be special? A key insight is suggested by research in comparative cognition, where there is much evidence of adaptive behavior in nonhumans that appears to require the equivalent of arithmetic operations by perceptual and cognitive systems (Gallistel, 1990, 2017; Grace et al., 2020). A good example is navigating by path integration, where individuals can take a convoluted outward journey when foraging

but return by most direct route to their home or nest. This behavior is found in many species from insects to mammals (Etienne & Jeffery, 2004; Srinivasan, 2015). According to Gallistel's computational-representational theory, organisms code environmental variables such as solar angle and distance travelled as real numbers in an engram and are capable of performing arithmetic operations with them (Gallistel, 2017, 2018; Langille & Gallistel, 2020). There are many other examples of behavioral phenomena that seem to involve computation, including associative learning (i.e., rates and probabilities of reward; Gallistel & Gibbon, 2000; Miller & Grace, 2003), behavioral decision making (Gallistel et al., 2007; Kyonka & Grace, 2008; Leon & Gallistel, 1998), timing (Balci et al., 2011) and cue integration (Cheng, Shettleworth, Huttenlocher, & Rieser, 2007; Wystrach, Mangan, & Webb, 2015). All these examples suggest that arithmetic may have a biological basis: Nonhumans are able to respond adaptively based on combinations of environmental variables that are represented as magnitudes by perceptual systems.

If natural selection has produced organisms that can behave as if they were able to perform algebraic operations on internally-represented magnitudes or quantities, then those operations – the non-symbolic precursors of addition and multiplication – may have evolved to be optimal. For example, these operations might be the simplest or most efficient way of combining magnitudes, and thus identifiable according to an appropriate criterion. Let X be a subset of magnitudes or quantities that can be internally represented by an organism. We assume that the organism is able to compare or discriminate the magnitudes in X (e.g., to detect which of two objects is closer); this structure is directly analogous to the mathematical concept of a *linear order*. Specifically, we equip X with a linear order $\leqslant_X$, which is structurally identical to the 'less than or equal to' ($\leq$) relation on $\mathbb{N}$; from $\leqslant_X$, we derive a partial order $\leqslant_{\mathcal{F}_X}$ on $\mathcal{F}_X$. Because order relations can allow the minimum (or maximum) element of a set (if it exists) to be determined, thus providing a criterion, we can ask: What

conditions are sufficient to identify uniquely the natural numbers and arithmetic in X and $\mathcal{F}_X$ using order-theoretic methods? If we can prove that qualitative conditions are sufficient for this, we can then ask whether those conditions have psychological meaning beyond their mathematical definition.

In Appendix A, we present a formal proof that given a countably infinite set X with order relation $\leqslant_X$ and technical assumptions, that four qualitative conditions – monotonicity, convexity, continuity and isomorphism, are sufficient uniquely to identify addition and multiplication over the real numbers $\mathbb{R}$. Thus the algebraic structure on which arithmetic is based can be derived from purely qualitative (i.e., non-algebraic) conditions. This result provides a rigorous explanation for why addition and multiplication are the fundamental arithmetic operations.

The mathematical argument is diagrammed in Figure 1. We start with minimal structure – a countably-infinite set X and the set $\mathcal{F}_X$ of binary operations on it. We impose a linear order $\leqslant_X$ on X and two assumptions regarding that order; (i) that there is no maximal element and (ii) that all down-sets are finite. These allow us to define an $\mathbb{N}$ -chain, $(X, \leqslant_X)$, which is a linearly ordered set that is isomorphic to the natural numbers equipped with its usual less-than order relation $(\mathbb{N}, \leq)$. We then show that the set of convex self-embeddings of $(X, \leqslant_X)$ under function composition is isomorphic to the natural numbers with addition $(\mathbb{N}, +)$. Thus, the additive structure over $\mathbb{N}$ can be obtained from the combination of monotonicity and convexity. We derive natural addition in an alternative way, by showing that if we start with the natural numbers and the less-than order relation $(\mathbb{N}, \leq)$, addition can be identified uniquely as the minimum element of the set of possible binary operations if monotonicity is assumed. Next, we obtain addition over the rational numbers $(\mathbb{Q}, +)$ in two ways; (i) as the rational convex hull of natural addition, and (ii) as the unique rational convex

extension of natural addition. We obtain real addition $(\mathbb{R}, +)$ in three ways; (i) as the unique real convex extension of rational addition, (ii) as the unique real monotonic extension of rational addition, and (iii) as the unique real continuous extension of rational addition.

Finally, we obtain multiplication over the reals $(\mathbb{R}, \times)$ from real addition via isomorphism².

Figure 1. Illustration of the major steps in the mathematical argument. The qualitative conditions (monotonicity, convexity, continuity and isomorphism) used for each step are shown. Algebraic structure increases from left to right.

These results constitute a formal mathematical proof that addition and multiplication over the real numbers – algebraic structure – can be obtained from purely qualitative conditions and without relying on any algebraic tools. They show that addition and multiplication indeed have a privileged status because they can be uniquely identified among all possible binary operations on the reals. The implication is that for any symbol system equipped with an appropriate order (i.e., an $\mathbb{N}$ – chain), conditions of monotonicity, convexity, continuity and isomorphism are sufficient to ensure that system will be arithmetic.

As such, these results are a construction of arithmetic, but unlike the axiomatic method typically used in the foundations of mathematics where addition and multiplication are defined as operations based on a formal logical system (e.g., Dedekind-Peano axioms),

the present approach might be described as the opposite: Starting with all possible operations, qualitative conditions are imposed as constraints until only addition and multiplication remain. The effect of these constraints is analogous to a sculptor's work that by chipping away reveals a statue hidden in a block of stone.

In the next section, we review research which shows that these conditions – monotonicity, convexity, continuity and isomorphism – are inherent to the perceptual systems of humans and nonhumans alike. They are pre-verbal and likely have a common evolutionary basis. We argue that these conditions have a psychological existence prior to their mathematical definition and correspond to core intuitions about the world, including magnitude, space and time, objects and motion, and relations between objects.

4. Psychological Status of Monotonicity, Convexity, Continuity and Isomorphism

The mathematical results in the Appendix show that the conditions of monotonicity, convexity, continuity and isomorphism (MCCI) are sufficient to identify addition and multiplication uniquely as arithmetic operations. Here we review evidence that MCCI are fundamental to the sensory and cognitive systems of nonhumans and humans. They characterize the nature of perceptual representations and/or how those representations are processed. For each we first provide a formal mathematical definition and then review relevant empirical findings.

Figure 2. Illustration of monotonicity (A), convexity (B), continuity (C) and isomorphism (D). See text for more explanation.

Monotonicity

The mathematical definition of monotonicity is:

- Given two ordered sets $(X, <_X)$ and $(Y, <_Y)$, a map $f: X \rightarrow Y$ is (strictly) **monotonically increasing** if, whenever $x <_X x'$ holds, $f(x) <_Y f(x')$ also holds. The map f is **monotonically decreasing** if, whenever $x <_X x'$ holds, $f(x') <_Y f(x)$ also holds. The map f is **monotonic** if it is either monotonically increasing or monotonically decreasing.

Monotonicity is an extension of the concept of order over a set to a function between two sets (see Figure 2A). A monotonic function is a mapping between the sets that

preserves order. The relevance for psychology becomes clear if the sets are real-world objectively-measured magnitudes (e.g., distance, duration, luminance, rate, number) and the representations of those magnitudes by perceptual systems. If the relationship between magnitudes and their representation is monotonic, then organisms will accurately perceive the ordinal information in the world – longer distances will be perceived as longer, a greater number of flowers will be perceived as greater, and so forth. Thus, monotonicity would be adaptive for any perceptual system.

Psychophysical research shows that for human observers, relationships between perceptual and real-world magnitudes are monotonic over a wide range of physical values (e.g., Stevens & Galanter, 1957). Frequently these studies use magnitude estimation tasks in which participants are asked to report the subjective magnitudes of stimuli on a numeric scale, and results are analyzed using psychophysical scaling techniques based on monotonic functions (i.e., logarithmic and power functions; see Gescheider, 1988, 2013 for review).

Monotonicity is also evident in nonhuman behavior. A vast research literature in conditioning, associative learning and animal cognition shows that behavior is monotonically related to real-world variables that can be measured in terms of time, space and number, as well as quantities derived from these such as rates or probabilities. For example, pigeons' rate of acquisition in Pavlovian conditioning depends on the ratio of the interval between food presentations (i.e., US-US) and the interval from the onset of the conditioned stimulus to food (CS-US) over two orders of magnitude (Gallistel & Gibbon, 2000; Gibbon et al., 1977), and similar results have been obtained with rats (Holland, 2000; Lattal, 1999). A key finding is scalar variability or Weber's law, where the standard deviation of responding increases proportionally with the mean (Akre & Johnsen, 2014; Gibbon, 1977). Weber's law is ubiquitous in animal cognition, seen in the distance

estimates of foraging insects (Cheng, Srinivasan & Zhang, 1999; Sommer & Wehner, 2004), discrimination of shoal sizes by fish (Gomez-Laplaza & Gerlai, 2011), and numerosity estimates by rhesus macaques (Jordan & Brannon, 2006), to name a few examples. Weber's law was first discovered in research on human tactile perception, and was the basis of Fechner's (1860) proposal for a logarithmic psychophysical function (Helen, Ross & Murray, 2018). The scalar variability associated with Weber's law suggests that not only are magnitudes represented monotonically, but their uncertainty as well.

Monotonicity has also been studied in terms of sequences of stimuli, where the general finding is that for both humans and nonhumans, monotonic sequences are far easier to learn than non-monotonic ones. Brannon (2002) found that 11-month old infants could discriminate the direction of monotonic change (increasing or decreasing) of sequences of squares that varied in number or non-numeric cues such as area, whereas 9-month old infants were only able to detect monotonic sequences of non-numeric cues (see also Suanda, Thompson, & Brannon, 2008). Brannon and Terrace (1998) showed that rhesus monkeys learned rapidly to order numerosities from 1 to 4 in ascending order with exemplars that differed in color, size and shape, and later testing confirmed their performance generalized to novel pairs of numerosities from 5 to 9. In contrast, Brannon and Terrace (2000) found that monkeys were unable to learn to order a non-monotonic numerosity sequence ($3 \rightarrow 1 \rightarrow 4 \rightarrow 2$).

Convexity

We give the formal definition of convexity in real 3-space ($\mathbb{R}^3$):

Given two points $p = (x, y, z)$ and $p' = (x', y', z')$ in $\mathbb{R}^3$, the real straight line segment between them, denoted $\ell_{\mathbb{R}}(p, p')$, is the set

$$\ell_{\mathbb{R}}(p, p') = \{(1 - t)(x, y, z) + t(x', y', z') : t \in \mathbb{I}_{\mathbb{R}}\}.$$

A subset X of $\mathbb{R}^3$ is **convex** (in $\mathbb{R}^3$) if, for each pair of points p and p' in X , the set $\ell_{\mathbb{R}}(p, p')$ is contained in X .

This mathematical definition is precise but can obscure an intuition of *betweenness* that is the basis for convexity. A less formal but more intuitive definition is: A set X is **convex** if, whenever x and z are in X and y is between x and z , y is also in X . Specifically, y is **between _{ℓ}** x and z if y lies on the line segment joining x to z . Figure 2B illustrates how convex and non-convex sets in the two-dimensional plane $\mathbb{R}^2$ differ in betweenness. In a convex set, for all possible pairs of points in the set, all points on the straight line segment connecting the pair are also in the set. Note that for real 3-space, $\mathbb{R}^3$, y being **between _{ℓ}** x and z means that y lies on the line segment joining x to z , as defined above. This is a geometric interpretation of ‘betweenness’. Note that an order-theoretic interpretation is also possible:

- Given a partially-ordered set (i.e., poset; see Appendix A) $(X, \leq)$, we say that y is **between _{$\leq$}** x and z if either $x \leq y \leq z$ or $z \leq y \leq x$ holds.

To see that these interpretations are similar, consider the real line equipped with its usual linear order $\leq$; this space admits both definitions - **between _{ℓ}** and **between _{$\leq$}** - of betweenness. Moreover, these interpretations agree on $\mathbb{R}$. Thus, a subset X of $\mathbb{R}$ is convex in the geometric (**convex _{ℓ}**) sense if, and only if, X is convex in the order-theoretic (**convex _{$\leq$}**) sense. Thus, betweenness – and convexity – has both a geometric and an order-theoretic interpretation.

The role of convexity in the geometric sense for perception of contours, shapes and objects was recognized by the early Gestalt psychologists (Wagemans et al., 2012). Convexity is important for distinguishing figure-ground relationships and for parsing objects into shapes by detecting contours (see Bertamini & Wagemans, 2013, for review). It plays a key role in perceptual completion (Liu, Jacobs & Basri, 1999), both in detecting the contours of an object behind an occluder (i.e., amodal completion; Singh & Fulvio, 2007; see Gerbino, 2020, for review) and perception of illusory contours (modal completion; Kanizsa & Gerbino, 1976, Kanizsa, 1979). Results indicative of modal completion with Kanizsa triangles have also been obtained with nonhuman primates (Feltner & Kiorpes, 2010).

The boundaries between convex and concave surfaces are important for identifying components of objects (Koenderink & Van Doorn, 1982; Biederman, 1987). There is an overall bias for perceiving objects as convex, as in ‘hollow-face’ and ‘hollow-potato’ illusions (Dobias, Papathomas, & Vlajnic, 2016; Gregory, 1970; Hill & Bruce, 1994; Hill & Johnston, 2007; Langer & Bulthoff, 2001), which appears even in 6-month old infants (Corrow, Mathison, Granrud, & Yonas, 2014). A bias for convexity has also been shown with 3D ‘polycube’ stimuli (consisting of random configurations of 10 individual cubes), where participants rated the likelihood that they would perceive a polycube as an object (Wörgötter, Sutterlütti & Tamosiunaite, 2015). It is also notable that the convexity bias is regarded as a top-down attentional effect (Baluch & Itti, 2011). Convexity biases have also been found with nonhumans (see Vallortigara, 2021, for review).

Convexity appears to be explicitly encoded by the visual system. Suzuki (2001, 2003; Suzuki & Cavanagh, 1998) showed that contrast aftereffects in neutral test stimuli were produced by brief presentations of convex or concave contours, and concluded that these were likely associated with high-level neural units that coded global orientation. Muller, Wilke and Leopold (2008) adapted Suzuki’s procedure and stimuli and showed that neurons

in the macaque visual cortex (ventral V4) were associated with adaptation to convexity. Neuroimaging studies with humans have shown that convex shapes are preferentially coded in the lateral occipital complex, a brain region that is linked to object recognition (Kourtzi & Kanwisher, 2001; Haushofer et al., 2008).

These results suggest that convexity in the geometric sense formalized in the mathematical definition above is related to perception of objects in 2D and 3D space. What about the order-theoretic definition, which applies to a single dimension? One possibility is that convexity in this sense relates to time perception, specifically of temporal intervals. Many species have a capacity for interval timing – that is, an ability to estimate the duration of an interval (Buhusi & Meck, 2005; Grondin, 2010). This is often studied in temporal bisection, which has been widely used with humans and nonhumans (e.g., rats, pigeons, mice; see Penney & Cheng, 2018, for review). For example, Church and Deluty (1977) rewarded rats if they pressed the left lever after a short duration tone and the right lever after a long duration tone, and subsequent testing with intermediate durations gave a psychometric function with the bisection point at the geometric mean. Much is known about the neural basis of timing, with the posterior parietal cortex implicated in the coding of interval duration (Leon & Shadlen, 2003; see Paton & Buonomano, 2018, for review).

Objective measures of time, such as by a clock, are linearly ordered. Because intervals in linear orders are defined by endpoints, interval timing requires an ability to group the moments between them into a perceptual whole. The moments that comprise an interval are a convex set under the order-theoretic definition above. Thus, interval timing suggests that convexity is inherent in our perception of time.

Continuity

In mathematical terms, continuity is understood in the context of functions (i.e., a map from $\mathbb{R}$ to $\mathbb{R}$), and is based on the $\epsilon - \delta$ definition, due to Weierstrass:

- A map $f: \mathbb{R} \rightarrow \mathbb{R}$ is **continuous at the point x_0** in $\mathbb{R}$ if, for each $\epsilon > 0$ there exists $\delta > 0$, such that for each real x , if $|x_0 - x| < \delta$ then $|f(x_0) - f(x)| < \epsilon$; in symbols,

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} (|x_0 - x| < \delta \Rightarrow |f(x_0) - f(x)| < \epsilon).$$

- The map f is **continuous** if f is continuous at *each* point x_0 in $\mathbb{R}$.

Although this definition is formally correct, it obscures the mathematical intuition that motivates it. Perhaps this is not surprising, because it took mathematicians nearly 200 years after Newton and Leibniz had invented calculus to achieve it. The mathematical intuition is captured by the following:

- A map $f: \mathbb{R} \rightarrow \mathbb{R}$ is **continuous** if, whenever x_0 and x are close, $f(x_0)$ and $f(x)$ are also close.

This is an informal definition because we have not specified what we mean by ‘close’, but the intuitive notion is sufficient for our current purposes. Figure 2C shows examples of both continuous and discontinuous functions. Continuous functions are ‘smooth’ and do not have any gaps, whereas a discontinuous function has at least one point where its behavior becomes unpredictable.

The mathematical notion of a function’s output being close when its inputs are close is underpinned by psychological intuition regarding the motion of objects in space and time. The motion of an object in 3D space (e.g., a ball) can be approximated by a continuous mathematical function, as defined above. We expect change in an object’s position in space

to be continuous rather than abrupt, as evidenced by our surprise when we are confronted with abrupt change. One key line of research has examined how infants react to abrupt change (e.g., Leslie, 1982, 1984; Oakes, 1994; Spelke, Kestenbaum, Simons, & Wein, 1995). For example, Spelke et al. (1995) found that 4-month old infants spent less time looking at an object that smoothly transited from A to D than at an object which appeared to transit smoothly from A to B, abruptly from B to C and then again smoothly from C to D. The extra time is taken here (as in other experiments) as an indication that the abrupt transition is surprising, suggesting that the expectation of continuity is innate. Infants and newly-hatched chicks show a preference for biological motion when viewing point-light displays (e.g., of a walking hen versus random motion), consistent with the view that there is an innate capacity to detect animate objects as revealed by their motion (Regolin, Tommasi, & Vallortigara, 2000; Simion, Regolin, & Bulf, 2008; Vallortigara, Regolin, & Marconato, 2005).

Another example of how continuity is deeply embedded in perception is the phi phenomenon first described by Wertheimer (1912). When visual stimuli in close proximity are presented in alternation at high frequency, an occluding shadow seems to move over them, although the stimuli themselves do not move. Phi is an illusion of apparent motion, distinct from ‘true’ apparent motion obtained at lower frequencies in which distinct images presented successively produce a sensation of continuous movement, as in stock-ticker displays or motion pictures (‘beta movement’; Steinman, Pizlo & Pizlo, 2000).

Neuroimaging studies also suggest that continuity is processed very early in visual cortex. In an fMRI study, Yi et al. (2008) found that face-selective cortical regions showed less activation after faces were presented in continuous rather than discontinuous trajectories, suggesting that the latter caused identical stimuli to be represented as distinct individuals. Alink et al. (2010) showed that visual stimuli produced reduced activation in primary visual cortex (V1) when their onset or motion direction could be predicted based on surrounding

visual information. They concluded their results were consistent with a ‘predictive processing’ model of brain function, which assumes that neural systems have evolved to anticipate future patterns of stimulation (Rao & Ballard, 1999; see also Clark, 2013; Friston, 2010; Friston et al., 2017; Keller & Mrsic-Flogel, 2018).

Isomorphism

A precise formal definition of isomorphism is given by category theory, which is an abstract branch of mathematics developed in the mid-20th century to study the nature of mathematical structures and their relations (Awodey, 2010; Eilenberg & Mac Lane, 1945; Mac Lane 2013). The definition of isomorphism is as follows:

- Given a category C , an isomorphism is a morphism $f: a \rightarrow b$ that has an inverse morphism $g: b \rightarrow a$, so that the composition $fg = 1_b$ and the composition $gf = 1_a$

This is very abstract because it can apply to any kind of mathematical structure, so it may be helpful to give a concrete example. Let C be the category which includes the multiplicative group of the positive reals, $\langle \mathbb{R}^+, \times \rangle$ and the additive group of the reals $\langle \mathbb{R}, + \rangle$. The logarithmic function, $\log: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a morphism (i.e., group homomorphism) because $\log(xy) = \log(x) + \log(y)$ for all $x, y \in \mathbb{R}^+$. The exponential function $\exp: \mathbb{R} \rightarrow \mathbb{R}^+$ is also a morphism because $\exp(x + y) = \exp(x) \times \exp(y)$ for all $x, y \in \mathbb{R}$. These morphisms are the inverses of each other because $\log(\exp(x)) = x$ and $\exp(\log(y)) = y$. Thus the logarithmic and exponential functions provide an isomorphism between the positive reals under multiplication and the reals under addition – that is, a mapping between two groups that preserves their structure and which can be inverted.

Figure 2D illustrates the concept of isomorphism in basic conceptual terms: Hat is to head, as glove is to hand, as shoe is to foot. These three relations are isomorphic to one another insofar as each preserves the same apparel $\leftrightarrow$ body part structure.

‘Isomorphism’ is derived from the Greek words *isos* –‘same’ or ‘equal’ – and *morphe* –‘form’ or ‘shape’. Thus, isomorphism means ‘same form’ or ‘same structure’. We argue that the mathematical concept of isomorphism is grounded in a psychological intuition of similarity or sameness. An ability to determine if a collection of objects or items are the same or different was described as the ‘keel and backbone’ of our cognition by James (1890), but has also been demonstrated in infants (Hespos, Gentner, Anderson, & Shivaram, 2021) and many nonhuman species including pigeons (Katz & Wright, 2006; Wasserman & Young, 2010), bees (Giurfa et al., 2001; Giurfa, 2021), crows (Smirnova et al., 2015), parrots (Pepperberg, 1987), and macaques and rhesus monkeys (Katz, Wright & Bachevalier, 2002; Wright et al., 2003). Ducklings have been shown to imprint on a sameness vs. difference relation (Martinho & Kacelnik, 2016; see also Versace et al., 2017, for similar results with chicks). Sameness/difference learning is important because it is necessary for categorization and concept formation – for example, a child who can recognize that an apple is similar to other apples and different from other kinds of fruit has a concept of ‘apple’ (Zentall, Galizio, & Critchfield, 2002). There is also considerable evidence of category learning in many nonhuman species (Smith, 2014; Smith et al., 2012), including insects (Avargues-Weber, Deisig, & Giurfa, 2011).

The concept of isomorphism is the basis for accounts of higher-order cognition such as analogical and relational reasoning. According to Gentner’s (1983) structure mapping theory, analogies are based on similarities between familiar and unfamiliar situations (i.e., source and target analogs; see Gentner & Markman, 1997, Gentner & Maravilla 2018, for review). According to this account, the source and target analogs can have very different

elements but the key aspect is that relationships between the elements in each system are similar. Gentner and Maravilla give an example of analogical reasoning: “Mitochondria are the furnace (or powerhouse) of a cell” (p. 186). The relational structure is similar for mitochondria and furnaces because both take fuel as an input and produce energy as an output. Structure mapping in Gentner’s (1983) theory is basically synonymous with isomorphism. Notably, isomorphism is used explicitly in Holyoak and Thagard’s (1989, 1996) constraint satisfaction theory of analogical mapping (as ‘structural consistency’), which is closely related to Gentner’s (1983) theory. These accounts have had significant impact in artificial intelligence research on algorithms for analogical reasoning (Gentner & Forbus, 2011). Thagard’s (1989) explanatory coherence theory builds on constraint satisfaction and isomorphism as structural consistency to provide an account of scientific reasoning (see also Thagard & Verbeugt, 1998).

Isomorphism is fundamental for scientific measurement, that is, the assignment of numbers to empirical phenomena. In the representational theory of measurement developed by Luce and colleagues (e.g., Krantz, Luce, Suppes & Tversky, 1971; Luce & Narens, 1985; Narens & Luce, 1986), a homomorphism of an empirical relational structure into a numerical relational structure defines a scale of measurement (e.g., ordinal, interval, ratio). This definition applies not only to extensive physical measurement for which a concatenation operation is available (e.g., measuring length with a ruler), but also situations where concatenation is not possible, such as the perceived loudness of a tone which depends on intensity and frequency (i.e., additive conjoint measurement; Luce & Tukey, 1964).

Isomorphism also plays a key role in the dance communication system of honeybees. When a foraging bee has discovered a food source and returns to the hive, it performs a figure-eight ‘waggle dance’ on the vertical wall of the hive that is observed by other bees. The dancer’s angle from vertical indicates the direction of the food source relative to the sun,

while the duration of the waggle phase indicates the distance (von Frisch, 1967). The dance is repeated multiple times in a stereotyped fashion, which allows the effect of errors in any individual run to be reduced (Preece & Beekman, 2014). The correspondences between the average angle of the dances and direction to the food source, and between the average duration of the dances and distance, are one-to-one mappings and so can be regarded as isomorphisms between the foraging bee's journey and the waggle dance. Observer bees are able to decode the information in the dance and fly to the indicated location (Riley et al., 2005). The waggle dance is regarded as the most sophisticated known example of non-primate communication.

Summary

Monotonicity, convexity, continuity and isomorphism (MCCI) have mathematical definitions and are instrumental in the construction of arithmetic presented here. But these conditions are more than just arbitrary formalisms – they play a pivotal role in how humans perceive the world, and are implicated in the functioning of neural systems that support perception. They have a psychological status that is prior to their formal mathematical definition. Monotonicity ensures the ordinal consistency of perceptual and physical magnitudes; convexity is essential for perception of objects and temporal intervals; continuity is linked to perceived motion in space and time; and isomorphism affords perception of similarity or sameness, which is the genesis of abstraction and higher cognition. These conditions might be described as principles of perceptual organization, core psychological intuitions that have considerable overlap with Gestalt concepts (e.g., convexity and figure-ground).

Available evidence suggests that nonhuman perceptual systems are also characterized by MCCI. Whether these qualities are universal across nonhuman species, considering differences in anatomy and kinds of stimuli that perceptual systems have evolved to be sensitive to, is as yet unclear. We have no idea what the subjective experience of animals is like (cf. Nagel, 1974); for example, echolocation in bats (Jones & Teeling, 2006), magnetoreception in birds (Wiltschko & Wiltschko, 2019), and sunlight polarization in insects (Homberg, Heinze, Pfeiffer, Kinoshita, & El Jundi, 2011). However because all species evolved under the same spatio-temporal conditions, it may well be that these properties are universal in nonhuman perception. If embodied in perceptual systems, monotonicity, convexity, continuity and isomorphism would promote the survival of a mobile organism, localized in time and space in a dynamic world full of objects, by enhancing its ability to represent that world.

5. Discussion

Why are addition and multiplication the basic operations of arithmetic? It would be good to have an answer that meets standards of scientific rigor, because arithmetic's importance can hardly be overstated. But we do not have a rigorous answer. Philosophy of mathematics has not given one, and neither have mathematical logic or the cognitive sciences. Arithmetic has been regarded as an *a priori*, necessary truth in many philosophical accounts, which tacitly assumes that an explanation is impossible or unenlightening. Mathematical logic provides a logical reconstruction of arithmetic— in effect, a simulation. This work has been tremendously important, eventually leading to computer science – but a simulation is not the *Ding an sich* (thing-in-itself), and Gödel showed that no logical system rich enough to support arithmetic could be both complete and consistent. Lakoff and Núñez (2000) claimed that the cognitive sciences, but not mathematics itself, could explain the origin of mathematics. However their ‘mathematical idea analysis’ sacrificed rigor by assuming the primacy of psychological concepts.

It is easy to understand why arithmetic seems to possess a necessity and ontological status independent of human experience. Our experience with the physical world confirms it, over and over again, at increasing levels of complexity and sophistication, from children playing with toys in the backyard to physicists confirming predictions in particle accelerators. Arithmetic is the foundation of higher mathematics, the language of our best scientific theories and basis for our knowledge of the physical world. We *know* arithmetic is true, but do not know *why* it is so. But to regard arithmetic as a necessary truth may have dissuaded us from seeking an explanation, as if there was nothing more we could learn about it. In Hilbert's words, engraved on his tombstone: *Wir müssen wissen. Wir werden wissen.* (We must know. We shall know).

To seek an explanation, we employed an approach based on mathematical psychology and comparative cognition. We reasoned that symbolic number and arithmetic may have evolutionary roots in a single system that supports both representations of quantities and operations with them analogous to arithmetic. If so, we might expect these operations to be optimal due to natural selection. Optimality implies identifiability according to an appropriate criterion, and here the mathematical part of our investigation began. We searched for qualitative conditions that might identify arithmetic operations. Starting with minimal structure (a set equipped with an order) and by applying an order-theoretic criterion of minimum element, we proved that assumptions of monotonicity, convexity, continuity and isomorphism (MCCI) were sufficient to identify addition and multiplication over the real numbers uniquely among the uncountably infinite class of possible operations. Because individually these conditions are simpler than the algebraic structure they produce, our results are a formal, rigorous explanation for why addition and multiplication are the fundamental operations of arithmetic. The implication is that given MCCI, any set of symbols with an order that is the basis for a combinatoric system will be equivalent to arithmetic.

These results provide a construction of the natural numbers and arithmetic that is distinct from the Dedekind-Peano axioms. The Dedekind-Peano axioms give a procedural or algorithmic construction – they specify how, when given a starting point (i.e., zero), all the natural numbers can be computed by iterative application of a successor function. Addition and multiplication can then be defined through recursion. In contrast, our approach can be described as structural rather than algorithmic. Given an $\mathbb{N}$ – chain and all binary operations on it, natural addition is identified as the structurally unique element when monotonicity and convexity are assumed. As defined in Appendix A, an $\mathbb{N}$ – chain is a linearly ordered set with a minimum element, no maximal element, and in which all downsets are finite. In effect, $\mathbb{N}$ – chains describe the order-theoretic structure of the natural numbers. In our

construction, the natural numbers with their additive structure emerge ‘all at once’, rather than from iterative computation. Our construction is related to the Dedekind-Peano axioms in that the successor function in the latter is equivalent to the existence of an $\mathbb{N}$ – chain.

This is a mathematical result, but we argue that MCCI have substantive psychological meaning that is prior to their formal definition. Empirical research in experimental psychology and comparative cognition suggests that monotonicity, convexity, continuity and isomorphism are all related to how humans and nonhumans perceive the world. Stimulus dimensions are often subjectively experienced as monotonic in magnitude (e.g., brightness, loudness, length), convexity characterizes object perception and temporal intervals, continuity describes motion of objects in space and time, and isomorphism refers to perception of sameness that underlies categorization and concept formation. We argue that these qualitative conditions are pre-verbal psychological intuitions that shape our perception of the world. They are biologically based, reflecting our shared evolutionary history with nonhumans, and likely related to (or part of) ‘core knowledge systems’ that have been studied in developmental science (Rugani, Fontanari, Simoni, Regolin, & Vallortigara, 2009; Spelke, 2017; Spelke & Kinzler, 2007; Spelke & Lee, 2012; Vallortigara, 2012, 2017).

This is fundamentally a Kantian view. According to Kant’s (1781/1908) transcendental idealism, synthetic *a priori* judgments define the necessary conditions for human experience. Our account naturalizes the Kantian intuition, identifying assumptions which are fundamentally psychological (and ostensibly, evolutionarily based), which not only comprise the scaffolding of human experience, but also arithmetic and the higher mathematics that follows. These assumptions have formal mathematical definitions but can also be studied empirically in research with humans and nonhumans (Dehaene & Brannon, 2010). The joint sufficiency of monotonicity, convexity, continuity and isomorphism for addition and multiplication opens up the possibility that arithmetic need not be regarded as a

necessary, *a priori* state of affairs imposed from without, but rather a natural consequence of how our perception is structured. The implication is that algebraic structure may be intrinsic or inherent in our perception of the world, or expressed differently, that our perceptual systems form representations of the world with algebraic structure.

Our mathematical investigation was prompted by the many examples of nonhuman behavior, such as path integration in insects (Srinivasan, 2015), which can be described as ‘computationally complex’ (Grace et al., 2020) or mediated by biological computation. Monotonicity, convexity, continuity and isomorphism are sufficient conditions to identify arithmetic in formal terms, and as shown above, these conditions describe how nonhumans and humans alike perceive the world. In the next sections we consider some possible implications of this result.

What is Biological Computation?

Gallistel (1990, 2017, 2018; Gallistel & King, 2010; Langille & Gallistel, 2020) has long been the foremost advocate of the view that adaptive behavior is mediated by arithmetic operations performed by neural systems on real-number representations of environmental magnitudes such as distance travelled or solar angle. According to his computational-representational theory, biological computation – the neural activity that supports adaptive behavior such as spatial navigation – is formally equivalent to real-number arithmetic, not merely analogous. Consequently, a key question for his account is how environmental variables might be coded as real numbers in the brain. Gallistel (2017) offered an intriguing analogy based on molecular biology – just as DNA encodes an organism’s phylogenetic history as a blueprint for its physical structure, so its ontogenetic history in terms of individual experience in the world might also be encoded at the molecular level. Although

how this occurs is unknown, it is plausible that there are mechanisms that support molecular-level computation. Akhlaghpour (2022) has proposed an RNA-based theory of universal computation. He shows that editing rules for RNA (e.g., cleavage and ligation) can in principle, provide models of Turing machine- or lambda calculus computation, that is, universal computation. He argues that given the survival advantage such computation would confer, it is likely that it would have evolved. Although his models are unverified at this stage, Akhlaghpour's work is important because by bridging combinatory logic and molecular biology, it suggests a possible basis for biological computation.

Figure 3 depicts an analogy between arithmetic and biological computation in terms of a structural isomorphism (after Holyoak & Thagard, 1989). Our mathematical results show that monotonicity, convexity, continuity and isomorphism are sufficient to identify addition and multiplication as arithmetic operations. The relation R (solid arrow, left; $\text{MCCI} \rightarrow \text{arithmetic}$) is causal because these conditions provide a formal explanation for why addition and multiplication are the fundamental arithmetic operations. The horizontal dashed arrows represent mappings between the target (right) and source (left) analogs. The upper arrow signifies the correspondence between MCCI as embodied in the perceptual systems of living organisms ('embodied MCCI ') and their mathematical definitions. The lower arrow shows a mapping between biological computation and arithmetic consistent with Gallistel's (1990, 2017) theory: Real addition and multiplication are a formalization of the operations that perceptual systems perform on represented magnitudes. In both cases, the mappings correspond to a process of abstraction – from the biological/non-symbolic/implicit to the conceptual/symbolic/explicit. This diagram makes clear the idea that arithmetic has a biological basis. If the analogy in Figure 3 is valid, it suggests there is a causal relation R' between embodied MCCI and biological computation. What are the implications of this?

Our mathematical results show that MCCI act as constraints on possible operations. If these conditions are applied, then any infinite set of symbols equipped with an order and which supports operations or mappings between two symbols and another symbol – that is, a combinatoric system – will be formally equivalent to arithmetic. Assume that the neural structures responsible for perception support a combinatoric system mediated at the molecular level by RNA-based processes similar to those described by Akhlaghpour (2022). If an organism’s perception of the world, characterized by MCCI, acts to constrain that system, then it will be equivalent to arithmetic. Admittedly, based on current knowledge it is unclear how high-level perceptual organization could drive the molecular biology of neural systems. However, there are many examples of ‘top-down’ attentional effects including convexity (Baluch & Itti, 2011), so it is plausible that a linkage between embodied MCCI and neural systems might be discovered that could enable biological computation. The analogy in Figure 3 suggests that it would be a good idea to search for it.

Figure 3. Depiction of analogy between the mathematical argument in Appendix A and B (MCCI -> arithmetic) and a possible linkage between MCCI as embodied in perceptual systems and biological computation.

Are Symbolic Number and Arithmetic Innately Determined?

As noted in the introduction, arithmetic is universal in that results agree across all human cultures that have developed it (e.g., $2 + 2 = 4$). Our account suggests that arithmetic is universal not because it is a necessary truth but because all humans have a perceptual system that is fundamentally the same, in which subjective experience of the world is characterized by monotonicity, convexity, continuity and isomorphism. We now consider the implications of this finding for the debate about whether representations of exact number have a biological basis and are thus innately determined. At issue is whether the acquisition of number-related words and concepts in childhood builds on the ANS and parallel individuation subsystems (Dehaene, 2009, 2011; Gallistel & Gelman, 1992; Gelman & Butterworth, 2005; Szklarski & Brannon, 2017) and possibly an innate ‘successor function’ or incrementing process (Gallistel & Gelman, 2000; Leslie, Gelman, & Gallistel, 2008), or depends on language and enculturation – for example, learning number words for small sets, and then algorithms for counting (Carey, 2011; Carey & Barner, 2019; Le Corre & Carey, 2007; Rips, 2017; Rips, Bloomfield, & Asmuth, 2008). Núñez (2017) proposed that the ability to make approximate judgments of quantity associated with number sense should be called ‘quantical’ cognition, and was distinct from exact numerical cognition, which required cultural mediation.

Recently, O’Shaughnessy, Gibson and Piantadosi (2021) have strongly argued for the cultural mediation of number concepts. They listed a set of predictions that they claimed

should be satisfied if such concepts were innately or biologically determined. Effectively, these predictions require numbers to show *universal features* across human cultures: Symbolic numbers should be found in all human groups, be easily learned and with similar developmental timing, have some consistency in symbolic representations, and that the historical development of number systems should exhibit the abstractions predicted by nativist theories (e.g., use of a successor function). They concluded that none of the predictions was supported: Not all human groups have developed numbers and symbol systems vary widely across those that have (Chrisomalis, 2010), numbers are difficult for children to learn (Baroody, 2006), and timing of learning depends on external factors such as the amount of data that children are exposed to (Levine, Suriyakham, Rowe, Huttenlocher, & Gunderson, 2010). O'Shaughnessy et al. (2021) concluded that symbolic number systems have developed in response to social and economic needs of particular cultures, rather than innate or biological causes. For example, 'modern' Hindu-Arabic numerals only became widely used in Europe after Leonardo of Pisa published *Liber Abaci* ('The Book of Calculation') in 1202, which showed that algorithmic calculation with Hindu-Arabic numerals was superior to existing methods with Roman numerals and abacus and was very influential among Venetian merchants and traders (Sigler, 2003).

Although the historical development of mathematics has clearly been affected by economic and cultural factors, O'Shaughnessy et al.'s (2021) proposal overlooks the universal features of symbolic number systems. Chrisomalis (2010) provided a structural typology of all known notation systems – over 100 different systems that have been used in the past 5,500 years – which associate 'numeral phrases' (i.e., strings of symbols) with particular exact numbers. He showed that these systems could be grouped into five categories, as shown in Table 1. Cumulative additive systems, such as Roman numerals, have one sign for each power of the base which are added to give the total value of that

power. Cumulative positional systems (e.g., Babylonian cuneiform) also use repeated signs to indicate the value of each power, but these are also multiplied by a place value. In ciphered additive systems (e.g., Greek alphabetic) there is only one sign for each power of the base, which uniquely specifies the multiple of that value. Multiplicative additive systems (e.g., Chinese traditional) have two signs for each power of the base (a unit sign and a power sign), which are multiplied together to give the value of that power). Finally, ciphered positional systems (e.g., Khmer and also modern Western numerals) have unique signs from one up to one less than the power of the base (e.g., 1, 2, 3, ... 9) and a zero sign; these signs are multiplied by the power indicated by position and added together to give the numerical value.

	Additive	Positional
Cumulative	<i>Classical Roman</i> 1434 = MCCCCXXXIII $1000 + (100 + 100 + 100 + 100) + (10 + 10 + 10) + (1 + 1 + 1 + 1)$	<i>Babylonian cuneiform</i> 1434 = $(10 + 10 + 1 + 1 + 1) \times 60 + (10 + 10 + 10 + 10 + 10 + 1 + 1 + 1 + 1) \times 1$
Ciphered	<i>Greek alphabet</i> 1434 = $((1 \times)1000 + 400 + 30 + 4)$	<i>Khmer</i> 1434 = $(1 \times 1000 + 4 \times 100 + 3 \times 10 + 4 \times 1)$
Multiplicative	<i>Chinese (traditional)</i> 1434 = 一千四百三十四 $(1 \times 1000 + 4 \times 100 + 3 \times 10 + 4)$	

Table 1 – Structural typology of numeral notation systems. Reproduced with permission from Chrisomalis (2010, p. 13).

What these systems all have in common is that arithmetic operations are embedded in them. Both addition and multiplication are necessary to unpack the value of number phrases in cumulative positional, ciphered additive, multiplicative additive and ciphered positional systems, while addition is required for cumulative additive systems. Thus, all the representational systems for exact number known to have ever been used by humans are based on the same arithmetic operations. Chrisomalis (2010) also organized notational systems into eight cultural phylogenies – Hieroglyphic, Levantine, Italic (i.e., Etruscan), Alphabetic, South Asian, Mesopotamian, East Asian and Mesoamerican – reflecting their emergence across different civilizational and geographic contexts. The implication is that concepts of symbolic number and arithmetic are linked; there is no evidence that notational systems for symbolic number have been developed anywhere without concurrent use of the same arithmetic operations. Arithmetic is universal to all symbolic number systems, regardless of cultural variation in their notational and linguistic expression.

This universality suggests that a single, integrated system underlies the development of both symbolic number and arithmetic, with an origin that transcends individual cultures, even if any given emergence might be linked to social or economic factors that are culturally specific. Our results point to monotonicity, convexity, continuity and isomorphism, as embodied in perception and cognition, as key pillars of that system.

Is Arithmetic Unique to Humans?

Penn, Holyoak, and Povinelli (2008) proposed that there was a discontinuity between human and nonhuman cognition in the degree to which the requirements of a physical symbol system (Fodor, 1975; Newell, 1980) could be approximated. Penn et al. (2008) based their argument on evidence from relational learning paradigms, which tested concepts of similarity

(sameness-difference), rules, and transitive inference. We agree with their conclusion, but suggest that symbolic arithmetic, which was not discussed by Penn et al., appears to be an even more striking discontinuity between human and nonhumans, especially because, as we have shown, there is good evidence that monotonicity, convexity, continuity and isomorphism also characterize nonhuman perceptual systems. The requirement for a physical symbol system is abstract representation, and numeral notation would clearly qualify as such. Yet only humans have developed notational systems for exact number that support arithmetic.

From a formalist perspective, it could be argued that biological computation is identical to arithmetic. For example, in Akhlaghpour's (2022) model, it would be possible to establish a one-to-one correspondence between states of an RNA-based biological system and natural arithmetic. The objects or elements of the biological system – different molecular configurations – would then operate according to the same laws as ordinary arithmetic. According to Knopp (1952), such a system could be considered a number system: “Every system of objects for which this is true is called a *number system*, because, in a few words, it is customary to call all those objects *numbers* with which one can operate according to the fundamental laws [of arithmetic]” (p. 5, italics in original).

However, in our view there is an important distinction between symbolic arithmetic and computation by a biological system, even if equivalent from a formalist perspective. Numeric symbols have semantic content that is independent of their physical representation. The equation ‘ $7 + 5 = 12$ ’ can be written on paper, inscribed in stone, or displayed on a laptop, and always has the same meaning. In contrast, if a representation of ‘ $7 + 5 = 12$ ’ existed as a configuration of molecules in the brain of a honeybee that was estimating distance while foraging, it seems unlikely that those molecules in the same configuration would have exactly the same meaning in a rat's brain. Only humans have created abstract

symbols (i.e., codes) for number that have invariant meaning across different representations. We do not understand the molecular coding scheme that enables biological computation (Gallistel, 2017). It is possible that a scheme that is invariant in this sense might eventually be discovered, or it could be more dynamic such that the meaning of a specific molecular configuration would depend on its context in a way that symbolic numbers do not.

Based on the analogy in Figure 3, it makes sense to regard symbolic arithmetic as grounded in the activity of living organisms, as a formalization or abstraction of biological computation, rather than to assume the latter is equivalent to arithmetic. From this perspective, it is remarkable to consider that it took humans thousands of years, with contributions from different civilizations, to discover and express in symbols the structure underlying biological computation, first as arithmetic and then as algebra (Hodgkin, 2005). Later, efforts to place mathematics on a firm logical foundation led to theoretical computer science, and ultimately to artificial neural networks that can simulate biological computation (e.g., path integration in the bee brain; Stone et al., 2017) – in effect, closing the circle if biological computation is regarded as the original basis for arithmetic. If algebraic structure is inherent in perception, as the interpretation of MCCI and our mathematical results suggest, it leads to a novel perspective on the history of mathematics. The historical development of mathematics was spurred by the need to solve practical problems and has greatly increased our knowledge of the world through its application in science, but this external trajectory is accompanied by an internal one in which mathematics can be regarded as the mind learning to express its fundamental nature in symbols.

The 'Unreasonable Effectiveness' of Mathematics

The applicability of mathematics in the physical sciences has been a source of wonder and speculation since antiquity. In modern times, Wigner's (1960) famous essay on the 'unreasonable effectiveness' of mathematics as a 'miracle... that we neither understand nor deserve' has prompted much discussion among scientists and philosophers (Bangu, 2009, 2016; Colyvan, 2001b; McDonnell, 2017; Steiner, 1998). Here we briefly sketch a perspective on 'Wigner's puzzle' based on the present findings, which is admittedly speculative like all other attempts to grapple with this issue.

Knowing why arithmetic is true is necessary for a full understanding of Wigner's puzzle, because arithmetic is indispensable for mathematics and the physical sciences. Advanced theories in physics (e.g., quantum electrodynamics, general relativity) use algebraic operations that are ultimately extensions or generalizations of addition and multiplication. Experiments can confirm that these theories are true in that they provide a valid description of physical phenomena, but cannot tell us why arithmetic itself is true.

The same argument would presumably apply to a future 'theory of everything' that could describe all physical phenomena in the universe. Our results suggest that arithmetic arises from the constraints (MCCI) that shape how our mind structures perception of the world. Thus, when we develop mathematical theories in physics, at the same time we are realizing ideas that are inherent or implicit in the mind. Steiner (1998) proposed that advances in 20th century physics were achieved because scientists had implicitly adopted an anthropocentric strategy, that is, they had assumed that the world could be understood by applying qualities and concepts meaningful to humans, such as analogy and beauty. The implications of our results for Wigner's puzzle are consistent with Steiner's anthropocentric view, but take it further.

Our account suggests that in metaphorical terms, the joint historical progression of mathematics and science is like the journey of a blindfolded person who can pull on a rope to guide them as they walk ahead. Because of the rope, the person *knows* they are headed in the right direction (i.e., the necessary *a priori* truth of arithmetic), but does not know the destination or how the journey will unfold. As their journey progresses, they gain more and more knowledge of both an internal, mental world (i.e., mathematics) and an external, physical world (i.e., sciences). The internal world is abstract and ephemeral, while the external one is solid and real. But to the person's amazement, there is a remarkable correspondence in that everything about the external world can seemingly be described perfectly by ideas from the internal one, some of which are known long before their application to the external world is clear (e.g., complex numbers, non-Euclidean geometry). To unpack the metaphor, if mathematics is inherent in the fundamental nature of our mind, then its 'unreasonable effectiveness' in the physical sciences suggests a nondualist view, that is, that the mind and the world are part of a common unity.

6. Conclusion

Mathematics is indispensable for science, and arithmetic is the foundation of mathematics. But we do not know where it comes from – why addition and multiplication are its fundamental operations. Arithmetic has been viewed as a necessary, *a priori* truth – we know it's true but not why. There is a remarkable irony in that we keep pushing the boundaries of our knowledge in the physical sciences, asking questions and insisting on rigorous explanations, using a mathematical language that ultimately derives from arithmetic, which we do not question because we have assumed there is nothing more to learn about it. Maybe we should. In Eddington's words, "We used to think that if we knew one, we knew

two, because one and one are two. We are finding that we must learn a great deal more about ‘and’”.

Footnotes

¹Note that this formulation uses second-order logic (specifically for axiom 5); first-order formulations of Peano arithmetic exist (in which axiom 5 is replaced by a first-order induction schema) but these formulations admit non-standard models of arithmetic (Skolem, 1934; Robinson, 2016).

²More precisely, when we refer to rational and real addition we mean non-negative rationals and reals ($\mathbb{Q} \geq 0, \mathbb{R} \geq 0$); when we refer to real multiplication we mean for reals greater than or equal to 1 ($\mathbb{R} \geq 1$). See Appendix A for details.

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Appendix A

In Appendix A we present constructions of natural addition (in sections A2 and A2.1), (non-negative) rational addition (in sections A3 and A3.1), (non-negative) real addition (in section A4), and real multiplication (on $\mathbb{R} \geq 1$; in section A5). Our starting point is the set $\mathbb{N}$ of natural numbers equipped with its usual less-than ordering. From here, we combine monotonicity, convexity, continuity, and isomorphism in various ways to arrive at addition and multiplication over the reals. It should be noted that results generalize from the starting point $(\mathbb{N}, <)$ to an abstract $\mathbb{N}$ – chain, which captures the purely order-theoretic structure (but not additive structure) of $\mathbb{N}$. Thus, our construction of natural addition is based on an $\mathbb{N}$ – chain but we present the results starting from $(\mathbb{N}, <)$ for sake of clarity. First, some preliminaries are given in A1.

A1. Preliminaries

To identify natural, rational, and real addition, we repeatedly employ two tools: (i) minimizing with respect to some order, and (ii) unique functional extensions. Here we provide a brief account of the requisite background.

A1.1 Posets

Let X be a non-empty set and let R be a binary relation on X ; that is, R is a subset of X^2 . The pair (X, R) is a **partially ordered set** (or **poset**) if R is **reflexive**, **anti-symmetric**, and **transitive**; that is, for each x, y, z in X , the following hold.

$(x, x) \in R$ (Reflexivity)

If $(x, y) \in R$ and $(y, x) \in R$, then $x = y$ (Anti-Symmetry)

If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ (Transitivity)

As is customary, we help ourselves to the more suggestive $\leq$ instead of R ; we write $x \leq x'$ to indicate that (x, x') is in the relation $\leq$; we write $x < x'$ as shorthand for $x \leq x'$ and $x \neq x'$; we write $x \leq y \leq z$ as shorthand for $x \leq y$ and $y \leq z$. Given a poset $(X, \leq)$ (X is the underlying set, $\leq$ partially orders X), an element m of X is a **minimal element** of $(X, \leq)$ if there is no m' in X such that $m' \neq m$ and $m' \leq m$. An element m of X is a **minimum element** of $(X, \leq)$ if $m \leq m'$ for each m' in X ; maximal and maximum elements are defined dually. Note that minimal elements need not be minimum, but minimum elements are necessarily minimal. Note also, that posets need not have minimal elements (think of the integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ with the usual less-than ordering), and if a minimum element exists, then it is unique; that is, it is the only one. The poset $(X, \leq)$ is a **linear order** if each pair (x, y) in X^2 is comparable; that is, either $x \leq y$ holds or $y \leq x$ holds. Given an element x in X , z in X is an (ordinal) **successor**¹ of x if $x < z$ holds and there is no y in X such that $x < y < z$ holds. A subset X' of X is a **downset** if it is closed downward under $\leq$; that is, if x in X' and $y \leq x$, then y is in X' . For each element x in X we have the **principal downset** $\downarrow x$ (generated by x) consisting of those members of X below x ; that is, $\downarrow x = \{y \in X : y \leq x\}$.

A1.2 Functional Extensions

Given a function $f: X \rightarrow Y$ and sets X', Y' such that $X \subset X'$ and $Y \subset Y'$, a function $g: X' \rightarrow Y'$ is an **extension** (of f) if, for each x in X , we have that $g(x) = f(x)$. Typically,¹ we are interested in a particular extension. For instance, f may have some property P , and we may require an extension g of f , with g also having property P . When luck is on our side, such a g exists. Ideally, there is a *unique* extension with the property P ; we will find ourselves in this position on several occasions.

A1.3 Notation

The set $\{0, 1, 2, \dots\}$ of natural numbers is denoted by $\mathbb{N}$. The set of non-negative rationals is denoted by $\mathbb{Q}$, and $\mathbb{Q}^3$ is the set of ordered triples of non-negative rationals. The rational unit interval $\{q \in \mathbb{Q}: 0 \leq q \leq 1\}$ is denoted by $\mathbb{I}_{\mathbb{Q}}$. The set of non-negative reals is denoted by $\mathbb{R}$, and $\mathbb{R}^3$ is the set of ordered triples of non-negative reals. The real unit interval $\{r \in \mathbb{R}: 0 \leq r \leq 1\}$ is denoted by $\mathbb{I}$. The natural addition function is denoted by $f_{\mathbb{N}}^+$. The (non-negative) rational addition function is denoted by $f_{\mathbb{Q}}^+$. The (non-negative) real addition function is denoted by $f_{\mathbb{R}}^+$.

A2 Natural Addition Via Monotonicity

We present two constructions of natural addition in A2 and A2.1. Each construction relies on explicit use of the natural numbers and their usual less-than ordering $(\mathbb{N}, <)$, although we show later that they can be based on an abstract $\mathbb{N}$ – chain, which has the order-

¹ Note that this *ordinal* successor concept is distinct from the *functional* successor concept in the Dedekind-Peano axioms.

theoretic but not the additive structure of the natural numbers. These constructions take place in the context of $(\mathbb{N}, <)$ because it is the canonical example of an $\mathbb{N}$ – chain and it makes the proofs easier to understand. We provide a (second order) axiomatization of an $\mathbb{N}$ – chain in A3, and then show in Appendix B that this axiomatization is categorical; that is, all $\mathbb{N}$ – chains are isomorphic to $(\mathbb{N}, <)$. This means that our constructions hold for abstract $\mathbb{N}$ – chains (i.e., those with arbitrary elements, not necessarily the natural numbers).

The goal of this section is to show that the addition operation occupies a structurally unique position amongst monotonic functions of two natural variables. Let $\mathcal{F}_{\mathbb{N}}$ denote the set of functions from $\mathbb{N}^2$ to $\mathbb{N}$; that is, those functions that take in ordered pairs of natural numbers and return a natural number. A map f in $\mathcal{F}_{\mathbb{N}}$ is **monotonic** if, for all natural numbers a, b , and c , with $a < b$, we have that $f(a, c) < f(b, c)$ and $f(c, a) < f(c, b)$. We let $\mathcal{M}_{\mathbb{N}}$ denote the set of monotonic functions in $\mathcal{F}_{\mathbb{N}}$. Next, we define a binary relation $\leq$ on $\mathcal{M}_{\mathbb{N}}$ by setting

$$f \leq h \Leftrightarrow \forall a, b \in \mathbb{N} (f(a, b) \leq h(a, b)).$$

Note that $\leq$ is reflexive, anti-symmetric, and transitive; thus, $(\mathcal{M}_{\mathbb{N}}, \leq)$ is a partial order.

Lemma 1. *For each map f in $\mathcal{M}_{\mathbb{N}}$ and each triple of natural numbers (a, b, k) , we have that $f(a, b) + k \leq f(a + k, b), f(a, b + k)$.*

Proof. Letting f and (a, b) be arbitrary members of $\mathcal{M}_{\mathbb{N}}$ and $\mathbb{N} \times \mathbb{N}$ respectively, we proceed by induction on k . When $k = 0$, we have that

$$f(a, b) + 0 = f(a, b) = f(a + 0, b) = f(a, b + 0),$$

and equality holds. Now suppose that $k > 0$, and that the result holds for $k - 1$. Then we have that

$$\begin{aligned}
f(a, b) + k &= (f(a, b) + (k - 1)) + 1 \\
&\leq f(a + (k - 1), b) + 1 \\
&\leq f(a + k, b),
\end{aligned}$$

where the first inequality follows from our inductive hypothesis, and the second from the fact that f is strictly increasing in the first argument. We also have that

$$\begin{aligned}
f(a, b) + k &= (f(a, b) + (k - 1)) + 1 \\
&\leq f(a, b + (k - 1)) + 1 \\
&\leq f(a, b + k),
\end{aligned}$$

where the first inequality follows from our inductive hypothesis, and the second from the fact that f is strictly increasing in the second argument. Hence, the result holds for each natural number k . As f and (a, b) were arbitrary, the result holds for each f and (a, b, k) in $\mathcal{M}_{\mathbb{N}}$ and $\mathbb{N}^3$. $\square$

Lemma 2. *For each function f in $\mathcal{M}_{\mathbb{N}}$ and each pair of natural numbers (a, b) , we have that $a + b \leq f(a, b)$.*

Proof. Let f and (a, b) be arbitrary members of $\mathcal{M}_{\mathbb{N}}$ and $\mathbb{N}^2$, respectively. As f is a map from $\mathbb{N}^2$ to $\mathbb{N}$, we have that $0 \leq f(0, 0)$. By Lemma 1, we have that $f(0, 0) + a \leq f(a, 0)$ and $f(a, 0) + b \leq f(a, b)$. Putting all this together, we see that

$$\begin{aligned}
a + b &= 0 + a + b \\
&\leq f(0, 0) + a + b \\
&= (f(0, 0) + a) + b \\
&\leq f(a, 0) + b \\
&\leq f(a, b),
\end{aligned}$$

which is the desired result. $\square$

Theorem 1. *The addition function $f_{\mathbb{N}}^+ : (a, b) \mapsto a + b$ is the minimum element in the poset $(\mathcal{M}_{\mathbb{N}}, \leq)$.*

Proof. First, observe that the addition function $f_{\mathbb{N}}^+ : (a, b) \mapsto a + b$ is a member of $\mathcal{M}_{\mathbb{N}}$. Next, by applying Lemma 2, we see that $f_{\mathbb{N}}^+(a, b) = a + b \leq f(a, b)$ for each f in $\mathcal{M}_{\mathbb{N}}$ and each pair (a, b) in $\mathbb{N} \times \mathbb{N}$. By definition, we now have that $f_{\mathbb{N}}^+ \leq f$, for each f in $\mathcal{M}_{\mathbb{N}}$. Hence, $f_{\mathbb{N}}^+$ is the minimum element in $(\mathcal{M}_{\mathbb{N}}, \leq)$. $\square$

A2.1 The Natural Numbers and Addition Via Monotonicity and Convexity

The previous section showed that natural addition was the minimum monotonic binary operation on the natural numbers. Here we show that the additive structure of the natural numbers is isomorphic to the space of convex self-embeddings of $(\mathbb{N}, <)$.

We require a few definitions. Given a partial order $(X, \leq)$ and elements x, y , and z in X , we say that y is **between** x and z if $x \leq y \leq z$ holds or if $z \leq y \leq x$ holds. A subset X' of X is **convex** (with respect to the order $\leq$) if X' is closed under betweenness; that is, X' is convex if, whenever x and z are in X' and y is between x and z we also have y in X' .² Given two partial orders $(X, \leq_X)$ and $(Y, \leq_Y)$, a map $f : X \rightarrow Y$ is **monotonic** if, for each x and x' in X , $x \leq_X x'$ implies $f(x) \leq_Y f(x')$. The map f is **injective** if, for each x and x' in X , $x \neq x'$ implies $f(x) \neq f(x')$. The map f is an **embedding** (of $(X, \leq_X)$ in $(Y, \leq_Y)$) if f is injective and monotonic; to indicate that f is an embedding we write $f : X \hookrightarrow Y$. If f is an embedding, then $(X, \leq_X)$ is isomorphic with its image under f in $(Y, \leq_Y)$; that is, $(f(X), \leq_{Y_{f(X)}})^3$ is an isomorphic copy of $(X, \leq_X)$ living inside $(Y, \leq_Y)$. Note that a surjective embedding is an

² This is an order-theoretic notion of convexity; we shall also use a geometric notion of convexity in A3.

³ The order $\leq_{Y_{f(X)}}$ is the restriction of the order $\leq_Y$ to the set $f(X)$.

isomorphism. The map f is a **convex** embedding if $f(X)$ is a convex subset of the partial order $(Y, \leq_Y)$.

The aim of this section is to demonstrate how addition of natural numbers emerges from order-theoretic considerations; in particular, from the convex self-embeddings of the order-theoretic structure of the natural numbers. Let $\mathbb{N}$ denote the set of natural numbers; that is, $\mathbb{N} = \{0, 1, 2, \dots\}$. We shall help ourselves to the usual order on $\mathbb{N}$; that is,

$$0 < 1 < 2 < 3 < \dots$$

Having established our starting point of $(\mathbb{N}, <)$ - the natural numbers equipped with their usual order - we now wish to understand the set of convex self-embeddings of the space $(\mathbb{N}, <)$.

A **self-embedding** of $(\mathbb{N}, <)$ is an injective map $f: \mathbb{N} \hookrightarrow \mathbb{N}$ that is monotonic - that is,

$$m < n \implies f(m) < f(n)$$

holds for each m and n in $\mathbb{N}$; we shall let $\text{Emb}(\mathbb{N}, <)$ denote the set of all such self-embeddings. Recall that a subset X of $\mathbb{N}$ is **convex** (with respect to the order $<$) if, whenever a and c are in X and $a \leq b \leq c$ holds in $(\mathbb{N}, <)$, we also have b in X ; that is, X is convex if X is closed under $<$ betweenness. We are interested in the convex self-embeddings of $(\mathbb{N}, <)$; that is, those embeddings f whose image $f(\mathbb{N})$ is a convex subset of $\mathbb{N}$. We shall let $\text{Emb}_C(\mathbb{N}, <)$ denote the set of convex self-embeddings of $(\mathbb{N}, <)$.

Lemma 3. *For each natural m , there exists a unique map f in $\text{Emb}_C(\mathbb{N}, <)$ such that $f(0) = m$.*

Proof. Let m be an arbitrary natural number, and consider the map $f: \mathbb{N} \rightarrow \mathbb{N}$ given by setting $f(n) = n + m$ for each natural number n . We claim that f is a convex self-embedding of

$(\mathbb{N}, <)$. To see that f is injective, note that if $n \neq n'$ then $f(n) = n + m \neq n' + m = f(n')$.

To see that f is monotonic, note that for any natural numbers n and n' , we have that

$$n \leq n' \Rightarrow f(n) = n + m \leq n' + m = f(n').$$

Hence, the map $f: n \mapsto n + m$ is in $\text{Emb}(\mathbb{N}, <)$. To see that f is convex, it suffices to observe that $f(\mathbb{N})$ is the upset $\uparrow m = \{n \in \mathbb{N} : m \leq n\}$, which is a convex set in $(\mathbb{N}, <)$. Hence, f is in $\text{Emb}_C(\mathbb{N}, <)$.

Having established the existence of a convex self-embedding taking 0 to m , we now establish the uniqueness of this map. Suppose that a convex self-embedding f of $(\mathbb{N}, <)$ is such that $f(0) = m$; we claim that $f(n) = n + m$ for each natural number n . To justify this claim, we proceed via induction on n . The base case - when $n = 0$ - is true by hypothesis, so suppose that $f(k) = k + m$ for each natural number k in the interval $[0, n]$, for some natural number n ; from this assumption we aim to show that $f(n + 1) = (n + 1) + m$. Observe that $n + m = f(n) < f(n + 1)$ must hold, because f is monotonic and injective; thus, we have either $(n + 1) + m = f(n + 1)$ or $(n + 1) + m < f(n + 1)$. If $(n + 1) + m < f(n + 1)$ holds, then some natural number $k \notin [0, n + 1]$ is such that $f(k) = (n + 1) + m$, and therefore we have

$$f(k) < f(n + 1);$$

such a k exists because f is injective and convex, and because the downset $\downarrow n$ is finite. As $k \notin [0, n + 1]$, we have that $n + 1 < k$; combined with $f(k) < f(n + 1)$, this violates the monotonicity of f . Thus, we must have that $f(n + 1) = (n + 1) + m$. By induction, we may conclude that $f(n) = n + m$ for each natural number n . $\square$

In light of the previous result, we may denote by f_m the unique map in $\text{Emb}_C(\mathbb{N}, <)$ sending 0 to m ; thus, we may write

$$\text{Emb}_C(\mathbb{N}, <) = \{f_m : m \in \mathbb{N}\}.$$

The set $\{f_m : m \in \mathbb{N}\}$ comes equipped with an implicit binary operation; namely, functional composition, which we denote $\circ$. The reason we can freely compose maps in $\{f_m : m \in \mathbb{N}\}$ is that each such map is from $\mathbb{N}$ to $\mathbb{N}$; that is, each map in $\{f_m : m \in \mathbb{N}\}$ agrees on its domain and codomain. Moreover, if we are given f and g in $\{f_m : m \in \mathbb{N}\}$, then the composite map $f \circ g$ is also in $\{f_m : m \in \mathbb{N}\}$.

Lemma 4. *If f and g are in $\text{Emb}_C(\mathbb{N}, <)$, then the composite map $f \circ g$ is in $\text{Emb}_C(\mathbb{N}, <)$.*

Proof. To see that $f \circ g$ is injective, observe that for any natural numbers n and n' we have that

$$n \neq n' \Rightarrow g(n) \neq g(n') \Rightarrow f \circ g(n) = f(g(n)) \neq f(g(n')) = f \circ g(n'),$$

where the first and second implications are due to the injectivity of g and f respectively; thus, $f \circ g$ is injective. To see that $f \circ g$ is monotonic, observe that for any natural numbers n and n' we have that

$$n \leq n' \Rightarrow g(n) \leq g(n') \Rightarrow f \circ g(n) = f(g(n)) \leq f(g(n')) = f \circ g(n'),$$

where the first and second implications are due to the monotonicity of g and f respectively; thus, $f \circ g$ is monotonic. As $f \circ g$ is both injective and monotonic, $f \circ g$ is in $\text{Emb}(\mathbb{N}, <)$.

To see that $f \circ g$ is convex, suppose that $a \leq b \leq c$ holds in $(\mathbb{N}, <)$ and that a and c are in $f \circ g(\mathbb{N})$; we require that b is also in $f \circ g(\mathbb{N})$. As a and c are in $f \circ g(\mathbb{N})$, there are natural numbers n_a and n_c such that

$$f(g(n_a)) = a \text{ and } f(g(n_c)) = c,$$

from which it follows that a and c are also in $f(\mathbb{N})$. As $a \leq b \leq c$ holds, and both a and c are in $f(\mathbb{N})$, there exists (by the convexity of f) a natural number k such that $f(k) = b$. As f is $\leq$ reflecting and $a \leq b \leq c$ holds, we have that $g(n_a) \leq k \leq g(n_c)$ holds. As $g(n_a) \leq k \leq g(n_c)$ holds, there exists (by the convexity of g) a natural number l such that $g(l) = k$, and then we have

$$f \circ g(l) = f(g(l)) = f(k) = b.$$

Thus, b is in $f \circ g(\mathbb{N})$ as required, and $f \circ g$ is convex; therefore $f \circ g$ is in $\text{Emb}_C(\mathbb{N}, <)$. $\square$

So in $\text{Emb}_C(\mathbb{N}, <)$, we have a structure of binary operation type; to indicate this (binary operation) aspect explicitly, we shall write $(\text{Emb}_C(\mathbb{N}, <), \circ)$ to indicate the set $\text{Emb}_C(\mathbb{N}, <)$ equipped with the binary operation $\circ$. Formally, the binary operation $\circ$ is the set of ordered triples of the form $(f, g, f \circ g)$ where f and g are in $\text{Emb}_C(\mathbb{N}, <)$; that is,

$$\circ = \{(f, g, f \circ g) : f, g \in \text{Emb}_C(\mathbb{N}, <)\}.$$

Now we know that $f \circ g$ is in $\text{Emb}_C(\mathbb{N}, <)$ when f and g are in $\text{Emb}_C(\mathbb{N}, <)$, it would be useful to understand the relationship between f and g on the one hand, and $f \circ g$ on the other.

Lemma 5. *For each pair (f_{m_1}, f_{m_2}) of maps in $\text{Emb}_C(\mathbb{N}, <)(= \{f_m : m \in \mathbb{N}\})$, we have that*

$$f_{m_1} \circ f_{m_2} = f_{m_1+m_2}.$$

Proof. Let m_1 and m_2 be arbitrary natural numbers, and let f_{m_1} and f_{m_2} be the maps in $\text{Emb}_C(\mathbb{N}, <)$ taking 0 to m_1 and m_2 respectively. For each natural number n , we have that

$$\begin{aligned}
f_{m_1} \circ f_{m_2}(n) &= f_{m_1}(f_{m_2}(n)) \\
&= f_{m_1}(n + m_2) \\
&= (n + m_2) + m_1 \\
&= n + (m_2 + m_1) \\
&= n + (m_1 + m_2) \\
&= f_{m_1+m_2}(n).
\end{aligned}$$

As $f_{m_1} \circ f_{m_2}$ and $f_{m_1+m_2}$ agree on each natural number n , it follows that $f_{m_1} \circ f_{m_2} = f_{m_1+m_2}$. $\square$

Now we are in a position to obtain natural addition.

Theorem 2. *The structures $(\mathbb{N}, +)$ and $(\text{Emb}_C(\mathbb{N}, <), \circ)$ are isomorphic.*

Proof. Consider the map $\psi: \text{Emb}_C(\mathbb{N}, <) \rightarrow \mathbb{N}$ given by setting $\psi(f_m) = m$ for each natural number m . The bijectivity of ψ follows from Lemma 3. To see that ψ is operation preserving, observe that for each pair (f_m, f_n) of maps in $\text{Emb}_C(\mathbb{N}, <)$, we have that

$$\psi(f_m \circ f_n) = \psi(f_{m+n}) = m + n = \psi(f_m) + \psi(f_n).$$

As ψ is operation preserving and bijective, ψ is an isomorphism. $\square$

Thus, we see that the additive structure of the natural numbers is the structure of the convex self-embeddings of the order-theoretic structure of the natural numbers; or put another way, we have algebraic (or arithmetic) structure emerging from the combination of monotonicity and convexity.

It is worth noting that the preceding developments did *not* require the explicit use of the natural numbers. We could have instead started with a more abstract structure. A linear order $(X, <)$ is an $\mathbb{N}$ – **chain** if (i) each element of X has an ordinal successor, (ii) $(X, <)$ has a minimum element, and (iii) all principal downsets are finite. In Appendix B, we prove that all $\mathbb{N}$ – chains are isomorphic; indeed, they are all isomorphic to $(\mathbb{N}, <)$.

In place of $(\mathbb{N}, <)$, the preceding developments could begin with a generic $\mathbb{N}$ – chain $(X, <)$. The product $X \times X$ inherits the product order from $(X, <)$ which gives rise to the set $\mathcal{M}_X$ of all monotonic maps from $X \times X$ to X . The set $\mathcal{M}_X$ is partially ordered in the way we ordered $\mathcal{M}_{\mathbb{N}}$, and the minimum element of $\mathcal{M}_X$ is isomorphic to natural addition. Alternatively, the space $\text{Emb}_C(X, <)$ of convex self-embeddings of $(X, <)$ is - under function composition - isomorphic to the additive structure of the naturals. Thus, the constructions of natural addition in A2 and A2.1 rest purely on the order-theoretic structure of an abstract $\mathbb{N}$ – chain, that is, they do not presuppose any arithmetic properties of the natural numbers.

A3 Rational Addition Via Convexity

In this section, we obtain the rational addition operation $f_{\mathbb{Q}}^+$ from the natural addition operation $f_{\mathbb{N}}^+$ via convexity in two distinct ways; minimizing against a partial order, and by a unique extension. Both methods are geometric in nature, and they take place in $\mathbb{Q}^3$.

The set-theoretic representation of $f_{\mathbb{N}}^+$ is

$$f_{\mathbb{N}}^+ = \{(m, n, f_{\mathbb{N}}^+(m, n)) : m, n \in \mathbb{N}\} = \{(m, n, m + n) : m, n \in \mathbb{N}\}.$$

Viewed this way, we see that $f_{\mathbb{N}}^+$ is a subset of $\mathbb{N}^3$, and therefore, also a subset of $\mathbb{Q}^3$. When we look at the set $f_{\mathbb{N}}^+$ embedded in $\mathbb{Q}^3$, we see that the points in $f_{\mathbb{N}}^+$ all lie in the same plane. Having made this observation, intuitively it is clear that to extend $f_{\mathbb{N}}^+$ to non-negative rational inputs, we need to ‘fill in the gaps’ in this plane. In order to make this intuition precise, we again invoke convexity; unlike the order-theoretic notion of convexity in A2.1, this version of convexity is geometric in nature. The definition below uses algebraic language, but both it and the order-theoretic definition in A2.1 are based on a similar intuition of ‘betweenness’ and are equivalent on the real number line (see §4).

Given two points $p = (x, y, z)$ and $p' = (x', y', z')$ in $\mathbb{Q}^3$, the *rational* straight line segment between them, denoted $\ell_{\mathbb{Q}}(p, p')$, is the set

$$\ell_{\mathbb{Q}}(p, p') = \{(1 - t)(x, y, z) + t(x', y', z') : t \in \mathbb{I}_{\mathbb{Q}}\}.$$

A subset X of $\mathbb{Q}^3$ is **convex** (in $\mathbb{Q}^3$) if, for each pair of points p and p' in X , the set $\ell_{\mathbb{Q}}(p, p')$ is contained in X .

Note that this definition uses rational addition to specify the points on the line segment between p and p' . This might appear to be a circular argument because we are trying to construct rational addition, and have used an algebraic definition of convexity. However, convexity is fundamentally a geometric intuition. We use the algebraic definition because it is more convenient for establishing the proof that rational addition can be obtained from natural addition by ‘filling in the gaps’ via convexity. Our proof uses algebraic language in the definition of convexity, but the results that rational addition is the rational convex hull of natural addition and the unique rational convex extension of natural addition do not rely on addition appearing in this definition. In A3.1 we give a definition of convexity based on isomorphism to show that geometric intuitions of ‘betweenness’ can be defined without any algebraic information.

Given a subset X of $\mathbb{Q}^3$, we let $\mathcal{C}_{\mathbb{Q}}(X)$ denote the set of all convex subsets of $\mathbb{Q}^3$ that contain X ; that is,

$$\mathcal{C}_{\mathbb{Q}}(X) = \{Y \subseteq \mathbb{Q}^3 : X \subseteq Y \text{ and } Y \text{ is convex}\}.$$

Note that $\mathbb{Q}^3$ is itself convex, and therefore $\mathbb{Q}^3$ is an element of $\mathcal{C}_{\mathbb{Q}}(X)$, irrespective of the choice of X . Note also that $\mathcal{C}_{\mathbb{Q}}(X)$ is partially ordered by the set inclusion relation $\subseteq$.

Lemma 6. *The poset $(\mathcal{C}_{\mathbb{Q}}(X), \subseteq)$ has a minimum element.*

Proof. Consider the set $\cap \mathcal{C}_{\mathbb{Q}}(X) = \{p \in \mathbb{Q}^3 : p \in Y, \text{ for each } Y \text{ in } \mathcal{C}_{\mathbb{Q}}(X)\}$ - the intersection of the sets in $\mathcal{C}_{\mathbb{Q}}(X)$ - and let p and p' be arbitrary members of $\cap \mathcal{C}_{\mathbb{Q}}(X)$. We wish to show that $\ell_{\mathbb{Q}}(p, p')$ is contained in $\cap \mathcal{C}_{\mathbb{Q}}(X)$. As each member of $\mathcal{C}_{\mathbb{Q}}(X)$ is convex and each member contains p and p' , it follows that each member contains $\ell_{\mathbb{Q}}(p, p')$. Hence, $\ell_{\mathbb{Q}}(p, p')$ is a subset of $\cap \mathcal{C}_{\mathbb{Q}}(X)$. As p and p' were arbitrary, we have that $\cap \mathcal{C}_{\mathbb{Q}}(X)$ is convex. As each member of $\mathcal{C}_{\mathbb{Q}}(X)$ contains X , we have that $\cap \mathcal{C}_{\mathbb{Q}}(X)$ also contains X . Hence, $\cap \mathcal{C}_{\mathbb{Q}}(X)$ is a member of $\mathcal{C}_{\mathbb{Q}}(X)$. Finally, for each Y in $\mathcal{C}_{\mathbb{Q}}(X)$, we have that $\cap \mathcal{C}_{\mathbb{Q}}(X) \subseteq Y$. Hence, $\cap \mathcal{C}_{\mathbb{Q}}(X)$ is the minimum element in $(\mathcal{C}_{\mathbb{Q}}(X), \subseteq)$. $\square$

For a subset X of $\mathbb{Q}^3$, the set $\cap \mathcal{C}_{\mathbb{Q}}(X)$ is the (rational) **convex hull** of X .

Lemma 7. *The convex hull of $f_{\mathbb{N}}^+$ in $\mathbb{Q}^3$ is $f_{\mathbb{Q}}^+$; that is, $\cap \mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+) = f_{\mathbb{Q}}^+$.*

Proof. The proof has two parts. First, we show that if a point $p = (x', y', z')$ in $\mathbb{Q}^3$ is such that $p \notin f_{\mathbb{Q}}^+$ - that is, $x' + y' \neq z'$ - then there exists a set X in $\mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+)$ such that $p \notin X$. Then we show that each point $p = (x, y, x + z)$ in $\mathbb{Q}^3$ is in every set X in $\mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+)$.

Let $p = (x', y', z')$ be a point in $\mathbb{Q}^3$ such that $x' + y' \neq z'$. Preserving generality, we may assume that $z' = x' + y' + \epsilon$, where $\epsilon > 0$ (and ϵ is rational). Now consider the set $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}] = \{(x, y, z) \in \mathbb{Q}^3 : z \leq x + y + \frac{\epsilon}{2}\}$. Note that $p = (x', y', z')$ is not a member of $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}]$, as $x' + y' + \frac{\epsilon}{2} < x' + y' + \epsilon = z'$. Also, note that $f_{\mathbb{N}}^+$ is contained in $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}]$, as $m + n \leq m + n + \frac{\epsilon}{2}$, for all natural numbers m and n . To see that $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}]$ is convex, let t be in $\mathbb{I}_{\mathbb{Q}}$, let $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ be points in $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}]$, and consider the point $(1 - t)p_1 + tp_2$, which, after some manipulation, can be rewritten as $((1 - t)x_1 + tx_2, (1 - t)y_1 + ty_2, (1 - t)z_1 + tz_2)$. We require that $(1 - t)z_1 + tz_2 \leq (1 - t)x_1 + tx_2 + (1 - t)y_1 +$

$ty_2 + \frac{\epsilon}{2}$. As (x_1, y_1, z_1) and (x_2, y_2, z_2) are in $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}]$, we have that $z_1 \leq x_1 + y_1 + \frac{\epsilon}{2}$ and $z_2 \leq x_2 + y_2 + \frac{\epsilon}{2}$. As $z_1, 1 - t, z_2, t \geq 0$, we also have that $(1 - t)z_1 \leq (1 - t)x_1 + (1 - t)y_1 + (1 - t)\frac{\epsilon}{2}$ and $tz_2 \leq tx_2 + ty_2 + t\frac{\epsilon}{2}$. From this, it follows that $(1 - t)z_1 + tz_2 \leq (1 - t)x_1 + tx_2 + (1 - t)y_1 + ty_2 + \frac{\epsilon}{2}$. As $t, (x_1, y_1, z_1)$, and (x_2, y_2, z_2) were arbitrary, we have that $f_{\mathbb{Q}}^+[\frac{\epsilon}{2}]$ is convex. Hence, the point $p = (x', y', z')$ is *not* in the convex hull of $f_{\mathbb{N}}^+$. As p was arbitrary, it follows that $f_{\mathbb{Q}}^+$ contains the convex hull of $f_{\mathbb{N}}^+$ - in symbols, $\cap \mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+) \subseteq f_{\mathbb{Q}}^+$. That completes the first part of the proof.

Now let x and y be (non-negative) rational numbers, and consider the point $p = (x, y, x + y)$ in $f_{\mathbb{Q}}^+$. We may write $x = \frac{a}{b}$ and $y = \frac{a'}{b'}$, where $a, a', b, b' \in \mathbb{N}$ and $b, b' \neq 0$. The point $(ab', a'b, ab' + a'b)$ is in $f_{\mathbb{N}}^+$, as is the origin $(0, 0, 0)$. Note that $\frac{1}{bb'}$ is rational and $0 < \frac{1}{bb'} \leq 1$. Hence, the point $\frac{1}{bb'}(ab', a'b, ab' + a'b)$ is in every convex set containing $f_{\mathbb{N}}^+$ - that is, it is in the convex hull of $f_{\mathbb{N}}^+$. All that remains is to verify that the point just named is $(x, y, x + y)$. Well,

$$\begin{aligned} \frac{1}{bb'}(ab', a'b, ab' + a'b) &= \left(\frac{ab'}{bb'}, \frac{a'b}{bb'}, \frac{ab' + a'b}{bb'} \right) \\ &= \left(\frac{a}{b}, \frac{a'}{b'}, \frac{ab'}{bb'} + \frac{a'b}{bb'} \right) \\ &= \left(x, y, \frac{a}{b} + \frac{a'}{b'} \right) \\ &= (x, y, x + y), \end{aligned}$$

and therefore $p = (x, y, x + y) \in \cap \mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+)$. As p was an arbitrary point in $f_{\mathbb{Q}}^+$, we now have the reverse inclusion, namely $f_{\mathbb{Q}}^+ \subseteq \cap \mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+)$. Hence, the convex hull of $f_{\mathbb{N}}^+$ (in $\mathbb{Q}^3$) is $f_{\mathbb{Q}}^+$. $\square$

Thus, by minimizing against convexity, we arrive at rational addition. Another way to obtain rational addition from natural addition is via a convex extension. A function $f: \mathbb{Q}^2 \rightarrow \mathbb{Q}$ is convex (in $\mathbb{Q}^3$) if the set $\{(x, y, f(x, y)) : x, y \in \mathbb{Q}\}$ is convex in $\mathbb{Q}^3$.

Lemma 8. *The function $f_{\mathbb{Q}}^+$ is the unique convex rational extension of $f_{\mathbb{N}}^+$.*

Proof. First, note that $f_{\mathbb{Q}}^+$ is a convex rational extension of $f_{\mathbb{N}}^+$; this follows from the previous result. To see that $f_{\mathbb{Q}}^+$ is the unique such object, suppose that $f: \mathbb{Q}^2 \rightarrow \mathbb{Q}$ is an extension of $f_{\mathbb{N}}^+$; that is, $f(m, n) = m + n$ for all naturals m and n . Furthermore, suppose that f is convex and $f \neq f_{\mathbb{Q}}^+$. As f is a convex extension of $f_{\mathbb{N}}^+$, we have that $f \in \mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+)$. As $\bigcap \mathcal{C}_{\mathbb{Q}}(f_{\mathbb{N}}^+) = f_{\mathbb{Q}}^+$, it follows that $f_{\mathbb{Q}}^+ \subseteq f$. As $f \neq f_{\mathbb{Q}}^+$, we have that $f_{\mathbb{Q}}^+ \subset f$, which contradicts the fact that f is a function. Thus, no such f exists, and the result holds. $\square$

A3.1 Convexity via Isomorphism

In A3 we arrived at rational addition via convexity in two ways. As noted earlier, our definition of convexity used rational addition and thus was algebraic. Here we show that geometric convexity does not rely on any arithmetic or algebraic concepts; indeed, we can ground it in symmetry/isomorphism alone. Because we perform this maneuver in the next section (in order to obtain real addition), we shall provide the justification for each now.

First, consider real 3-space equipped with the Euclidean metric d_E ; that is, for each pair $((x, y, z), (x', y', z'))$ of points in $\mathbb{R}^3$, we have

$$d_E((x, y, z), (x', y', z')) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

Given the metric d_E , we can define a ternary (3-place) relation B on $\mathbb{R}^3$ by setting

$$B(\alpha, \beta, \gamma) \Leftrightarrow d_E(\alpha, \beta) + d_E(\beta, \gamma) = d_E(\alpha, \gamma),$$

for each α, β , and γ in real 3-space. Note that $B(\alpha, \beta, \gamma)$ holds precisely when β lies on the straight-line segment $\overline{\alpha\gamma}$; hence, when $B(\alpha, \beta, \gamma)$ holds, we say β is **between** α and γ . Thus, $B(\alpha, \beta, \gamma)$ holds precisely when $\beta = (1 - t)\alpha + t\gamma$ for some $t: 0 \leq t \leq 1$. From the relation B , we define the *strict* betweenness relation B^* by setting

$$B^*(\alpha, \beta, \gamma) \Leftrightarrow (B(\alpha, \beta, \gamma) \text{ and } \alpha \neq \beta \neq \gamma).$$

Given the relation B^* in $\mathbb{R}^3$, we can define convexity. A subset X of $\mathbb{R}^3$ is **convex** if, whenever α and γ are in X and $B^*(\alpha, \beta, \gamma)$ holds, β is also in X ; that is,

$$X \text{ is convex} \Leftrightarrow \forall \alpha, \beta, \gamma \left((\alpha, \gamma \in X \text{ and } B^*(\alpha, \beta, \gamma)) \Rightarrow \beta \in X \right).$$

Put another way, X is convex if X is closed under betweenness.

Next, we consider the set $I_{\mathbb{R}^3}$ of **isometries** of the metric space $(\mathbb{R}^3, d_E)$; that is, $I_{\mathbb{R}^3}$ is the set of bijective maps $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that preserve the metric d_E . Note that $I_{\mathbb{R}^3}$ contains: (i) rotations around lines; (ii) reflections in planes; (iii) translations. Moreover, $I_{\mathbb{R}^3}$ contains all (finite) compositions of rotations reflections and translations.

Now that we have our hands on $I_{\mathbb{R}^3}$, we shall discard the metric d_E and deal only with the set $I_{\mathbb{R}^3}$ of isometries. The reason for discarding the metric is that it too relies on the very arithmetic machinery we are seeking to construct. The approach we are taking - namely, extracting the automorphism group of the space while ignoring the generating metric - has historical precedent; indeed, this type of approach was outlined by Felix Klein in 1872 in the so-called *Erlangen Program*.

Our task is to define the relation B^* using just the set $I_{\mathbb{R}^3}$ of isometries. Note that if we are given three distinct points α, β and γ , then $B^*(\alpha, \beta, \gamma)$ holds in $\mathbb{R}^3$ if, and only if, (i)

α, β , and γ are collinear, and (ii) $d_E(\alpha, \beta), d_E(\beta, \gamma) < d_E(\alpha, \gamma)$; thus, our task reduces to (i) defining collinearity via the isometries, and (ii) ordering distances via the isometries. A concept we shall rely on for both these tasks is the **fix** - written $\text{Fix}(f)$ - of an isometry f ;

$$\text{Fix}(f) = \{\alpha \in \mathbb{R}^3 : f(\alpha) = \alpha\}.$$

First, we deal with collinearity. Observe that α, β , and γ are collinear - written $\overline{\alpha\beta\gamma}$ - if, and only if, each isometry fixing both α and β also fixes γ ; that is,

$$\overline{\alpha\beta\gamma} \Leftrightarrow \forall f \in I_{\mathbb{R}^3} (\alpha, \beta \in \text{Fix}(f) \Rightarrow \gamma \in \text{Fix}(f)).$$

Thus, we can indeed define collinearity via the isometries.

Next, we must order distances using just the isometries. Before we do that, we define - using just the isometries - when distances are the same;

$$d_E(\alpha, \beta) = d_E(\alpha', \beta') \Leftrightarrow \exists f \in I_{\mathbb{R}^3} (f(\alpha) = \alpha' \text{ and } f(\beta) = \beta').$$

Note that if such a map f exists, then all we know is that the distances $d_E(\alpha, \beta)$ and $d_E(\alpha', \beta')$ are the same; we do *not* know what this distance is. Conversely, if there is no map f in $I_{\mathbb{R}^3}$ such that $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$, then all we know is that the distances $d_E(\alpha, \beta)$ and $d_E(\alpha', \beta')$ are distinct; we know neither what these distances are, nor which of them is larger.

Next, given distinct points α and β , we define the **midpoint** - written $\alpha \cdot \beta$ - of α and β : it is the unique point such that (i) it is collinear with α and β , and (ii) there exists an isometry f such that $f(\alpha \cdot \beta) = \alpha \cdot \beta$, $f(\alpha) = \beta$, and $f(\beta) = \alpha$. As collinearity is definable via the isometries, so too are midpoints.

We need to be able to recognize a rotation around a line in $\mathbb{R}^3$. A map f in $I_{\mathbb{R}^3}$ is a rotation around a line if, and only if, there exist distinct points α and β in $\text{Fix}(f)$ and there exists a map $g(\neq f)$ in $I_{\mathbb{R}^3}$ such that $\text{Fix}(f) = \text{Fix}(g)$.

Now suppose that: (i) α, β, γ , and δ are distinct points in $\mathbb{R}^3$; (ii) the midpoints $\alpha \cdot \beta$ and $\gamma \cdot \delta$ are distinct; (iii) f is a rotation with the midpoints $\alpha \cdot \beta$ and $\gamma \cdot \delta$ in $\text{Fix}(f)$; (iv) $f(\alpha) = \beta, f(\beta) = \alpha, f(\gamma) = \delta$, and $f(\delta) = \gamma$. Note that all of this is definable using just the isometries. When $\alpha, \beta, \gamma, \delta$, and f are arranged in this manner, we can order the distances $d_E(\alpha, \beta)$ and $d_E(\gamma, \delta)$. To see this, observe that both the line segments $\overline{\alpha\beta}$ and $\overline{\gamma\delta}$ are orthogonal to the unique line $\text{Fix}(f)$ containing both the midpoints $\alpha \cdot \beta$ and $\gamma \cdot \delta$. Thus, it follows that $d_E(\alpha, \beta) < d_E(\gamma, \delta)$ holds if, and only if, there is no point ϵ in $\text{Fix}(f)$ such that $d_E(\delta, \epsilon) = d_E(\alpha, \alpha \cdot \beta)$. All of this definable using just the isometries; hence, we can order these distances when the points in question are arranged in this way. Note that all possible distance pairs can be realized by points arranged in this way. Thus, if we are given four points not so well arranged, we can ask if the distances match those of well-positioned points, and we can do this using just isometries. Hence, we can order all the distances.

Given the preceding developments, we can say that three distinct points α, β , and γ are collinear and that $d_E(\alpha, \beta), d_E(\beta, \gamma) < d_E(\alpha, \gamma)$; that is, we can say when $B^*(\alpha, \beta, \gamma)$ holds. Thus, we can define convexity in $\mathbb{R}^3$ using just the isometries. To obtain convexity in $\mathbb{Q}^3$, we simply help ourselves to the restriction $B_{\mathbb{Q}}^*$ of B^* to rational 3-space; that is,

$$B_{\mathbb{Q}}^* = \{(\alpha, \beta, \gamma) \in B^*: \alpha, \beta, \gamma \in \mathbb{Q}^3\}.$$

As one might now expect, a subset X of $\mathbb{Q}^3$ is **convex** if it is closed under $B_{\mathbb{Q}}^*$.

Thus, the proof in A3 obtained rational addition via an algebraic definition of convexity, but as we have shown here, convexity can be defined purely in terms of the

isometries of the space and is therefore an inherently geometric concept. This is a particular instance of a more general mathematical phenomenon whereby algebraic language is used to describe inherently non-algebraic ideas (usually for purposes of convenience). In such situations, it is easy to fall into the trap of thinking that the underlying concepts are algebraic when they are not. As isometries are instances of isomorphisms, we can say that our use of convexity relies on isomorphism.

As promised earlier, we now address the concerns about using algebraic machinery to define a metric, then discarding the metric and keeping the automorphism group. The Euclidean metric d_E - as defined above - appears to rely on algebraic machinery. We say ‘appears’, because the same metric can be obtained in an algebra-free manner. Imagine real 3 –space co-ordinatized in the usual manner - that is, by three (mutually orthogonal) copies of $\mathbb{R}$. Now take a copy of the non-negative real line and use it as a ruler with which to measure the straight-line distance between points. The resulting metric is the Euclidean metric d_E , and no algebraic machinery is required.

The above algebra-free definition of d_E prompts the question of why we use a definition that incorporates algebra when the underlying structure is not inherently algebraic. The reason is that an algebraic definition makes it easier to construct proofs; indeed, this is why we used algebraic language in the proof of Lemma 7.

A4 Real Addition Via Convexity, Monotonicity, & Continuity.

In this section, we obtain the real addition operation $f_{\mathbb{R}}^+$ from the rational addition operation $f_{\mathbb{Q}}^+$ via convexity in two distinct ways; minimizing against a partial order, and by a

unique extension. We also obtain $f_{\mathbb{R}}^+$ from $f_{\mathbb{Q}}^+$ via unique monotonic and unique continuous extensions.

With respect to convexity, obtaining $f_{\mathbb{R}}^+$ from $f_{\mathbb{Q}}^+$ is almost identical to the situation in the previous section. We embed the set

$$f_{\mathbb{Q}}^+ = \{(q, p, f_{\mathbb{Q}}^+(q, p)) : q, p \in \mathbb{Q}\} = \{(q, p, q + p) : q, p \in \mathbb{Q}\}$$

in $\mathbb{R}^3$ and observe that the points in $f_{\mathbb{Q}}^+$ all lie in the same plane. Then we ‘fill in the gaps’.

There are some minor details that differ from the previous section. Given two points $r = (x, y, z)$ and $r' = (x', y', z')$ in $\mathbb{R}^3$, the *real* straight line segment between them, denoted $\ell_{\mathbb{R}}(r, r')$, is the set

$$\ell_{\mathbb{R}}(r, r') = \{(1 - t)(x, y, z) + t(x', y', z') : t \in \mathbb{I}\}.$$

A subset X of $\mathbb{R}^3$ is **convex** (in $\mathbb{R}^3$) if, for each pair of points r and r' in X , the set $\ell_{\mathbb{R}}(r, r')$ is contained in X . Given a subset X of $\mathbb{R}^3$, we let $\mathcal{C}_{\mathbb{R}}(X)$ denote the set of all convex subsets of $\mathbb{R}^3$ that contain X ; that is,

$$\mathcal{C}_{\mathbb{R}}(X) = \{Y \subseteq \mathbb{R}^3 : X \subseteq Y \text{ and } Y \text{ is convex}\}.$$

Again, we can partially order $\mathcal{C}_{\mathbb{R}}(X)$ by the set inclusion relation $\subseteq$. We then have the real analogue of Lemma 6.

Lemma 9. *The poset $(\mathcal{C}_{\mathbb{R}}(f_{\mathbb{Q}}^+), \subseteq)$ has a minimum element.*

Proof. Consider the set $\cap \mathcal{C}_{\mathbb{R}}(f_{\mathbb{Q}}^+)$; from here, the proof is structurally identical to that of Lemma 6. $\square$

For a subset X of $\mathbb{R}^3$, the set $\cap \mathcal{C}_{\mathbb{R}}(f_{\mathbb{Q}}^+)$ is the (real) **convex hull** of X .

Lemma 10. *The convex hull of $f_{\mathbb{Q}}^+$ in $\mathbb{R}^3$ is $f_{\mathbb{R}}^+$; that is, $\cap \mathcal{C}_{\mathbb{R}}(f_{\mathbb{Q}}^+) = f_{\mathbb{R}}^+$.*

Proof. The proof of this is structurally similar to that of Lemma 7. First, show that if a point $r = (x', y', z')$ in $\mathbb{R}^3$ is such that $r \neq f_{\mathbb{R}}^+$, then there exists a set X in $\mathcal{C}_{\mathbb{R}}(f_{\mathbb{R}}^+)$ with $r \notin X$; this can be done as it was in the proof of Lemma 7. Second, show that each point of the form $r = (x, y, x + y)$ in $\mathbb{R}^3$ is in every set X in $\mathcal{C}_{\mathbb{R}}(f_{\mathbb{R}}^+)$; this requires a different approach to that employed in the proof of Lemma 7, and we leave the details to the reader. $\square$

Thus, by minimizing against convexity, we arrive at real addition. As was the case when moving from natural to rational addition, we can get from rational to real addition via a convex extension. A function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is convex (in $\mathbb{R}^3$) if the set $\{(x, y, f(x, y)): x, y \in \mathbb{R}\}$ is convex in $\mathbb{R}^3$.

Lemma 11. *The function $f_{\mathbb{R}}^+$ is the unique convex real extension of $f_{\mathbb{Q}}^+$.*

Proof. The proof is structurally identical to that of Lemma 8. $\square$

Thus we arrive at real addition via a unique convex extension. It is worth noting that real addition can be obtained directly from natural addition via convexity in both ways; that is, the (real) convex hull of $f_{\mathbb{N}}^+$ is $f_{\mathbb{R}}^+$, and $f_{\mathbb{R}}^+$ is the *unique* convex real extension of $f_{\mathbb{N}}^+$. That concludes the use of convexity. We now return to monotonicity.

Note that $f_{\mathbb{Q}}^+$ is monotonic; that is, $p < q$ implies $k + p < k + q$ and $p + k < q + k$ for all rationals p, q and k .

Lemma 12. *The function $f_{\mathbb{R}}^+$ is the unique monotonic real extension of $f_{\mathbb{Q}}^+$.*

Proof. First, note that $f_{\mathbb{R}}^+$ is a monotonic real extension of $f_{\mathbb{Q}}^+$. To establish uniqueness, suppose that f is a monotonic real extension of $f_{\mathbb{Q}}^+$. Let q and ι be (respectively) a rational

and an irrational number, and suppose that $f(q, \iota) \neq q + \iota$. Preserving generality, we may assume that $f(q, \iota) = q + \iota + \epsilon$, where $\epsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational q' such that $\iota < q' < \iota + \epsilon$. As f is an extension of $f_{\mathbb{Q}}^+$ and (q, q') is a rational pair, we have that $f(q, q') = f_{\mathbb{Q}}^+(q, q') = q + q' < q + (\iota + \epsilon) = f(q, \iota)$. As $\iota < q'$ and $f(q, q') < f(q, \iota)$, f is not monotonic, which contradicts our hypotheses on f . Hence, $f(q, \iota) = q + \iota$ for each rational q and each irrational ι .

Now let ι and r be (respectively) an irrational and a real number (r can be either rational or irrational). Furthermore, suppose that $f(\iota, r) \neq \iota + r$. Preserving generality, we may assume that $f(\iota, r) = \iota + r - \epsilon$, where $\epsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational q such that $\iota - \epsilon < q < \iota$. Note that $f(q, r) = q + r$ - if r is rational, this follows from f being an extension of $f_{\mathbb{Q}}^+$, if r is irrational, this follows from the argument above. Hence, we have that $f(q, r) = q + r > (\iota - \epsilon) + r = f(\iota, r)$. As $q < \iota$ and $f(q, r) > f(\iota, r)$, f is not monotonic, which contradicts our hypotheses on f . Hence, $f(\iota, r) = \iota + r$ for each irrational ι and each real r . Now it follows that $f(r, r') = r + r'$ for each real pair (r, r') - that is, $f = f_{\mathbb{R}}^+$. $\square$

Finally, we obtain real addition from rational addition via continuity.

Lemma 13. *The function $f_{\mathbb{R}}^+$ is the unique continuous real extension of $f_{\mathbb{Q}}^+$.*

Proof. First, note that $f_{\mathbb{R}}^+$ is a continuous real extension of $f_{\mathbb{Q}}^+$. Uniqueness follows from the fact that a map $f: \mathbb{Q}^2 \rightarrow \mathbb{R}$ has *at most one* continuous real extension $\hat{f}: \mathbb{R}^2 \rightarrow \mathbb{R}$. $\square$

Thus, we may obtain real addition from rational addition by exploiting any one of convexity, monotonicity, or continuity. Moreover, each of these options uniquely determine the result.

A5 Real Multiplication Via Isomorphism

In this section, we show that addition on non-negative reals is ‘the same as’ or isomorphic to multiplication on the reals.

For our purposes, a **structure** is a pair $(X, \circ_X)$, where X is a set and $\circ_X$ is a binary operation on X ; that is, a map $\circ_X: X^2 \rightarrow X$. Given two structures $(X, \circ_X)$ and $(Y, \circ_Y)$, a map $f: X \rightarrow Y$ is **operation-preserving** if, for each x and x' in X , we have that

$$f(x \circ_X x') = f(x) \circ_Y f(x').$$

The map $f: X \rightarrow Y$ is an **isomorphism** if f is bijective and operation-preserving. The structures $(X, \circ_X)$ and $(Y, \circ_Y)$ are **isomorphic**, written $X \cong Y$, if there exists an isomorphism $f: X \rightarrow Y$. Recall that $\mathbb{R}$ is the set of non-negative reals; we now just write $+$ for addition on $\mathbb{R}$. We also let $\mathbb{R}_{\geq 1}$ denote the set of those reals greater than or equal to one, and multiplication on this set is written $\times$.

Lemma 14. *The structures $(\mathbb{R}, +)$ and $(\mathbb{R}_{\geq 1}, \times)$ are isomorphic.*

Proof. First, we define a map $f: \mathbb{R} \rightarrow \mathbb{R}_{\geq 1}$ by setting $f(r) = 2^r$ for each non-negative real r . Note that $2^r \geq 1$ whenever $r \geq 0$. To see that f is injective, suppose that $f(r) = f(r')$ - that is, $2^r = 2^{r'}$ for non-negative r and r' . Then we have that $2^{r-r'} = \frac{2^r}{2^{r'}} = 1$, from which it follows that $r - r' = 0$ and so $r = r'$. To establish that f is surjective, we note that f is continuous, unbounded, and $2^0 = 1$. Surjectivity now follows from the intermediate value theorem. Hence, f is bijective. To see that f is operation-preserving, note that for non-negative reals r and r' , we have that $f(r + r') = 2^{r+r'} = 2^r \times 2^{r'} = f(r) \times f(r')$. Hence, f is an isomorphism and we may write $(\mathbb{R}, +) \cong (\mathbb{R}_{\geq 1}, \times)$. $\square$

Thus, we obtain real multiplication from real addition via a structural isomorphism.

Of course, there was nothing special about the base of 2 in the map f above; any map of the form $r \mapsto \alpha^r$ (with $\alpha > 1$) will also be an isomorphism. What is interesting, and less obvious, is the fact that all isomorphisms from $(\mathbb{R}, +)$ to $(\mathbb{R}_{\geq 1}, \times)$ are of the form $r \mapsto \alpha^r$ for some $\alpha > 1$. Note that as addition and multiplication are both associative, an isomorphism f also satisfies a more general version of operation preservation; namely,

$$f\left(\sum_{i=1}^n x_i\right) = \prod_{i=1}^n f(x_i),$$

for each natural $n \geq 2$ and each $(x_1, \dots, x_n)$ in $\mathbb{R}^n$.

Lemma 15. *Each isomorphism from $(\mathbb{R}_{\geq 0}, +)$ to $(\mathbb{R}_{\geq 1}, \times)$ is a map of the form $r \mapsto \alpha^r$, where $\alpha > 1$.*

Proof. Suppose that f is an isomorphism from $(\mathbb{R}_{\geq 0}, +)$ to $(\mathbb{R}_{\geq 1}, \times)$ and consider $f(1) = \alpha$. To see that $\alpha > 1$, note that $f(0) = f(0 + 0) = f(0) \times f(0)$, so $f(0) = 1$, and as f is injective we must have that $f(1) = \alpha > 1$. Now consider $f\left(\frac{1}{n}\right)$, where $n \geq 2$. By (general) operation preservation, we have that

$$\alpha = f(1) = f\left(\sum_{i=1}^n \frac{1}{n}\right) = \prod_{i=1}^n f\left(\frac{1}{n}\right) = \left(f\left(\frac{1}{n}\right)\right)^n,$$

and so $f\left(\frac{1}{n}\right) = \alpha^{\frac{1}{n}}$. From this, we obtain that

$$f\left(\frac{m}{n}\right) = f\left(\sum_{i=1}^m \frac{1}{n}\right) = \prod_{i=1}^m f\left(\frac{1}{n}\right) = \left(f\left(\frac{1}{n}\right)\right)^m = \left(\alpha^{\frac{1}{n}}\right)^m = \alpha^{\frac{m}{n}},$$

for any rational $\frac{m}{n}$. Finally, suppose that ι is a positive irrational and that $f(\iota) \neq \alpha^\iota$.

Preserving generality, we may assume that $\alpha^\iota < f(\iota)$. The map $r \mapsto \alpha^r$ is continuous and

unbounded, so by the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists a rational q such that $\alpha^\iota < \alpha^q < f(\iota)$.

Note that $\iota < q$, because $\alpha > 1$, and therefore $0 < q - \iota$. Hence, $f(q - \iota)$ is defined. Now observe that

$$\begin{aligned}\alpha^q &= f(q) \\ &= f(\iota + (q - \iota)) \\ &= f(\iota) \times f(q - \iota),\end{aligned}$$

and so $f(q - \iota) = \frac{\alpha^q}{f(\iota)} < 1$, as $1 < \alpha^q < f(\iota)$. But this contradicts the fact that $f(r) \geq 1$ for each $r \geq 0$. Hence, $f(\iota) = \alpha^\iota$ for each positive irrational ι . That covers the entire domain of f and we conclude that $f(r) = \alpha^r = (f(1))^r$ for each real α . $\square$

Appendix B

In A2.1 we introduced the concept of an $\mathbb{N}$ – chain; that is, a linear order with successors, a minimum element, and finite principal downsets. As we alluded to, $\mathbb{N}$ – chains are categorical, by which we mean that all $\mathbb{N}$ – chains are isomorphic – we now prove this. The first task is to show that every $\mathbb{N}$ – chain is countable; that is, the underlying set can be put into bijective correspondence with the set $\mathbb{N}$ of natural numbers. Before we start, we require one more definition. Given an $\mathbb{N}$ – chain $(X, \leq)$, we define a map $\ell: X \rightarrow \mathbb{N}$ by setting

$$\ell(x) = |\downarrow x| - 1$$

for each element x of X . Recall that $\downarrow x = \{z \in X: z \leq x\}$, and this is a finite set because $(X, \leq)$ is an $\mathbb{N}$ – chain and all its principal downsets are finite. Intuitively, $\ell(x)$ is the $\leq$ distance from the minimum element in $(X, \leq)$ to x .

Lemma B1. *Given an $\mathbb{N}$ – chain $(X, \leq)$, the map $\ell: X \rightarrow \mathbb{N}$ is a bijection; that is, if x and y are distinct elements of X , then $\ell(x) \neq \ell(y)$, and for each natural number n there exists an x in X with $\ell(x) = n$.*

Proof. First, suppose that x and y are distinct elements of X ; we require that $\ell(x) \neq \ell(y)$. As $x \neq y$, we have (by comparability) that precisely one of $x < y$ or $y < x$ holds; preserving generality, we may assume that $x < y$ holds. For each z in $\downarrow x$, we have (by definition) $z \leq x$. As $x < y$ holds, we obtain (by transitivity) $z \leq y$, and therefore z is in $\downarrow y$; thus, the downset $\downarrow x$ is a subset of the downset $\downarrow y$. As $x < y$ holds, we have that $y \in \downarrow y$ and $y \notin \downarrow x$; thus, the downset $\downarrow x$ is a *proper* subset of the downset $\downarrow y$. As the downsets $\downarrow x$ and $\downarrow y$ are

finite sets, and $\downarrow x \subset \downarrow y$ holds, we have $|\downarrow x| < |\downarrow y|$ and therefore $\ell(x) = |\downarrow x| - 1 < |\downarrow y| - 1 = \ell(y)$. As x and y were arbitrary, we have that ℓ is injective.

To establish the surjectivity of ℓ we shall proceed via induction. First, letting x_0 denote the minimum element of $(X, \leq)$, we have $\downarrow x_0 = \{x \in X: x \leq x_0\} = \{x_0\}$, and so

$$\ell(x_0) = |\downarrow x_0| - 1 = |\{x_0\}| - 1 = 1 - 1 = 0.$$

Now let n be an arbitrary positive integer, and suppose that there is an element x of X such that $\ell(x) = |\downarrow x| - 1 = n - 1$. As $(X, \leq)$ is an $\mathbb{N}$ -chain, x has a (unique) successor x' . As $x < x'$ holds, we have that $\downarrow x$ is a proper subset of $\downarrow x'$;⁴ hence, $\ell(x) + 1 \leq \ell(x')$. We claim that $\downarrow x \setminus \downarrow x' = \{x'\}$. Note that $x' \in \downarrow x'$, but $x' \notin \downarrow x$ (because $x < x'$); hence, $x' \in \downarrow x' \setminus \downarrow x$. Now consider any element $z (\neq x')$ of X . As x' is the successor of x , the two cases to consider are (i) $z \leq x$ and (ii) $x' < z$. If $z \leq x$ holds, then $z \in \downarrow x$ and $z \in \downarrow x'$, and therefore $z \notin \downarrow x' \setminus \downarrow x$. If $x' < z$ holds, then $z \notin \downarrow x'$, and therefore $z \notin \downarrow x' \setminus \downarrow x$. In each case we have $z \notin \downarrow x' \setminus \downarrow x$, and therefore $\downarrow x' \setminus \downarrow x = \{x'\}$; from this, it follows that $|\downarrow x| = |\downarrow x'| - 1$, and therefore $n = \ell(x) + 1 = |\downarrow x| = |\downarrow x'| - 1 = \ell(x')$. By induction, we have that for each natural number n , there is an element x of X such that $\ell(x) = n$; thus, ℓ is surjective, and combined with the injectivity, we obtain bijectivity of ℓ . $\square$

Theorem B1. *Every $\mathbb{N}$ -chain is isomorphic to $(\mathbb{N}, \leq)$.*

Proof. Let $(X, \leq_X)$ be an arbitrary $\mathbb{N}$ -chain, and let x_0 be the minimum element of $(X, \leq_X)$. We require a bijective monotonic map from X to $\mathbb{N}$; as luck would have it, we are already in possession of such a map. Consider the map $\ell: X \rightarrow \mathbb{N}$ given by setting $\ell(x) = |\downarrow x| - 1$ for each element x of X . By the previous result, ℓ is bijective.

⁴ The argument used in the injectivity part of the proof works here.

To see that ℓ is monotonic, suppose that $y \leq_x z$ holds in $(X, \leq_x)$; we require that $\ell(y) \leq \ell(z)$ holds in $(\mathbb{N}, \leq)$. As $y \leq_x z$ holds, we have that one of $y <_x z$ or $y = z$ holds. If $y <_x z$ holds, then $\downarrow y$ is a proper subset of $\downarrow z$, and as these are finite sets, we have that $\ell(y) = |\downarrow y| - 1 < |\downarrow z| - 1 = \ell(z)$; thus $\ell(y) \leq \ell(z)$ holds. If $y = z$ holds, then we have $\ell(y) = \ell(z)$; thus, $\ell(y) \leq \ell(z)$ holds. In either case, we have $\ell(y) \leq \ell(z)$, and ℓ is monotonic. As ℓ is both bijective and monotonic, it follows that ℓ is an order isomorphism. $\square$

Our categorical description of $\mathbb{N}$ – chains is written in a second-order language; the offending article is the requirement that all principal downsets are finite. When we say ‘a principal downset is finite’ we are saying ‘its size is either 0 or 1 or 2 or 3 or 4...’ and this expression is infinite in length, not something that is permitted in a first-order language.

We are encroaching on logical and model-theoretic ground, so we may as well outline the logical relationships between the extra requirements - beyond being a linear order - that an $\mathbb{N}$ – chain must satisfy.

First, we shall show that the requirement that there be a minimum element is actually implied by the other requirements.

Theorem B2. *If $(X, \leq)$ is a linear order in which each principal downset is finite, then $(X, \leq)$ has a minimum element.*

Proof. Suppose that $(X, \leq)$ is a linear order in which each principal downset is finite. Choose an element x of X (it does not matter which element we choose), and consider the principal downset $\downarrow x = \{y \in X : y \leq x\}$ generated by x . By hypothesis, $\downarrow x$ has n elements for some positive integer n . The set $\downarrow x$ inherits the order $\leq$ from $(X, \leq)$, so we can linearly order $\downarrow x$ as follows:

$$x_0 < x_1 < \cdots < x_{n-1} (= x).$$

Note that $\downarrow x$ has a minimum element, which we denote by x_0 ; we claim that x_0 is also the minimum element for $(X, \leq)$ - that is, $x_0 \leq y$ holds for each y in X . To see that this is the case, observe that for an arbitrary element y of X we have (by comparability) that precisely one of the following holds (i) $x_0 = y$, (ii) $x_0 < y$, (iii) $y < x_0$. In case (i), we have $x_0 = y$ and therefore $x_0 \leq y$ holds. In case (ii), we have $x_0 < y$ and therefore $x_0 \leq y$ holds. In case (iii), we have $y < x_0$ and therefore $y \leq x_0 \leq x$ holds, and by transitivity we obtain $y \leq x$ and therefore $y \in \downarrow x$, and this contradicts the fact that x_0 is the minimum element of $\downarrow x$. Thus, we must be in case (i) or (ii) and $x_0 \leq y$ holds; from this it follows that x_0 is the minimum element of $(X, \leq)$. $\square$

Next, we shall relax the requirement that all elements have a successor; what we are left with is a linear order in which each principal downset is finite, and therefore (due to the previous result) there is a minimum element. Note that every $\mathbb{N}$ - chain satisfies these requirements, but so too does any finite linear order, which we can obtain by taking a downset $\downarrow x$ generated by any element x in an $\mathbb{N}$ - chain. The reader is encouraged to convince themselves that such a downset - that is, a finite linear order - is not an $\mathbb{N}$ - chain.

Finally, we shall relax the requirement that each principal downset is finite; what we are left with is a linear order in which (i) there is a minimum element and (ii) each element has a successor. Note that each $\mathbb{N}$ - chain satisfies these requirements, but unlike the previous scenario, we cannot just pass to a substructure of an $\mathbb{N}$ - chain to obtain something that both satisfies our desired requirements and is not itself an $\mathbb{N}$ - chain. Instead, we shall take an existing $\mathbb{N}$ - chain and add something extra.

Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be $\mathbb{N}$ - chains, with the underlying sets X_1 and X_2 disjoint; that is, $X_1 \cap X_2 = \emptyset$. From $(X_1, \leq_1)$ and $(X_2, \leq_2)$ we wish to define a new linear order; the

underlying set will be the union $X_1 \cup X_2$ of X_1 and X_2 and the order $\leq$ on $X_1 \cup X_2$ is defined by setting

$$x \leq x' \Leftrightarrow \begin{cases} x, x' \in X_1 & \text{and } x \leq_1 x', \text{ or} \\ x, x' \in X_2 & \text{and } x \leq_2 x', \text{ or} \\ x \in X_1 & \text{and } x' \in X_2. \end{cases}$$

Thus, if x and x' are both from X_1 , we order them using $\leq_1$; if x and x' are both from X_2 , we order them using $\leq_2$; and everything in X_1 is less than everything in X_2 . The minimum element of $(X_1 \cup X_2, \leq)$ is the minimum element of $(X_1, \leq_1)$. What we have done is essentially take $(X_1, \leq_1)$ and put $(X_2, \leq_2)$ on the end of it to obtain the linear order $(X_1 \cup X_2, \leq)$. The reader is welcome to verify the claim that $(X_1 \cup X_2, \leq)$ is a linear order. Note that each element x in $X_1 \cup X_2$ has a $\leq$ successor: if x is in X_1 , then the $\leq$ successor of x in $(X_1 \cup X_2, \leq)$ is the $\leq_1$ successor of x in $(X_1, \leq_1)$; similarly, if x is in X_2 , then the $\leq$ successor of x in $(X_1 \cup X_2, \leq)$ is the $\leq_2$ successor of x in $(X_2, \leq_2)$. Hence, $(X_1 \cup X_2, \leq)$ satisfies our desired requirements, now we just need to show that it is not an $\mathbb{N}$ – chain. To see that $(X_1 \cup X_2, \leq)$ is not an $\mathbb{N}$ – chain, take an element x_2 from X_2 (it does not matter which element is chosen) and consider the $\leq$ downset $\downarrow x_2$. The $\leq$ downset $\downarrow x_2$ is the subset of $X_1 \cup X_2$ consisting of those elements x such that $x \leq x_2$. As x_2 is in X_2 , it follows from the definition of $\leq$ that

$$x \leq x_2 \Leftrightarrow \left(x \in X_1 \text{ or } \left(x \in X_2 \text{ and } x \leq_2 x_2 \right) \right).$$

In other words, the $\leq$ downset $\downarrow x_2$ is the union of the $\leq_2$ downset $\downarrow_2 x_2$ and the set X_1 ; that is,

$$\downarrow x_2 = \left(\downarrow_2 x_2 \right) \cup X_1.$$

As $(X_1, \leq_1)$ is an $\mathbb{N}$ – chain, it follows from Lemma B1 that the set X_1 is countably infinite; hence, the $\leq$ downset $\downarrow x_2$ is also infinite. Thus, the linear order $(X_1 \cup X_2, \leq)$ contains principal downsets that are not finite, and therefore $(X_1 \cup X_2, \leq)$ is not an $\mathbb{N}$ – chain.

As noted above, we can omit the requirement that $\mathbb{N}$ – chains have a minimum element, as this is implied by the principal downsets being finite. As a final thought, we note that ordinal successors in the $\mathbb{N}$ – chain definition can be replaced by the requirement that there are no maximal elements; in simple terms, an $\mathbb{N}$ – chain is a linear order in which every element has something above it, and finitely many things below it.