Omitted Response Treatment Using a Modified Laplace Smoothing for Approximate Bayesian Inference in Item Response Theory

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Abstract

This article applies the approach of adding artificially created data to observations to stabilize estimates to the problem of treating missing responses for cases in which students choose to omit answers to questionnaire items or achievement test items.

This addition of manufactured data is known in the literature as Laplace smoothing or the method of data augmentation priors. It can be understood as a penalty added to a parameter's likelihood function. This approach is used to stabilize results in the National Assessment of Educational Progress (NAEP) analysis and implemented in the MGROUP software program that plays an important role in generating results files for NAEP.

The modified data augmentation approach presented here aims to replace common missing data treatments used in IRT so it can be understood as special deterministic cases of data augmentation priors that add fixed information to the observed data, either by conceptualizing these as adding a fixed form to the likelihood function to constant represent prior information or by understanding the augmentation as a conjugate prior that 'emulates' non-random observations.

Keywords: TIMSS, PIRLS, NAEP, Data Augmentation, Latent Regression, IRT

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Introduction

Not responding to a question on a test can have many reasons. Students may be unsure or not know the response, or they may plan to go back later and answer, or they ran out of time or felt that the question had more than one correct answer. Not all non-response is directly related to a lack of understanding or not knowing the answer. It can be shown that treating missing responses as incorrect produces a bias in ability estimates of students who do not answer all questions. Moustaki & Knott (2000), Glas & Pimentel (2008), Rose, von Davier, & Xu (2010), Rose, von Davier, & Nagengast (2017) and others describe model-based approaches to handle non-response properly by introducing a latent dimension besides the ability variable(s) that quantifies the tendency to omit responses. Ulitzsch, von Davier, & Pohl (2020) present a model that shows how time spent on the question, non-response, and guessing can be reconceptualized as response processes and measured alongside the ability that is the target of inferences. These authors showed how item-level non-response, random guessing, and fast, effortless responding are related phenomena that can be jointly modeled as a function of a latent variable that quantifies cognitive engagement with the assessment material.

These approaches are particularly useful in low-stakes assessments, where students are asked to take an assessment without being offered individual feedback. At the same time, the results are used to measure the skill distributions on the educational system level and to study differences between policy-relevant subpopulations (e.g., von Davier et al., 2006; Rutkowski et al. , 2013). In this domain, the treatment of non-response as incorrect is slowly replaced by more appropriate approaches (reference PISA 2015 report, PIAAC report, NAEP, Lord 1980).

In related work, von Davier et al., (2019) showed how an ill-advised but very common missing data treatment in widely used intelligence tests introduces a bias in ability estimation: Discontinue rules are applied when developmental IQ tests are used, which present items in the order of increasing difficulty. A common approach is to stop testing and treat all subsequent responses as incorrect that follow a string of 3, 4, or maybe 5 observed incorrect answers. von Davier et al., (2019) showed that this introduces bias and local dependencies among items treated this way, thus jeopardizing measurement quality. This can be avoided, as missingness is ignorable because the observed data completely determines the missing data mechanism.

Nevertheless, practitioners in measurement often hold on to misguided and often statistically inferior practices, as many of the more appropriate methods involve complex latent variable models that appear too complex to implement into practice. As stated by Plank (1950):

"A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up familiar with it."

This depressing outlook is often shortened by others citing Plank (1950) as: "Science progresses one funeral at a time." The physicist credited with inventing (discovering?)

quantum mechanics did not think older scientists could learn new things. Interestingly, there is empirical evidence for this view (Azoulay, 2019).

The same, unfortunately, holds for the treatment of omitted responses observed when students take a test and do not answer all questions. While it is well established that treating non-response as incorrect introduces bias in the estimates of achievement, most practitioners use plausibility and familiarity arguments rather than statistical and logical derivations to hold on to familiar rather than optimal practices.

Therefore, the present study frames the treatment of non-response in the larger context of approximate Bayesian estimation (Greene, 2007) or data augmentation priors. This approach is well documented in other applications such as logistic regression, and is also known under the label Laplacian Smoothing in NLP (Manning et al. 2008), and used (but not well documented in NAEP's MGROUP (e.g., Mislevy et al. 1992; Thomas, 1993; von Davier & Sinharay, 2007) software. We will discuss and describe the MGROUP approach to fill that gap and then move to present a simple and easily implemented treatment of omitted responses embedded in the data augmentation framework. This approach is simple to implement and accounts for item-level non-response of test takers. This article presents a simple alternative that can be implemented without complex modeling efforts and relies solely on rescoring omitted response indicators and incorporating these into the measures of student ability.

Laplace Smoothing as the Original Data Augmentation

A method known as Laplace smoothing is a commonly applied technique to produce smoothed pseudo-counts for naive Bayesian classifiers used in statistical natural language processing (Manning et al., 2008).

Assume a categorical random variable X with a multinomial probability distribution π, with image $X \in \{0, ..., M\}$ and probabilities $\pi = \left(\pi_1, ..., \pi_M\right)$ and that only a small number of observations $x_1, ..., x_I$ with $I < k \times M$, for some (small) constant k, are available to estimate the probabilities π_c. In these cases, it is not uncommon to observe categories i with $x_i = 0$ (empty cells) which would lead to vanishing estimates of $\hat{\pi}_i = \frac{x_i}{\sum_j x_j} = 0$. In order to avoid such boundary values, Laplace smoothing is used. Essentially, all observed counts are augmented by adding small positive constants $x_i^* = x_i + a_i$ to each observed count and to estimate $\hat{\pi}_i^* = \frac{x_i^*}{\sum_j x_j^*}$ instead of the naive estimate without smoothing away the empty cells.

In the most simple case of a binary variable, this Laplace Smoothing works as follows: Let f_0, f_1, with $I = f_0 + f_1$ denote the observed frequencies of the binary outcomes $X \in \{0, 1\}$. Then estimate π by

$$\frac{\hat{\pi}_0}{1-\hat{\pi}_0} = \frac{\hat{\pi}_0}{\hat{\pi}_1} = \frac{f_0+b}{f_1+a}$$

or equivalently

$$\hat{\pi}_1 = \frac{f_1+b}{f_1+b+f_0+a}$$

for some constants $a > 0,\ b > 0$. These *smoothing constants* are added to all observed frequencies and act as if they provided additional sample size. It is equivalent to drawing an additional 'ideal' sample of size $N_{ab} = a + b$ with $\pi_0 = \frac{a}{a+b}$ and adding these cases to the observed data. In the case above, the relation to the beta distribution and conjugate priors is obvious. Many authors have pointed it out in introductory texts on Bayesian statistics (e.g. Iversen, 1984).

Laplace smoothing is a commonly used technique. Clogg et al. (1991), for example, used this approach at a very large scale for imputations in sparse logistic regressions. Greenland 2007, and Greenland & Christensen 2001 and Discacciati et al. (2015) describe data augmentation in logistic regression as approximate Bayesian analysis via penalized likelihood. The next section shows how a similar data augmentation is used in latent regression IRT models (Mislevy et al. 1992; Thomas, 1993; von Davier & Sinharay, 2007) as implemented in operational analyses in NAEP, TIMSS, PIRLS, and other assessments such as PISA.

Data Augmentation in NAEP's Plausible Values Software MGROUP

In combination with the above, additional data augmentation is being used on a large scale in the assessment industry. An implementation of a data augmentation prior is being used in estimating achievement distributions in the National Assessment of Educational Progress (NAEP). This is not well documented, as it is only available in the software used to report NAEP data. Thomas (1993) made the source code of the NAEP software MGROUP available through the Journal of Computational and Graphical Statistics (http://jcgs.stat.rice.edu/) website. The software was made available on the journals' website and distributed as F77 source code compressed with the *shar* Unix archiver v.122, and contains the following statement by the author of the article and ETS:

The MGROUP program computes parameter estimates for a multivariate multiple regression model where the "Y" variables are subject to measurement error represented by item response models generated by the PARSCALE program. The MGROUP program also produces multiple imputations for subsequent analyses. The files to run a small simulated data set are included. The program is used extensively by the National Assessment of Educational Progress (NAEP) and the Adult Literacy Survey.

The program is difficult to use and is being updated frequently. Anyone wanting to run the program is encouraged to contact the NAEP group at Educational Testing Service for documentation and the most recent version of the program.

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MGROUP was developed by Educational Testing Service.

MGROUP software is provided "AS IS" - the entire risk as to the results and performance of the program is assumed by the user. ETS does not warrant, guarantee or make representations regarding the use of, or the results obtained with the program in terms of correctness and reliability or legality. In no event shall ETS be liable for any loss of profit or damages.

NAEP uses the MGROUP software for latent regression IRT models (Sheehan, 1985; Mislevy et al., 1992; Thomas, 1993; von Davier & Sinharay, 2007). ETS also produces a graphical user interface PC version of the MGROUP software that can be obtained upon request and is commonly referred to as DGROUP (Rogers et al. 2006). MGROUP was extended to handle multiple variants of the e-STEP (von Davier & Sinharay, 2007; von Davier & Yu, 2003). The algorithm implemented in MGROUP involves an evaluation of the log-likelihood of all respondents' observed responses following the 3PL or other IRT models and determines the likelihood function for each test taker. In this calculation of the likelihood function, a modification is introduced that can be described as a variant of Laplace smoothing. A provision is made to prevent monotone likelihood functions (e.g., Heinze and Schemper, 2002), which would lead to non-existing (infinite) estimates when maximizing such a function. While IRT models may use Firth-Warm-like estimators can be used to ensure finite estimates (Firth, 1993; Warm, 1989; Magis & Verhelst, 2016) the MGROUP approach chooses to add two constant item responses (and associated Rasch item functions) that are the same for all test takers. More specifically, one very easy item is added and assigned a correct response for all test takers, as well as one very difficult item which is assigned an incorrect response for all test takers.

This *augments* the data by two items and establishes a *data augmentation prior* aiming at modifying or penalizing the estimation of latent regressions in the NAEP MGROUP approach. Unfortunately, to the author’s knowledge, the use of this data augmentation approach is not described in the NAEP technical documentation. It was available for inspection - in a rather indirect way - through the work of Thomas (1993), which included the source code of an early MGROUP version as supplemental material to the 1993 article.

Equivalent Python code to the original NAEP data augmentation Fortran 77 code (found in the archive's source file SETQPT.f) is given below:

```
import numpy as np

qpmin = -5.0

qpmax =  5.0

step  =  0.1

qpts = list(np.arange(qpmin,qpmax+step,step))

p = [  1/(1+np.exp(1.7*(qpmin-x))) for x in qpts]

q = [1-1/(1+np.exp(1.7*(qpmax-x))) for x in qpts]

plki = np.exp(np.log(p)+np.log(q))
```

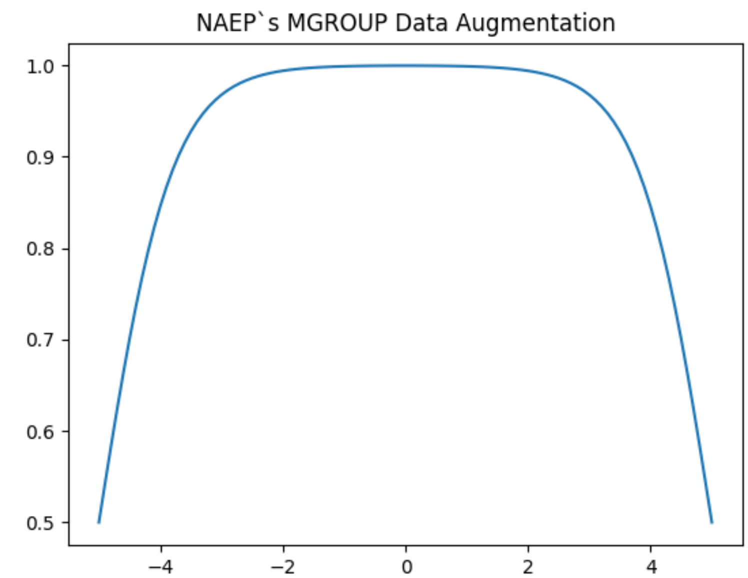
This code adds terms to the likelihood function that are equivalent to two (deterministic, always correct, and always incorrect, respectively) Rasch items with difficulties that are equivalent to the smallest and largest interval points used in the numerical integration of the algorithm needed for estimation. This can be expressed as an improper prior that has the form

$$\pi(\theta) = p_{\alpha} q_{\omega} = \frac{1}{1+\exp[1.7(\alpha-\theta)]}\left[1 - \frac{1}{1+\exp[1.7(\omega-\theta)]}\right]$$

where $\alpha = qpmin$ is the minimum of the numerical integration range and $\omega = qpmax$ is the maximum of the integration range.

The resulting data augmentation prior can be visualized as provided in Figure 1, showing how this summand acts as a penalty function that reduces the likelihood of small and large theta estimates.

Figure 1: *NAEP's MGROUP data augmentation prior based on two Rasch items added to the likelihood function of the person ability estimate.*



Note that in Figure 1 above, this data augmentation term was presented in its normalized form

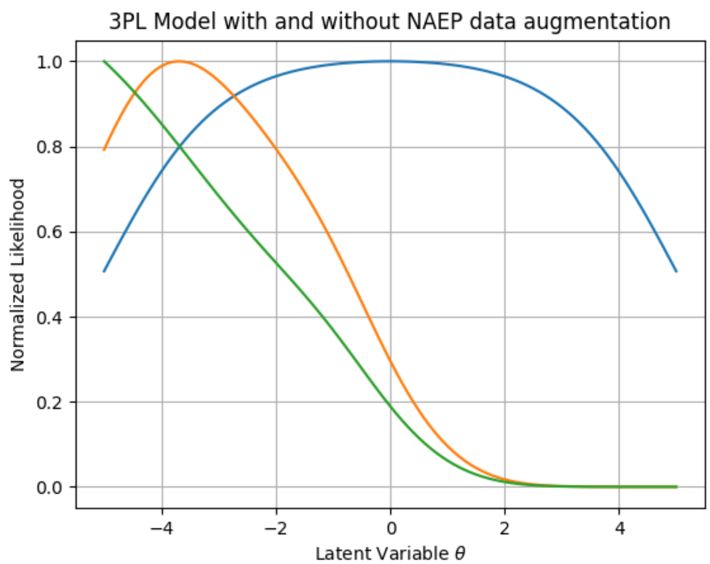
$$\pi^{*}(\theta) = \frac{\pi(\theta)}{\max_{\theta}\{\pi(\theta)\}}.$$

The two deterministic Rasch items used in NAEP MGROUP add Rasch item functions as pseudo-information to all response vectors, making a monotone likelihood function much less likely, if not impossible, to ensure that MGROUP's calculations will yield finite ability estimates for any observed response vector.

Figure 2 shows how this augmentation of the likelihood by two Rasch items prevents monotone likelihood functions in the 3PL. Monotone (and sometimes multimodal) likelihood functions can be frequently observed when applying the 3PL IRT model. They are rarely if ever, observed in the Rasch model and 2PL IRT model. The data example is a 5-item scale. A test taker answered two items correctly and three incorrectly. In this example, a monotone likelihood function is observed, leading to a negative infinite estimate if the 3PL were used. Still, the data augmentation with the two Rasch items described above leads to a function where the maximum is finite (and is found around $\theta \approx -3.8$).

It is helpful to develop an intuition of how using the 3PL leads to the threat of observing monotone likelihoods, even for non-extreme response patterns. Monotone likelihoods can only be observed in the case of extreme total scores when the Rasch model or 2PL is used. The 3PL has a much higher incidence of these cases. The usual treatment of missing data as incorrect does not alleviate this issue. Rather, it amplifies the effect, as low-scoring individuals will be penalized by assuming that additional incorrect response terms are part of the likelihood function.

Figure 2: *Effect of NAEP's MGROUP data augmentation prior on the likelihood function of a 5-item 3PL model. The response vector is [0,1,0,0,1], and the item parameters are provided in the appendix.*



The green line in Figure 2 represents the 3PL likelihood function based on the five items only. It can be seen that the function is monotone, and a maximum likelihood approach would lead to an infinite estimate. The orange line in Figure 2 represents the product of the 5-item likelihood and the 2-item data augmentation prior. It can be observed that this line represents a unimodal function with a maximum value (likelihood plus 2 Rasch-item based) estimate of around -4 to -3.8.

To build intuition as to why the 3PL (which allows a non-zero lower asymptote for all items) leads to monotone likelihoods even for non-extreme response patterns, the following calculations are helpful:

Let $c^* = \prod_{i=1}^{I} c_i > 0$ denote the product of all guessing parameters in the 3PL IRT model.

Note that this number is larger than zero, $c^* > 0$, and does not vanish unless $\exists i$: $c_i = 0$, i.e. unless at least one guessing parameter is zero. This number is non-zero, describing the probability of getting all items correct for a person approaching infinitely low ability, $\theta \rightarrow -\infty$. This non-vanishing c^* asymptote *is the culprit* that can lead to monotone likelihood functions, which in turn lead to (negative) infinite ability estimates in the 3PL even for non-extreme response patterns (i.e., even for respondents who answered one or the other item correctly).

For any response pattern, $X_j = \left(x_{j1},..., x_{jI}\right)$, define

$$c^*_{X_j} = \prod_{i:x_{ji}=1} c_i > 0$$

and

$$c^*_{X_j}(\theta) = \prod_{i:x_{ji}=1} P\left(X_{ji} = 1|\theta\right) > 0.$$

and

$$d^*_{X_j}(\theta) = \prod_{i:x_{ji}=0} P\left(X_{ji} = 0|\theta\right) > 0.$$

The first is the product over the guessing parameters for those items that were answered correctly, i.e., all i for which $x_{ji} = 1$, the second is the product of all probabilities for the same correctly answered set of items for a given ability level, and the third is the product of all probabilities of incorrect responses for a given ability level.

Now for the Rasch model and 2PL IRT models, we have a vanishing lower asymptote with decreasing ability, i.e.,

$$lim_{\theta\rightarrow-\infty} c^*_{X_j|1,2PL}(\theta) = 0$$

whereas, for the 3PL we obtain

$$lim_{\theta\rightarrow-\infty} c^*_{X_j|3PL}(\theta) = c^*_{X_j} > 0$$

with

$$\forall\theta: c^*_{X_j}(\theta) \geq c^*_{X_j}.$$

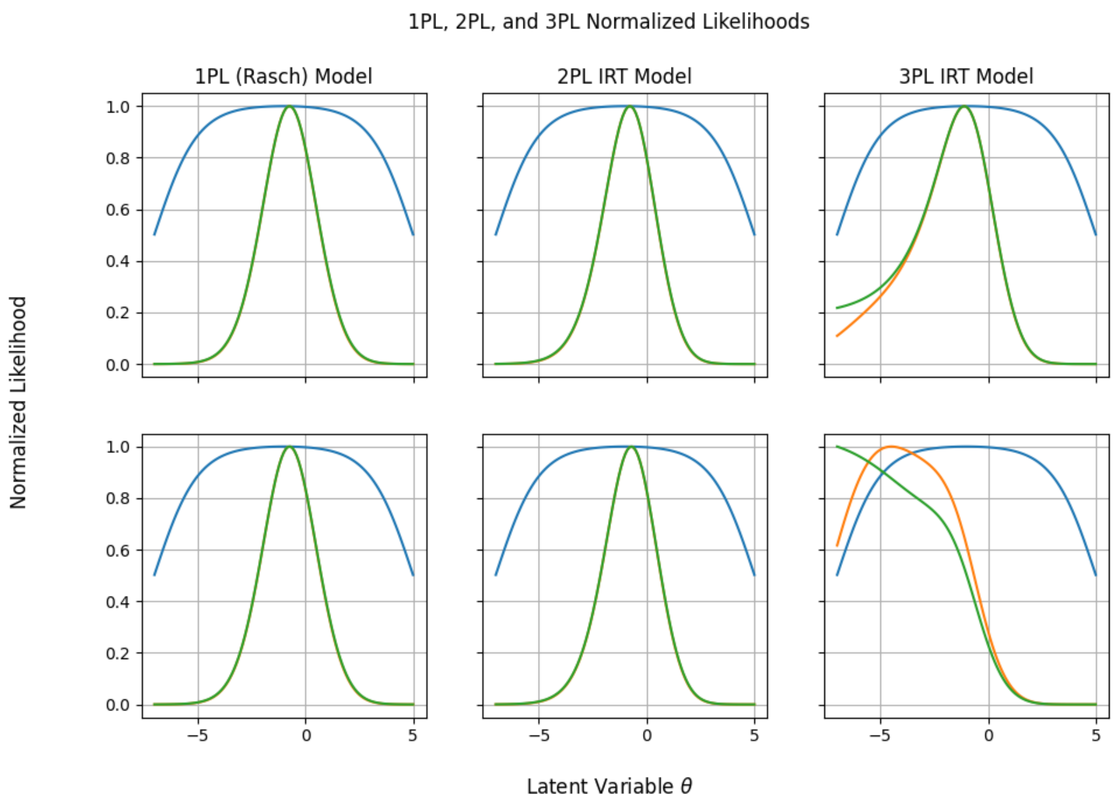
If, for $\theta_1 < \theta_2 \rightarrow -\infty$, we have

$$c^*_{X_j}\left(\theta_1\right)d^*_{X_j}\left(\theta_1\right) \geq c^*_{X_j}\left(\theta_2\right)d^*_{X_j}\left(\theta_2\right)$$

then this 3PL likelihood exhibits a monotone increase as $\theta\rightarrow -\infty$. The likelihood function depicted in green in Figure 2 is an example. Another example is given below for a less extreme case where there is still a monotone increase for the 3PL. For comparison, the 1PL and 2PL are also shown. These do not exhibit this behavior as the lower asymptote of these models must be $lim_{\theta \to -\infty} c^{*}_{X_j|1,2PL}(\theta) = 0$.

Without the data augmentation, the 3PL model would not infrequently lead to monotone likelihood functions resulting in undefined (negative infinite) ability estimates, especially for test takers with very few correct responses. This is why ETS's software MGROUP (developed originally for the NAEP assessment, which uses the 3PL IRT model) implements this data augmentation prior by default, without an option to disable it, to the authors' knowledge.

Figure 3: *Effects of NAEP's data augmentation prior on likelihood functions of the 1PL, 2PL and 3PL IRT models with particular emphasis on monotone 3PL likelihoods. The first row of plots is the likelihood function for a total score of 2 (out of 5 possible) correct answers and responses [1,0,1,0,0], and the second row depicts likelihood functions for [1,1,0,0,0].*



It can be seen utilizing the mathematical arguments put forward in this section and the examples shown in Figures 2 and 3 that this form of augmentation plays a crucial role when the 3PL is used. At the same time, it has a negligible effect on 1PL and 2PL likelihood functions, even when the number of items is small.

From Laplace Smoothing to Omitted Response Treatments in IRT

In its most simple form, data augmentation has been around for quite some time. Laplace smoothing was introduced above, and the following equivalency can be shown based on the binomial distribution. Consider the term that depends on parameter $p = \pi_0$:

$$p^{f_1+b}(1-p)^{f_0+a} = p^{f_1}(1-p)^{f_0}p^{b}(1-p)^{a}.$$

The usual line of argumentation calls out the beta distribution as a conjugate prior distribution for the binomial here (e.g. Iversen, 1984).

Also, the above identity has other applications. Consider the following two cases:

1) The number of observations is one, $I = 1$: Assume $0 \leq a, b \leq 1$. Then we have
$$p^{f_1+b}(1-p)^{f_0+a} = p^{x}(1-p)^{x-1}.$$

2) The number of observations is zero, $I = 0$: Assume $a + b = 1$. Then we have
$$p^{f_1+b}(1-p)^{f_0+a} = p^{b}(1-p)^{a}.$$

Note that both cases can be combined by defining the following trichotomous variable:

$Y = 0$	if	$I = 1, X = 0$
$Y = 1$	if	$I = 1, X = 1$
$Y = M$	if	$I = 0$

Then we can write, using $q = 1 - p$, the probability of Y

$$P(Y) = \left[p^{X}q^{1-X}\right]^{I}\left[p^{b}q^{a}\right]^{1-I}.$$

This is a function of either one of the two parameters $p = 1 - q$ and $a = 1 - b$. The response probabilities are p and q if the response was observed, i.e., if $I = 1$. If the response was not observed, the smoothed probability given by $p^{b}q^{a}$, using smoothing constants a and b with $a + b = 1$ are used.

Application to Missing Item Response Data

Obviously, $I = 0$ implies that no observation was made, and $I = 1$ describes that exactly one binary outcome was observed. This is directly applicable to the data types seen in repeated binary outcomes, either in panel data or item response data collected in educational and psychological testing.

Let us now assume we have probabilities p_{ij} for multiple binary variables observed on multiple respondents, x_{ih}, $i = 1, ..., I$, $j = 1, ..., N$. Assume that the p_{ij} are following an IRT model, i.e.,

$$p_{ij} = \left(1 + exp\left(- a_i\left(\theta_j - \beta_i\right)\right)\right)^{-1}$$

where θ_j denotes the person ability parameter and a_i, b_i are the item (slope and difficulty) parameters.

Then, p_{ij} is commonly associated with the probability of 1, the correct response, and $q_{ij} = 1 - p_{ij}$ denotes the probability of 0, the incorrect response.

Then we can define customarily used omitted response treatments based on the special cases above.

Omitted Treated as Incorrect Response

If $Y = M$, setting $a = 1, b = 0$ yields

$$P_{ij}(Y = M) = p_{ij}^{f_1+b}\left(1 - p_{ij}\right)^{f_0+a} = \left(1 - p_{ij}\right) = q_{ij},$$

which equals the probability of an incorrect response for person j on item i.

Lord's Fractionally Correct Scoring of Omitted Responses

If $Y = M, f_0 + f_1 = 0$ and with setting $a = \frac{k-1}{k}$, $b = \frac{1}{k}$ for any integer k yields

$$p_{ij}^{f_1+b}\left(1 - p_{ij}\right)^{f_0+a} = p_{ij}^{\frac{1}{k}}\left(1 - p_{ij}\right)^{\frac{k-1}{k}}.$$

Typically, this approach is used for selection-items or multiple-choice (MC) items, and k is chosen to be equal to the number of response options on the item, so $k \in \{4, 5\}$ are commonly used for MC items.

Data Augmentation as Used in Response Propensity Models

The treatment of omits as incorrect equates a left-out response with an incorrect answer. This is known to cause bias in the ability estimates derived from the IRT model. Mislevy & Wu, 1996, Moustaki & O'Muircheartaigh (2000), Moustaki & Knott, 2000; Glas & Pimentel, 2008; Rose et al., 2010; Rose et al., 2017, Sachse et al., 2019, among others, presented models and evidence that acknowledge that the decision to omit may not be directly associated to the ability, but rather constitute a separate dimension of individual differences.

These models can be written in the data augmentation framework, embedded in our span of cases 1 and 2 as discussed above. The definition needs a bit of an extension in this case, as the prior may not be the same as the $p_{ij} = 1 - q_{ij}$. In the response propensity models, an additional Bernoulli process is assumed so that

$$P_{ij}(Y) = p_{ij}^{\delta_{Y1}} q_{ij}^{\delta_{Y0}} r_{ij}^{1-\delta_{YM}} s_{ij}^{\delta_{YM}}$$

with $\delta_{AB} = 1_{\{A=B\}}$ denoting the Kronecker delta or indicator function.

The response propensity models typically assume that the s_{ij} also follow an IRT model, with parameters η_j, c_i, d_i so that

$$s_{ij} = \left(1 + exp\left(- d_i\left(\eta_j - c_i\right)\right)\right)^{-1}$$

and $r_{ij} = 1 - s_{ij}$. The d_i are the slope parameters of the response indicator item functions, the c_i is the response indicator difficulty parameter, and the η_j parameterize the response propensity of test taker j. This means that every respondent is characterized by a two-dimensional latent variable that includes the components θ_j, η_j.

Extension to Multiple Responses

Consider a response vector $y_{1j}, ..., y_{Ij}$ defined for each test-taker $j = 1, ..., N$. Assume $y_{ij} \in \{0, 1, M\}$ for all $i = 1, ..., I$ and $j = 1, ..., N$ encoding whether a correct or incorrect response was observed or whether the response was missing.

Assuming local independence, we can separate the probability of the set of responses and omissions into two products. We can write the joint probability for the set of responses and omits as follows:

- Omitted as Incorrect Approach

$$P\left(y_{1j}, ..., y_{Ij} | \theta_j\right) = \left(\prod_{y_{ij} \neq M} p_{ij}^{\delta_{y_{ij}1}} q_{ij}^{\delta_{y_{ij}0}}\right)\left(\prod_{y_{ij} = M} q_{ij}\right)$$

- Omitted as Fractionally Correct Approach

$$P\left(y_{1j}, \ldots, y_{Ij} | \theta_j\right) = \left(\prod_{y_{ij} \neq M} p_{ij}^{\delta_{y_{ij}1}} q_{ij}^{\delta_{y_{ij}0}}\right)\left(\prod_{y_{ij}=M} p_{ij}^{\frac{1}{K}} q_{ij}^{\frac{K-1}{K}}\right)$$

- Response Propensity Model

$$P\left(y_{1j}, \ldots, y_{Ij} | \theta_j\right) = \left(\prod_{y_{ij} \neq M} p_{ij}^{\delta_{y_{ij}1}} q_{ij}^{\delta_{y_{ij}0}}\right)\left(R_j \prod_{y_{ij}=M} \frac{s_{ij}}{r_{ij}}\right) \text{ where } R_j = \prod_{i=1}^{I} r_{ij}.$$

The above expressions share a common structure. They show that the missing data treatment in all cases discussed above can be written as a product of two terms. The left-hand term contains the likelihood of observed responses

$$P\left(y_{1j}, \ldots, y_{Ij} | \theta_j\right) = \left(\prod_{y_{ij} \neq M} p_{ij}^{\delta_{y_{ij}1}} q_{ij}^{\delta_{y_{ij}0}}\right)$$

which equals the treatment of omits if not observed responses are considered *ignorable* missing data (e.g., Glas & Pimentel, 2008; Rose et al., 2010).

The right-hand term differs with respect to how omitted responses are treated. These missing data treatments are structurally similar to a data augmentation prior that is used only if omits are present in the data.

The most extreme variant is scoring *omitted responses as incorrect* responses, as it produces a monotone prior $\prod_{y_{ij}=M} q_{ij} = \prod_{y_{ij}=M} P\left(X_i = 0 | \theta_j\right)$ that decreases monotonic in θ. This can lead to or worsen cases of monotone likelihood and will exacerbate the issues experienced with undefined, negative infinite, ability estimates due to monotone likelihood functions (e.g. Heinze & Schemper, 2002). Monotone likelihood functions lead to issues when estimating regression coefficients in logistic regression and latent regression models as used in MGROUP and MIRT (see the section above about data augmentation in MGROUP).

In line with this observation, it has been noted by multiple authors (e.g. Rose et al., 2010, Ulitzsch et al., 2020) that omitted response treatments that count every missing response as an incorrect response leads to biased θ estimates.

The *fractionally correct* approach (Lord, 1980) produces a penalty term for the likelihood function that equals

$$\left(\prod_{y_{ij}=M} p_{ij}^{\frac{1}{K}} q_{ij}^{\frac{K-1}{K}}\right) = \left(\prod_{y_{ij}=M} P\left(X_i = 1 | \theta_j\right)^{\frac{1}{K}} P\left(X_i = 0 | \theta_j\right)^{\frac{K-1}{K}}\right)$$

as the *data augmentation* prior applies if responses were not observed/omitted. This term provides a unimodal prior under mild conditions, especially if guessing parameters are not large and the number of missing items is larger than four or so.

The response propensity approach introduces the term

$$\left(R_j \prod_{y_{ij}=M} \frac{s_{ij}}{r_{ij}}\right) \text{with } R_j = \prod_{i=1}^{I} r_{ij}$$

which is a penalty similar to the fractionally correct and the all incorrect approach in that it includes estimates for all items that were omitted. However, it differs from these in that it involves the introduction of an additional latent variable which makes the response propensity approach a multidimensional item response theory (MIRT) model, which increases model complexity by effectively doubling the number of parameters that need to be estimated.

Common to all right-hand terms is that these *augment* the observed response likelihood by data that is not directly observed.

A Response Propensity Counts Model

In the proposed model, the idea is to avoid estimating an additional latent response propensity variable and to also reduce the number of parameters needed to describe the missing data process. Model parsimony appears to be desirable in the context of missing data treatment, as missingness at the item level appears to be a rare event, while the observation of general response incompleteness (whether any response was omitted or not) is a much more common event.

The idea is to estimate the weight of evidence of whether responses were omitted or not. Rather than mimicking the behavior of the other non-response treatments, the proposed approach does not assume all omitted responses equate to incorrect responses. Instead of a deterministic treatment that assumes all omitted responses are fractionally or completely incorrect (implying that low ability as the underlying cause of missingness), the aim is to estimate the extent to which missing responses are related to ability. However, instead of assuming a separate dimension (as in the work of Moustaki et al., 2000; Glass et all., 2008; Rose et al., 2010), and item-level probabilities for non-response, the RP counts model aggregates the number of missing responses and defines a data augmentation based on this aggregate for the ability variable θ.

More specifically, let

$$N_{MC} = \sum_{i \in \Omega_{MC}} 1_{y_{ij}=M}$$

$$N_{CR} = \sum_{i \in \Omega_{CR}} 1_{y_{ij}=M}$$

denote a random variable that represents how many responses were omitted by respondent j among the $y_{1j}, \ldots, y_{Ij}$, counted separately for multiple choice (MC) and constructed response items (CR).

We may use a statistical model for count data if the number of omits is informative. The Rasch Poisson Counts Model is one option that can be considered. In this model

$$P\left(N_{MC} | \theta_j\right) = exp\left(-\mu_{jMC}\right)\frac{\left(\mu_{jMC}\right)^{N_{MC}}}{N_{MC}!}$$

with $\mu_{jMC} = T_j S_{MC}$ and negatively associated $T_j = exp\left(-\theta_j\right)$ and $S_{MC} = exp\left(b_{MC}\right)$.

Note that the model assumes a strict monotonic association of the probability of non-response counts and the negative ability, since it is assumed to be likely that higher non-response rates are associated with lower ability levels.

Suppose it is sufficient to know whether or not omissions have occurred at all, and the number of omits is less informative than whether they occurred. In that case, one may simplify further and use dichotomized *complete response indicators* defined for each item type

$$R_{MC} = 1_{N_{MC}=0}$$

and

$$R_{CR} = 1_{N_{CR}=0}$$

.

These *response completeness indicators* encode whether a respondent produced missing responses or not separately for multiple-choice (MC) and constructed response (CR) items. It is proposed to use separate response indicators for different item types, as the levels of non-response commonly tend to differ substantially across item types.

The probability of these two binary response indicator variables is then modeled as dependent on the latent ability variable θ. Using a flexible IRT item function is appropriate here since the dependency of response completeness on the latent trait can vary considerably across test forms and item formats. Here we propose to use the 3PL or the 4PL IRT model so that

$$P\left(R_{MC,j} | \zeta_{MC}, \theta_j\right) = c_{MC} + \left(d_{MC} - c_{MC}\right)\frac{1}{1+exp\left(a_{MC}\left(b_{MC}-\theta_j\right)\right)}$$

where

$$\zeta_{MC} = \left(a_{MC}, b_{MC}, c_{MC}, d_{MC}\right)$$

and similarly for

$$P\left(R_{CR,j}|\zeta_{CR}, \theta_j\right).$$

Concatenating the response variables $\left(y_{1j}, \ldots, y_{Ij}\right)$ and the two binary response completeness variables R_{MC}, R_{CR} yields the *augmented* response vector $\left(y_{1j}, \ldots, y_{Ij}, r_{MCj}, r_{CRj}\right)$. If all respondents answer all items, the two variables r_{MCj}, r_{CRj} are constant $r_{MCj} = r_{CRj} = 1$ across test-taker and hence uninformative. However, if the random variables R_{MC}, R_{CR} are not deterministic, and if they moreover depend on the ability variable θ, additional information is gained, and the estimated 4PL item functions quantify the weight of the evidence of the response completeness indicators, which can be utilized for estimation of the latent trait.

Augmenting the data by these response completeness indicators, and estimating abilities with indicator item functions (represented by item parameters ζ_{MC}, ζ_{CR} here) defines a data augmentation prior.

$$\hat{\theta} = arg\,max\left(L\left(\theta|y_1, \ldots, y_I, r_{MC}, r_{CR}, \zeta_{MC}, \zeta_{CR}\right)\right)$$

with

$$L\left(\theta|y_1, \ldots, y_I, r_{MC}, r_{CR}, \zeta_{MC}, \zeta_{CR}\right) = \left(\prod_{y_i \neq M} p_i^{\delta_{y_i 1}} q_i^{\delta_{y_i 0}}\right) p_{r_{MC}} p_{r_{CR}}.$$

This data augmentation prior penalizes omitted responses only if these are indeed informative for the estimation of ability θ. In the above equation, we use

$$p_i = P\left(Y_i = 1|\theta, \zeta_i\right) = 1 - q_i$$

and

$$p_{r_{MC}} = P\left(1|\zeta_{MC}, \theta_j\right)^{r_{MC}} P\left(0|\zeta_{MC}, \theta_j\right)^{1-r_{MC}}$$

and

$$p_{r_{CR}} = P\left(1|\zeta_{CR}, \theta_j\right)^{r_{CR}} P\left(0|\zeta_{CR}, \theta_j\right)^{1-r_{CR}}$$

to denote the IRT-based response probabilities as well as the completeness probabilities given the ability θ.

In contrast to a-priori determining that any omitted variable contains the same amount of information as an incorrectly answered variable, the omit counts propensity model estimates the dependency of the omit counts distribution on ability and penalizes the likelihood function used to estimate ability proportionally to the information the omit counts contain about respondent's ability parameter.

The 4PL-IRT model appears to be the appropriate choice for the binary indicators. A model with a lower and upper asymptote can ensure that a maximally flexible item function is estimated.

It is important to note that in many cases, the *complete response* indicators are *very easy* items, as the proportion of non-response tends to be quite small in many applications. However, MC and CR items are known to have very different base rates of non-response. CR items tend to have a higher propensity of non-response, while MC items are responded to by a much larger proportion of test-takers in many cases (e.g. Rose et al., 2010).

Therefore, this approach is based on defining separate *response completeness* indicators for CR and MC items. This separation does allow us to estimate the level of association between ability and response completeness in a way that best reflects whether and how much item-level non-response depends on the proficiency of the test takers.

A New Approach to Non-Response Treatments

As has been illustrated in the sections above, data augmentation priors have been used to *reign in* monotone likelihood functions that can be frequently observed in the 3PL IRT model, but only rarely (for extreme response patterns) in the 1PL and 2PL IRT Model. The non-response treatment approach proposed here builds on the data augmentation approach but replaces the treatment of non-response as incorrect answers by an estimated data augmentation prior that depends on whether students respond all items or not, and on how much this response completeness is associated with the latent ability variable.

The proposal is to retain the data augmentation (using two deterministic Rasch items) as it is implemented in NAEP MGROUP, but to combine these with the response completeness indicators defined R_{MC}, R_{CR} and estimated item functions separately for item types. This produces a data augmentation prior that includes (for 2 item types) four data dependent groups

$$\pi_{r_{MC} r_{CR}}(\theta) = p_{r_{MC}} p_{r_{CR}} p_{\alpha} q_{\omega}$$

$$= P\left(r_{CR} | \zeta_{CR}, \theta\right) P\left(r_{MC} | \zeta_{MC}, \theta\right) \frac{1}{1+\exp[1.7(\alpha-\theta)]} \left[1 - \frac{1}{1+\exp[1.7(\omega-\theta)]}\right]$$

which leads to the (augmented) likelihood function

$$L_{aug}\left(\theta | y_1, \ldots, y_I, r_{MC}, r_{CR}, \zeta_{MC}, \zeta_{CR}\right) = \left(\prod_{y_i \neq M} p_i^{\delta_{y_i 1}} q_i^{\delta_{y_i 0}}\right) \pi_{r_{MC} r_{CR}}(\theta)$$

while this augmentation may seem like it is adding a lot to what was observed (the actual correct and incorrect responses), it turns out that it is adding fewer assumptions about how much a missing (omitted) response contributes in terms of information about the latent variable θ. The only addition that is ‘made up’ are the two Rasch items already part of the currently used latent regression model implemented in MGROUP.

This approach can be properly applied also to factored likelihoods as described in Thomas (1993). In that case, it is assumed that k scales are part of the model and that

$$L\left(\theta | y_1, \ldots, y_I, r_{MC}, r_{CR}, \zeta_{MC}, \zeta_{CR}\right) = \prod_{k=1}^{K} L\left(\theta_k | y_{1(k)}, \ldots, y_{I(k)}, r_{MC,K}, r_{CR,K}, \zeta_{MC,K}, \zeta_{CR,K}\right)$$

so that a set of completeness indicators $r_{MC,K}, r_{CR,K}$ is defined separately for each of the k scales. Then we define

$$\pi_{r_{MCk} r_{CRk}}\left(\theta_k\right) = p_{r_{MCk}} p_{r_{CRk}} \left[p_{\alpha k} q_{\omega k}\right]^{\frac{1}{k}}$$

so that

$$\text{for } \theta_1 = \ldots = \theta_K \text{ we have } \prod_{k=1}^{K} \pi_{r_{MCk} r_{CRk}}\left(\theta_k\right) = p_\alpha q_\omega \prod_{k=1}^{K} p_{r_{MCk}} p_{r_{CRk}}.$$

The MGROUP augmentation is currently not consistent if the same data is analyzed with different subscale selections. As a single (unidimensional) scale, two Rasch items would be added by the MGROUP augmentation, and as a multidimensional scale, two times the number of subscales would be the number of data augmentation items are added. In the latter case, the MGROUP augmentation currently yields

$$\pi\left(\theta_1 \ldots \theta_K\right) = \prod_{k=1}^{K} p_{\alpha k} q_{\omega k}.$$

That is, for a model with a K-factored likelihood, the MGROUP data augmentation adds the equivalent of $K\ *\ 2$ Rasch items to the likelihood function. In the approach proposed here, this term is replaced by

$$\pi^{\frac{1}{K}}\left(\theta_1 \ldots \theta_K\right) = \prod_{k=1}^{K} \left[p_{\alpha k} q_{\omega k}\right]^{\frac{1}{K}}$$

so that the equivalent of 2 Rasch items is retained representing the weight of the penalty added by the data augmentation prior.

As a researcher who is somewhat familiar with the Rasch model (e.g., von Davier, 2016; von Davier & Carstensen, 2006) and one who has followed research regarding the identifiability (or better, the lack thereof) of the 3PL IRT model it seems somewhat ironic to see this way of augmentation implemented by default: One the one hand the 3PL is being pushed as the appropriate IRT model for achievement tests using MC items, on the other hand, the likelihood function of every respondent is augmented by two made-up Rasch model items since the algorithm to estimate latent regression IRT using a fixed parameter 3PL model would otherwise not yield proper results. Figure 3 shows that the data augmentation prior has very little effect on the 1PL and 2PL models, while for the 3PL model, the augmentation by 2 Rasch items leads to a penalized likelihood function that is more regular, and more similar to the 1PL and 2PL likelihood functions than to the 3PL without augmentation.

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