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Estimation as analogy-making: Evidence that preschoolers'
analogical reasoning ability predicts their numerical
estimation

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Abstract

All humans and many animals can represent approximate quantities of perceptual objects nonlinguistically by using the Approximate Number System (Dehaene, 1997/2011). Early in life, children in numerate societies also learn to describe this system using number words. How do linguistic representations of number become related to nonlinguistic representations of number? We hypothesize that the analogical process of structure mapping (Gentner, 1983) helps children to form mappings between the linguistic and nonlinguistic number systems on the basis of structural similarities between the two systems. To test this, we tested and analyzed 47 four-and-five year olds' performance on estimation and analogy tasks. We found that analogical reasoning ability uniquely predicted several components of estimation performance, even when controlling for other domain-general cognitive skills. This provides strong evidence that analogical processes are uniquely related to the development of early estimation.

We rarely have the time or inclination to count all of the quantities that we might want to keep track of (e.g., coins in your car, the number of chips you just ate); fortunately, we can estimate! The ability to provide numerical estimates that is, to provide a linguistic numerical label for a numerical quantity develops in the first few years of life (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Le Corre & Carey, 2007; Lipton & Spelke, 2005), and predicts math success (Booth & Siegler, 2006, 2008; Moore & Ashcraft, 2015; Siegler & Booth, 2004). Much of the existing research on numerical estimation has thus far focused on understanding what estimation tasks can tell us about the development of numerical competence. However, while estimation undoubtedly requires numerical knowledge, because estimation requires connecting the linguistic and nonlinguistic number systems to one another, it serves as an excellent test case for understanding the more general question of how language becomes related to perception during development (Sullivan & Barner, 2012). In the present study, we ask whether one domain-general cognitive ability the ability to relate disparate domains to one another analogically supports early estimation. When we estimate, we label a perceptually available quantity (e.g., a set of dots; a series of beeps) with a number word in other words, we describe our nonlinguistic numerical representations by using linguistic representations of number. One nonlinguistic number system is the evolutionarily ancient Approximate Number System (ANS), which allows us to roughly quantify e.g., the number of objects in a set (albeit with systematic error; it cannot represent quantities like “exactly 56”; Dehaene, 1997/2011). In contrast, our linguistic number system precisely represents numerical quantity in a culturally- mediated and uniquely human format (e.g., via number words or numerals). Because the linguistic and nonlinguistic number systems differ radically in representational format, recent work has focused on understanding the mappings between these systems, and has done so by studying the development of estimation ability (Sullivan & Barner, 2012, 2014a).

How do the linguistic and nonlinguistic number systems become mapped onto one another?

While it is theoretically possible that the linguistic and nonlinguistic number systems become related to one another via associative learning (e.g., by associating words like eight with ANS representations of ‘about 8 things’; Lipton & Spelke, 2005), such a learning process would require substantial experience with specific word-magnitude pairings (which four-year-olds, who are learning these mappings, almost certainly lack), and has been empirically demonstrated to play only a minimal role in connecting number words to the ANS (Lyons, Ansari, & Beilock, 2012; Sullivan & Barner, 2012, 2014a). One alternative possibility is that children recruit structure mapping (Gentner, 1983; Gentner & Markman, 1997; Gentner & Namy, 2006), an analogical process, to form holistic mappings between their linguistic and nonlinguistic number systems, based on the structural similarities between these two systems (Sullivan & Barner, 2012, 2014a). For example, a child might compute an analogy based on the shared ordinal structure of the linguistic and nonlinguistic number systems: they might notice that the word sixty comes after the word thirty in the count list, just like sets containing 60 items appear more numerous than sets containing 30 items (in standard analogical notation, this might be presented as $\text{sixty} : \text{thirty} :: 60 : 30$). This, in turn would lead to the inference that sixty should be mapped onto larger sets than thirty is (see also, Lyons & Beilock, 2011). Or, a child might compute an analogy based on the relative-distance structural relations encoded within the linguistic and nonlinguistic number systems: they might notice that the word sixty comes twice as far along in the count list as the word thirty, and therefore infer that sixty should be mapped onto a quantity that is twice as large as that referred to by thirty. In this sense, estimation could be supported by analogical inferences about the relation between the structures of the linguistic and nonlinguistic number systems (for additional information about how analogy might support still other components of estimation, see Thompson & Opfer, 2010). There is substantial evidence consistent with the view that analogical reasoning supports the connection between linguistic and non-linguistic representations of number during estimation. First, when participants are given misleading feedback (e.g., by being shown that a particular quantity is 30, or

by being told that the largest quantity that they will encounter during an estimation task is 750), they recalibrate their estimates for all but the smallest numbers (Izard & Dehaene, 2008; Sullivan & Barner, 2012, 2014a). In other words, most number word mappings are mutually constraining when the mapping for one number is altered, estimates for most other numbers are affected. These calibration effects should emerge if participants have a holistic, structure mapping between their linguistic and nonlinguistic number systems, and not if item-by-item associations underlie number word mappings (Sullivan & Barner, 2012, 2014a,b). Second, even when participants are not given explicitly misleading feedback, recent work has shown that their estimates are mutually constraining, such that estimation error (e.g., over- or under-estimating) carries forward from trial to trial (Sullivan, Juhasz, Slattery, & Barth, 2011; Vul, Barner, & Sullivan, 2013). Again, this suggests that participants use structure mapping to relate the mapping for the current trial to mappings deployed on earlier trials, such that mappings are internally consistent.

Finally, a signature of the use of analogy during estimation is that estimates are internally consistent, even when they are inaccurate. Internally consistent estimates emerge when estimates (and therefore mappings between number words and the ANS) are not mutually contradictory. In other words, in order for estimates to be internally consistent, they must make sense with respect to one another. For example, estimates that are in the correct order relative to one another (e.g., larger number words are used to label larger quantities), or that are linearly related to one another, are internally consistent, even if they happen to be inaccurate. Children's early estimates exhibit two markers of an internally consistent structure mapping. First, children are able to provide internally consistent estimates (e.g., ordinal estimates that are in the correct order relative to previous trials) long before they provide accurate estimates, and even for quantities outside the child's stable count range (Barth, Starr, & Sullivan, 2009; Sullivan & Barner, 2014a,b). Second, children's estimates are often predictively related to one another (e.g., are non-random), even when they are wildly inaccurate (Barth et al., 2009; Berteletti et al., 2010; Booth & Siegler, 2006, 2008). This suggests

that the earliest mappings between the linguistic and nonlinguistic number systems are unlikely to be based on veridical pairings of number words to magnitudes (this view would predict that only accurate estimates should be internally consistent, and that estimates for unmapped magnitudes should be random), and instead are based in the relation between the structures of the two number systems (Thompson & Opfer, 2010). While these studies are consistent with the view that analogy underlies estimation, previous work has not directly tested the relation between estimation and domain-general analogical reasoning skill.

If estimation requires children to analogically relate their linguistic and nonlinguistic number systems to one another, then it is likely that children who are most able to compute analogies will also be most likely to recruit structure mapping when estimating. For this reason, we tested 4- and 5-year-old children's analogical reasoning skill, and asked whether it uniquely predicted the elements of estimation that have been argued to be analogical in nature (i.e. the effect of calibration, ordinality scores, and linear r^2), while controlling for a host of other non-analogical factors (age, counting ability, memory, and Raven's Progressive Matrices).

Methods

1.1. Participants

Participants were recruited from private preschools in a suburban, middle-class city in the Northeast of the United States. Parents provided written consent and children provided verbal assent. In total, 57 children were tested. Because data collection was constrained by the academic school year, we planned to stop data collection once we tested 50 participants or 2 weeks before the end of the academic school year, whichever happened later.

Ten participants were excluded from analysis; four for being outside of our desired age range, and six for failing to complete all tasks. The remaining 47 four- and five-year-olds ($M_{\text{age}} = 58.3$ mos.; $\text{Med}_{\text{age}} = 58$ mos.; range = 48–70 mos.) completed all tasks and were included in all analyses.

1.2. Materials and procedure

Ethical approval was secured from the Participant Review Board at Skidmore College. Participants were tested in a relatively quiet area of their preschool. Participants were seated in front of a MacBook Pro Laptop computer and completed two computerized tasks and five non-computerized tasks. Testing sessions lasted approximately 30–45 min. While most of the participants ($n = 42$) completed all tasks in a single session, a subset ($n = 5$) required two sessions to complete all tasks. After completing the study, participants were congratulated and offered a sticker, and teachers and parents were debriefed via a newsletter.

There were two blocks of testing. One block consisted of the number and memory tasks: the (1) Uncalibrated and (2) Calibrated estimation tasks, (3) the color and number Memory Task, and (4) the counting task. The other block consisted of the analogy tasks and the general intelligence task: (1) the Conceptual and (2) Number Analogy tasks and (3) Raven's Progressive Matrices Set A. So that two children could be tested at a time, approximately half of the participants completed the Estimation/Memory block first, and half completed the Analogy/General Intelligence block first. Within a given block, participants completed tasks in the fixed order described below.

1.2.1. Estimation tasks

The purpose of the Uncalibrated and Calibrated estimation tasks was to measure whether providing misleading feedback about the range of numbers being estimated would induce global changes in estimation performance; previous research has taken such effects of calibration as evidence of the role of analogy during estimation (Sullivan & Barner, 2012, 2014a). As described earlier, if estimates are made by deploying analogical structure mapping, they should exhibit internal consistency and be mutually constraining; if they are mutually constraining, then feedback about the mappings for one number should influence estimates for any other number mapped via analogical structure mapping.

During the Uncalibrated and Calibrated estimation tasks, participants were told that they

would see pictures of dots and were then asked how many dots they saw. During the Uncalibrated task, participants were given no further instruction; during the Calibrated estimation task, participants were first told that the biggest number of dots that they would see would be 750 (Sullivan & Barner, 2012, 2014a). All participants were asked to repeat the instructions to the experimenter to ensure comprehension. On each of the 44 trials, participants saw a single dot array that was presented for 1 s, followed by a 250 ms back mask of white noise. Arrays were presented on a black background and contained colored dots (color varied between trials and included white, cyan, magenta, green, red, yellow, and blue) that varied in (a) item size and (b) total area occupied by the array. Arrays contained between 4 and 360 dots. Participants gave verbal estimates, which the experimenter recorded. After every five trials, participants saw an image of an animal in order to maintain interest.

1.2.2. Memory tasks

The experimenter told the participant that they would hear a story about “a dog named Miranda who goes on an adventure in the woods” and would have to remember as much of the story as they could. Within the story, five types of animals were mentioned (ladybugs, owls, deer, snakes, and turtles), each a different color (green, orange, red, purple, blue) and each in different quantities (2, 5, 7, 8, 17). The story lasted approximately 30 s, after which the participant was asked, for each animal, whether they remembered the animal (yes/no question), what color they were (free response), and how many animals of that type were mentioned (free response).

1.2.3. Counting assessment

Participants were asked to count as high as they could. They were encouraged to continue counting until they reached 100 or until they made an error, which was recorded by the experimenters.

1.2.4. Conceptual analogy task

This task was designed in order to test children’s domain-general analogical reasoning.

Participants were told they were going to play a card game wherein they needed to “complete the pattern as best as they could”. They were shown a 24” x 5” cardboard panel with three 3.5” x 3.5” pictures attached and a Velcro spot for one more (Fig. 1).

Fig. 1. Schematic of the Conceptual Analogy (top) and Number Analogy (bottom) task. The materials are not depicted to scale; in the actual experiment, the analogy board and cards were presented on a table.

The board was set up to visually mimic the analogical format $A : B :: C : D$ (“A is to B as C is to D”, or in common language “A goes with B just like C goes with D”). In the leftmost position, there was a picture of an object (e.g., fish). Immediately to its right, there was a picture of an object or transformation relevant to that object (e.g., a fishbowl). The right-hand side of the board contained a picture of a new object (e.g., dog). The experimenter then presented four other “answer” pictures that the child could use to fill in the blank; pictures were presented in a square and in random order. The four choices related to the third object (dog) in one of four specific ways (Goswami & Brown, 1990; Rattermann & Gentner, 1998): the correct answer (dog house) contained the correct object and the correct transformation. Incorrect answers included the correct object with an incorrect transformation (dog treat), the incorrect object with the correct transformation (bird nest), and an object that resembled the correct object (wolf). Thus, only one of the pictures correctly

completed the analogy (e.g., dog house) by applying the highlighted transformation ('home for') to the correct object (dog).

During each trial, the experimenter narrated the analogy by saying "X goes with Y just like A goes with... B, C, D, E". For example, in one trial participants heard "loaf of bread goes with sliced bread, just like lemon goes with. . .[tennis ball, sliced lemon, squeezed lemon, slice of cake]." To minimize the reliance on linguistic cues to solve the task, we varied the words used to describe each relation. For example, in cases where cards depicted similar causal transformations (e.g., destruction), whenever possible, different words labeled the cards (e.g., "broken guitar" vs. "smashed egg" vs. "destroyed piano").

Prior to the start of the task, two example trials (with corrective feedback) were given to make sure the participant understood how to correctly complete the analogy. If the participant chose the incorrect answer during the example, the experimenter gave encouraging but negative feedback (e.g., "that's a good guess, but there's a better answer.") and asked the child to try again until the child gave the correct answer. Following the two examples, the participant completed 10 analogy trials by selecting the card they believed best completed the pattern. During the test phase, all answers elicited a neutral-positive feedback (e.g., "thank you").

1.2.5. Number analogy task

This task was designed to test participants' numerical analogical reasoning. The materials and instructions were similar to the Conceptual Analogy task; however, instead of pictures of objects, children completed patterns containing numerals (Fig. 1). For each trial, the child saw a cardboard panel with 3 numerals on it (configured identically to the Conceptual Analogy task), and was presented with 4 numerals as possible answers. Each pattern required the participant to understand not just the ordinality of numbers, but also their relative distance. For example, the experimenter would say, "2 goes with 4 just like 20 goes with. . ." The correct answer (40) was the correct number (20) correctly transformed (multiplied by 2). Incorrect answers included the correct

number (20) incorrectly transformed (400; 20 was multiplied by 20 instead of by 2); the incorrect number (4) correctly transformed (8, which is 4 multiplied by 2); a number that resembled the correct number (200, which visually resembles 20). In this task, two trials depicted 1:2 proportions, two depicted 1:3 proportions, and two depicted 1:4 proportions, such that the correct transformation was always to multiply the given object by either 2, 3, or 4 (note that even infants can reason about ANS transformations at a 1:2 ratio; Xu & Spelke, 2000). One practice number analogy board was presented, and incorrect responses were given negative feedback: “That’s a good guess, but that’s not quite right. I think there’s an even better answer.” After correctly completing the practice trial, participants completed six test trial analogy boards.

1.2.6. Raven’s Progressive Matrices A

Children were then instructed that they would see a pattern with a piece missing and would be asked to pick the piece that completed the pattern. Twelve patterns were shown, one at a time, in black and white on an 8” x 11” sheet of printer paper. Participants had to choose one piece out of six options that best completed the pattern. Patterns increased in difficulty and complexity over the course of the task (Raven, 2000).

Results

2.1. Data management

For all estimation analyses, we excluded absurd responses (0,1, non-numerical responses; $n = 52$). For all analyses that could be influenced by outliers (all except our Ordinality analyses), we also excluded very large estimates (estimates greater than 10 times the presented numerosity; $n = 140$), and very small estimates (estimates less than the presented numerosity/10); $n = 782$). The remaining 3245 datapoints (77% of data) were considered for all analyses.

We had several measures of estimation performance. First, we calculated Proportion Absolute Error (PAE), which captures the accuracy of participants’ responses, and has been shown to predict a host of math measures (Castronovo & Göbel, 2012; Link, Nuerk, & Moeller, 2014;

Lyons, Price, Vaessen, Blomert, & Ansari, 2014; Siegler & Booth, 2004). If analogy supports estimation, then the mappings between particular number words and the ANS representations they describe should not be formed via fixed associations (Lyons et al., 2012; Sullivan & Barner, 2012). Thus, the analogical view of estimation predicts that estimates should be most constrained by other estimates (as opposed to by stable word-to-ANS mappings), and this predicts that mappings are likely to be malleable and context-dependent (Sullivan & Barner, 2012; Vul et al., 2013). In other words, there is no reason to believe that a participant who relies heavily on analogical processes when estimating should provide particularly accurate (or inaccurate) estimates consistent with this, young children have shown evidence of providing structurally-consistent estimates long before they provide accurate estimates (Barth et al., 2009; Sullivan & Barner, 2014b). Thus, at the outset, we did not predict that PAE would be related to analogical reasoning ability, because analogically based estimates need not be accurate. PAE was calculated by computing the absolute value of the difference between the actual number of dots and the child's estimate (for each trial) and dividing this score by the number of dots.¹ For example, if the child saw 50 dots and estimated it as seventy, the PAE for that trial would be $(|50-70|/50)$ or 0.4. Thus, smaller scores indicated better performance; perfect performance would be indicated by a score of 0. In order to de-confound any effects of calibration on accuracy, we only computed PAE for the Uncalibrated estimation task. Participants' estimates were quite inaccurate: mean PAE was 0.82 (range: 0.431–2.517).

We next constructed two estimation measures (Ordinality, Linear r^2) that we expected to be related to analogical reasoning ability because they tracked the internal consistency of estimates. A coarse form of internal consistency is the preservation of Ordinality a child who preserves ordinality (described in detail below) preserves the relative ordering of estimates such that large numerosities are consistently labeled with larger number words. A more fine-grained form of internal consistency is the linearity of estimates, such that estimates are not only properly ordered relative to one another, but also appropriately spaced relative to one another. The idea here is that one way for

children (who are inaccurate estimators) to generate internally consistent estimates is to rely on an analogically-based structure mapping between the linguistic and nonlinguistic number system. Thus, there should be a relationship between analogical reasoning skill and performance on estimation measures that index internal consistency.

We first assessed internal consistency by calculating each child's Linear r^2 . This measure has been previously reported to predict math outcomes (Gunderson, Ramirez, Beilock, & Levine, 2012; Kolkman, Kroesbergen, & Leseman, 2013; Moore 1 In number-line estimation tasks, the typical formula for PAE is $|\text{number} - \text{estimate}|/100$. Here, 100 represents the maximum possible estimate (e.g., Booth & Siegler, 2006). Unlike number-line estimation tasks, our task did not have an upper bound, and therefore it was not possible to calculate PAE using the standard formula. The formula used in the present study captures the amount of error in participants' estimates, and is also consistent with what we know about the ANS (e.g., that the amount of error in estimation should scale up with numerosity; Whalen, Gallistel, & Gelman, 1999). For this reason, it may not be possible to directly compare our PAE value with those previously reported in the number-line literature, although, as in previous literature, larger PAE scores indicate less accurate performance. (Ashcraft, 2015; Siegler & Booth, 2004; Booth & Siegler, 2006, 2008), and captures the internal consistency of children's estimates. Why would children's estimates be internally consistent? One possibility is that very accurate estimates will be, by their nature, internally consistent. However, our children did not provide accurate estimates (see PAE analyses above). Alternatively, even inaccurate estimates can be internally consistent, as long as the child uses some sort of stable structure mapping to support their estimates (Sullivan & Barner, 2014b). By this view, high levels of internal consistency, as indexed by the child's Linear r^2 , should only emerge in inaccurate estimators if the child has a strong structural understanding of the relation between the linguistic and nonlinguistic number systems, and if this structural understanding is built on knowledge of the relative distance between numbers (e.g., that 40 is twice as large as 20). We calculated Linear r^2 by

fitting a line predicting each child's estimate from the number being estimated (for the Uncalibrated condition only), and reported the r^2 value of the best fit line. As with all r^2 values, the possible range of scores was 0–1 (with 1 indicating perfect internal consistency), and our mean Linear r^2 was 0.42 (range 0.044–0.905).

Next, we calculated each child's Ordinality score (Sullivan & Barner, 2014a,b), which measures the extent to which the child's estimates obeyed the ordinal structure of the count system. To do this, we coded each trial's estimate for whether it was in the correct direction relative to the previous estimate; for example, if a larger number of dots was shown on trial n than on trial $n-1$, the estimate would be labeled as ordinal if and only if the child provided a larger estimate on trial n than on trial $n-1$. We then calculated the proportion of ordinal responses for each participant, and reported the proportion of ordinal responses in the Uncalibrated condition ($M = 0.720$, range = 0.488–0.976).

Finally, we tested whether participants were influenced by calibration; in other words, we asked whether our misleading feedback ("the largest number of dots you will see is 750") influenced participants' estimates (e.g., Izard & Dehaene, 2008). By the view that analogy supports estimation, misleading feedback about a single number should only influence other estimates if mappings are mutually constraining and structurally-based. The effect of calibration was calculated as in previous work (Sullivan & Barner, 2012, 2014a): we constructed a model in which each participants' estimates were predicted by the number of dots presented, whether they were completing the Calibrated or Uncalibrated portion of the estimation task, and the interaction term (of number presented and calibration). Participants were coded as being influenced by calibration if there was either a main effect of calibration (suggesting that their estimates of the same numbers differed systematically when they were Calibrated, compared to when they were Uncalibrated) or an interaction (suggesting that their estimates of only the largest numbers were influenced by calibration). In this way, the effect of calibration was a binary measure.

Performance on the Conceptual Analogy ($M = 0.54$; range = 0–1), Number Analogy ($M = 0.324$; range = 0–0.833), Raven's ($M = 0.473$, range = 0.167–0.833), number memory ($M = 0.175$, range = 0–0.600), and color memory ($M = 0.506$, range = 0.200–1.00) tasks were all calculated by taking the proportion of correct responses for each child.² Participants' highest count was simply the highest number they counted to without making a mistake ($M = 34.360$, range = 7–100)

2.2. Analyses

For all analyses, we standardized each variable by converting each child's score to a z-score. This allowed us to more directly compare predictive relations across measures and scale.

For each estimation outcome, we constructed a linear model predicting the estimation measure of interest from each of our two analogical reasoning measures (Conceptual Analogy, Number Analogy) separately. No previous study has attempted to directly relate individual differences in analogical reasoning ability to individual differences in estimation ability, and so these univariate correlations are an important and easily interpretable way to test whether there is a relation between analogical reasoning and estimation. We next constructed a larger model that predicted our estimation measures of interest from all of the analogical reasoning measures that had significantly predicted the estimation outcome in the univariate models, with additional terms for each of our control measures (counting, age, number memory, color memory, and Raven's). This allowed us to test whether any of our analogy measures uniquely predicted estimation outcomes.

We report standardized parameter estimates (B), standard errors (SE), and p-values, and when applicable, we report 95% confidence intervals. For all non-binomial models, we also calculated Bayes Factors (BF) using the BayesFactor package in R (Morey & Rouder, 2015); unless otherwise noted, Bayes Factors were calculated from univariate (one predictor predicting one outcome) analyses.

We first tested whether any of our analogical reasoning measures was related to PAE. Because PAE indexes accuracy, we did not predict that it would be related to analogical reasoning

skill. Surprisingly, our univariate analyses revealed a relation between Number Analogy and PAE (Number Analogy: $B = -0.296$, $SE = 0.142$, $p = 0.043$; $BF = 1.61$), suggesting that for every 1 SD increase in Number Analogy score, there was an estimated 0.296 SD decrease in PAE (CI: -0.583 , -0.009 ; note that lower PAE scores are better (Fig. 2)). There was no such relation between PAE and Conceptual Analogy ($B = -0.143$, $SE = 0.148$, 2 Note that previous work relating number memory to estimation also used the r^2 value of line of best fit for the linear model predicting the number that the child recalled from the number that the child actually had to remember (Thompson & Siegler, 2010). To be consistent with previous literature, we also calculated our score for number memory in this fashion; however this score correlated poorly with color memory performance ($B = 0.12$, $p > 0.05$; compare to the memory scores reported in main analyses, $B = 0.34$, $p = 0.02$). This lack of correlation is troubling because we would expect any two memory measures with the same source (i.e. children were asked to remember color and number from a single story) to be correlated with one another. Importantly, when we used Thompson and Siegler's (2010) method, none of the results reported here changed.

Fig. 2. Relation between raw scores on the Number Analogy task and raw PAE (lower scores on PAE indicate better performance). Note that this predictive relation disappears when controlling for other, non-analogical tasks. Each point is a child. ($p = 0.339$; $BF = 0.42$).

When we constructed a linear model predicting PAE from Number Analogy and all control tasks, we found that Number Analogy no longer contributed to the model ($B = 0.282$, $SE = 0.162$, p

= 0.089; CI: 0.61, 0.05), and no control tasks predicted PAE (all $B < 0.20$, all $p > 0.20$, all $BF < 0.81$). This suggests that it is unlikely individual differences in analogical reasoning predict estimation accuracy.

We next tested whether Linear r^2 was related to analogical reasoning skill. As described earlier, internal consistency in estimates (independent from the accuracy of those estimates) is a hallmark of structure mapping, the analogical process that potentially underlies word-to-ANS mappings. More specifically, children who have a nuanced understanding of the relative distance between numbers (e.g., that 40 is not just bigger than 20, but is actually twice as big as 20) should be best able to provide internally consistent estimates, because their structural understanding of the number systems is relatively sophisticated. Thus, we predicted a priori that Linear r^2 should be related to analogical reasoning skill and might be especially related to children's ability to complete number analogies (which also required an understanding of the precise distance between numerals, and not just their relative ordering). In the univariate analysis, we found that Number Analogy was significantly related to Linear r^2 ($B = 0.352$, $SE = 0.140$, $p = 0.015$; $BF = 3.53$ (Fig. 3)), such that for every 1 SD increase in Number Analogy score, there was an estimated 0.352 SD increase in Linear r^2 (CI: 0.07, 0.63). However, we found no such relation between Conceptual Analogy and Linear r^2 ($B = 0.057$, $SE = 0.149$, $p = 0.704$; $BF = 0.31$). When we constructed a linear model predicting Linear r^2 from Number Analogy and all control tasks, we found that Number Analogy still contributed to the model ($B = 0.399$, $SE = 0.163$, $p = 0.019$; CI: 0.07–0.73), and no control tasks predicted Linear r^2 (all $B < 0.15$, all $p > 0.40$; all $BF < 0.41$). These data suggest that there is a strong relation between 4- and 5-year-olds' ability to complete number analogies (e.g., 22: 44 :: 220) and their ability to provide internally consistent linear estimates.

Next, we considered whether our analogical reasoning measures predicted the ordinality of participants' estimates. Ordinality captures the internal-consistency of the ordering of estimates, and can detect internal consistency even in estimates that are wildly inaccurate or that do not appear to

represent the relative distance between numbers accurately. For example, it is still possible for a child who doesn't consistently correctly represent the relative distance between numbers (e.g., treats the distance between 8 and 10 as equivalent to the distance between 10 and 90) to correctly order their estimates, and doing so would demonstrate an understanding of the structural relation between the linguistic and nonlinguistic number

Fig. 3. Relation between raw scores on the Number Analogy task and raw Linear r^2 (high Linear r^2 indicates highly internally-consistent estimates). Each point is a child.

Fig. 4. Relation between raw scores on the Conceptual Analogy (top panel) and Number Analogy

(bottom panel) task and Ordinality. Each point is a child.

systems (e.g., both are ordered systems). We found that both analogical reasoning tasks predicted Ordinality (Number Analogy: $B = 0.288$, $SE = 0.143$, $p = 0.050$, $BF = 1.47$; Conceptual Analogy: $B = 0.325$, $SE = 0.141$, $p = 0.026$, $BF = 2.36$), such that a 1 SD increase in Number Analogy predicted a 0.29 SD increase in Ordinality (CI:0.0005, 0.58), while a 1 SD increase in Conceptual Analogy predicted a 0.325 (CI:0.04, 0.61) SD increase in Ordinality (Fig. 4).

When considering the model predicting Ordinality from all of the analogy measures and all control measures, Raven's predicted Ordinality ($B = 0.515$, $SE = 0.133$, $p < 0.001$; CI: 0.246, 0.783), while Number Analogy ($B = 0.24$, $SE = 0.14$, $p = 0.10$) and Conceptual Analogy ($B = 0.026$, $SE = 0.148$, $p = 0.861$) did not. No other control measures added any predictive value to the model (all $B < 0.25$, $p > 0.2$, all $BF < 1.87$). Thus, it appears as though Raven's is the strongest predictor of Ordinality.

2.2.1. Calibration analyses

Our main goal was to test whether performance on analogical reasoning tasks was related to the likelihood that a child would be affected by calibration. However, because the effects of calibration in children have only been demonstrated in one previous paper (Sullivan & Barner, 2014a), we first sought to determine the extent to which our dataset replicated previously reported calibration effects. To do this, we used the lme4 package in R to predict participants' estimates from the number that participants saw (Numerosity), whether they were calibrated (Calibrated vs. Uncalibrated), and their interaction, with subject as a random factor (Bates, Machler, Bolker, & Walker, 2015). As in Sullivan and Barner (2014a), there was an effect of Numerosity ($B = 0.905$, $SE = 0.04$, $t = 20.91$) and an interaction of Calibration and numerosity ($B = 0.276$, $SE = 0.066$, $t = 4.202$) such that calibration influenced estimates for large, but not small, numerosities. Considering individual participants, 22 (47%) of our participants were affected by calibration (compared to 60%

of 5-year-olds in Sullivan & Barner, 2014a). Using the same analytic process as was reported in Sullivan and Barner (2014a), we also assessed the smallest number at which there was an effect of calibration as in Sullivan and Barner (2014a), the smallest number at which there was an effect of calibration was 12. Thus, our dataset replicated the effect of calibration reported in Sullivan and Barner (2014a), but with younger children.

In order to assess our main research question, we next tested whether any of our analogical reasoning tasks predicted the likelihood that a given child would be affected by calibration. As previously reported, participants' estimates should only be influenced by calibration if they have mapped their linguistic number system onto their nonlinguistic number system via structure mapping, a holistic, analogy-based process (Sullivan & Barner, 2012, 2014a). By adulthood, the vast majority of people are influenced by calibration which suggests that analogical structure mapping is important for mature estimation; Sullivan & Barner, 2012). However, because around half of five-year-olds are influenced by calibration (Sullivan

Fig. 5. Mean performance on the Conceptual Analogy task in the Calibrated and Uncalibrated conditions. Error bars are 95% CIs.

& Barner, 2014a), we suspected that individual differences in analogical reasoning skills might

predict whether children in our dataset would be influenced by calibration. Because our calibration measure was binary (the child either was, or was not, influenced by calibration), we tested this by constructing binomial logit models predicting calibration from each of our analogical reasoning measures. The likelihood of being influenced by calibration was not predicted by our Number Analogy task ($B = 0.07$, $SE = 0.30$, $p = 0.80$), but was predicted by our Conceptual Analogy task ($B = 0.718$, $SE = 0.341$, $p = 0.035$, Odds Ratio: 2.05; CI: 1.1, 4.3). A post-hoc Welch's two samples t-test also confirmed that mean scores on the Conceptual Analogy task differed for participants who were calibrated, when compared to those who were not influenced by calibration ($t(44.981) = 2.297$, $p = 0.036$) (Fig. 5). However, when constructing a model predicting calibration from Conceptual Analogy and all of our control tasks, the predictive relation between Conceptual Analogy performance and calibration became only marginally significant ($B = 0.726$, $SE = 0.391$, $p = 0.064$; OR: 2.066), with the 95% confidence interval showing odds ratios that just barely overlapped with 1 (CI: 0.994–4.756). No other control measures added predictive value to the model (all $B < 0.35$, all $p > 0.3$). Thus, while our data are consistent with the view that individual differences on the Conceptual Analogy task are related to the likelihood of being influenced by calibration, when controlling for other factors like memory and Raven's, we cannot be certain that this effect is uniquely predictive.

2.3. Experiment 1 discussion

We had four main findings. First, estimation accuracy, as indexed by Percent Absolute Error (PAE), was not reliably predicted by analogical reasoning performance, when controlling for other factors. Second, domain-specific numerical analogical reasoning skills appeared to be related to Linear r^2 , suggesting that an analogically based understanding of the relative distance between numbers supports estimation. Third, while both our domain-general Conceptual Analogy and our domain-specific Numerical Analogy tasks correlated with estimation Ordinality, Raven's Progressive Matrices was by far the best predictor of Ordinality. Fourth, after replicating previous

work (e.g., Sullivan & Barner, 2014a) showing that misleading feedback calibrates children's estimates, we showed that there was a potential predictive relation between performance on the domain-general Conceptual Analogy task and the likelihood that a child's estimates would be influenced by misleading feedback.

2.4. General discussion

Estimation requires mapping two numerical systems that differ radically in representational format (the linguistic vs. nonlinguistic number system) to one another. We hypothesized that children map these two systems to one another using analogical structure mapping. To test this, we asked whether individual differences in analogical reasoning ability predicted preschoolers' estimation ability. We found that individual differences in four- and five-year-olds' analogical reasoning ability predicted several measures of estimation ability. These data are broadly consistent with the view that analogical reasoning supports estimation (Carey, 2009; Sullivan & Barner, 2014a,b; Thompson & Opfer, 2010), and provide the first empirical evidence that performance on analogical reasoning tasks uniquely predicts estimation performance. These data are important for at least two reasons. First, they speak to the broader question of how we connect language to perception, suggesting that analogical processes may be fundamental to the formation of early word-to-perception mappings. Second, estimation performance has long been linked to math outcomes; by demonstrating that numerical estimation is deeply related to analogical reasoning, we raise important questions for understanding early mathematical education.

The starting point for the present study was the idea that children who have very limited experience with numbers are unlikely to have sufficient experience with specific word-magnitude mappings to support estimation. Consistent with this view, we found that children's estimates were wildly inaccurate, even though they were frequently internally consistent. While accurate estimates are necessarily internally-consistent and obey the underlying structure of the number systems, in the absence of accuracy, we need an alternative explanation for this internal consistency. The use of an

analogically based structure mapping to support estimation would predict exactly what we found: that internally-consistent estimates arise prior to accurate estimates. In what follows, we outline additional evidence from the present study that analogical reasoning is uniquely related to estimation performance.

First, we consider evidence about the relationship between the internal consistency of children's estimates and their analogical reasoning ability. We had two main measures of estimation internal-consistency: Ordinality and Linear r^2 . Our Ordinality measure indexed the extent to which children provided estimates in the same direction relative to previous estimates. Ordinality correlated with performance on both our Conceptual and Number Analogy tasks; however, when control measures were included in our model, our in-house analogy tasks no longer uniquely predicted Ordinality; instead, Raven's (and nothing else) did. How can we interpret this finding? On the one hand, it may be the case that preschoolers with better problem-solving abilities are better at completing many tasks (including estimating and Raven's), thus driving the predictive relation between Raven's and Ordinality. Against this view, Raven's didn't predict other estimation outcomes. Alternatively, several researchers have described Raven's as fundamentally an analogical reasoning task (Carpenter, Just, & Shell, 1990; Lovett, Forbus, & Usher, 2007): in order to solve the task, participants must understand the relations between visually presented items, and apply these relations to a new context (e.g., filling in the blank). This process is conceptually very similar to other analogical reasoning tasks (e.g., "Foot goes with sock just like hand goes with _____"; in fact, Raven's was correlated with our Conceptual Analogy task: $B = 0.361$, $SE = 0.139$, $p = 0.013$). Thus, we cannot rule out the possibility that Raven's predicts Ordinality precisely because it requires analogical reasoning skills. Specifically, Raven's requires children to relate items to one another based on size, number, and relative spatial configuration these are precisely the sorts of domain-general analogical relations that might support the development of an ordinality-based structure mapping between number language and the ANS.

Our second measure of estimation internal-consistency (Linear r^2) was robustly predicted by our Number Analogy task, and this remained the case even in models containing all of our control measures. Estimates will only be internally consistent (in the absence of accuracy) if children are able use the relative distance between numbers (e.g., that 60 is twice 30) to support their estimates (Sullivan & Barner, 2014b). Importantly, it is exactly this sort of relative-distance information that was tested in our Number Analogy task (e.g., $20 : 40 :: 200 : \text{?}$). Thus, we provide strong evidence that the ability to complete numerical analogies uniquely predicts the linearity of estimates (as indexed by Linear r^2) in preschoolers. This makes sense unlike Ordinality (which only requires tracking relative magnitude, a type of relation that could be measured by a number of domain-general analogical reasoning tasks, including Raven's), children must have number-specific knowledge of the relative distance between numerals (as assessed in our Number Analogy task) in order to provide consistently linearly increasing estimates. In this sense, it seems that domain-specific numerical analogy (and not just analogical reasoning in general) could support the development of Linear r^2 . From an educational perspective, this is important because Linear r^2 has been shown to correlate with math performance (Booth & Siegler, 2006, 2008; Gunderson et al., 2012; Kolkman et al., 2013; Moore & Ashcraft, 2015; Siegler & Booth, 2004).

Finally, we tested whether analogical reasoning performance predicted whether a child was influenced by calibration during an estimation task. The influence of calibrating feedback on estimation (Sullivan & Barner, 2012, 2014a) has been argued to demonstrate that structure mapping underlies estimation. Here, we replicated previous work demonstrating that calibration influences 5-year-olds' estimates (Sullivan & Barner, 2014a), and, new to this paper, showed that 4-year-olds are also influenced by calibration. We also found that performance on our Conceptual Analogy task predicted whether a child would be calibrated; while this predictive relation was only marginally significant when control tasks were included in the model, our best estimates suggested that for every one standard deviation increase in performance on the Conceptual Analogy task, the child

was twice as likely to be influenced by calibration. Taken together, these data suggest that analogical reasoning is deeply related to the process of numerical estimation. These findings open up exciting new directions for understanding both the development of numerical competence in childhood and how children relate language to perception during development.

The present study is consistent with the view that numerical estimation is supported by and related to analogical reasoning ability. However, it is important to consider the extent to which the analogical abilities that support estimation are domain- general vs. domain-specific. With respect to the present study, for example, children's analogical reasoning was assessed using three tasks that differed in their emphasis on the quantitative: the Numerical Analogy task assessed domain-specific numerical analogical skills; Raven's (when thought of as an analogical reasoning task) assessed more general quantitative and spatial analogical skills; while the Conceptual Analogy task assessed domain-general non-quantitative analogical reasoning skills. Interestingly, individual differences in each of these measures were related to individual differences in different components of estimation. While, as described above, measures like Linear r^2 seem especially likely to be supported by domain-specific numerical analogical skill, it is less clear whether domain-general or domain-specific analogical reasoning skills support other components of estimation development. Future research should aim not only to replicate the relations reported in the present study, but also to better understand the extent to which domain-general analogical reasoning skills may support estimation.

This study raises several exciting directions for future research, along with a number of theoretical questions that must be addressed in the coming years. First, while we have demonstrated that analogical reasoning is related to numerical estimation, the role that analogical reasoning plays in the acquisition of the meanings of early number words remains open. When children initially learn to count, they lack meanings for the words in their count list. These meanings are then populated, one-by-one, over the course of many months (Wynn, 1992), until the child recognizes

the Cardinality Principle, and can use counting to produce the appropriate quantities (e.g., a set of 3 fish when asked for three) for every number in their count list. Some have proposed that the acquisition of these number meanings may be facilitated by some “wild analogy” (Carey, 2004). Using tasks similar to those presented in the present paper (which successfully obtained analogical reasoning measures even in very young children), future research should consider the ways in which the acquisition of early number meanings might be supported by analogy.

More generally, the study of analogical reasoning is plagued by a beautiful problem: analogy is a very powerful and yet, as currently theorized, very general process. At the most basic level, analogy is the extension of a relation within one domain to a new domain. Under this definition, the catalogue of tasks that recruit “analogical reasoning” can be construed as infinitely broad. This breadth is a challenge for scientists, and, in future years, it will be important for those interested in the role of analogy in cognitive processes to articulate precisely which processes underlie analogical reasoning (in general), and which structural and comparative computations underlie performance on given tasks (in specific).

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