Deterministic Input, Noisy Mixed Modeling for Identifying Coexisting Condensation Rules

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Abstract

In cognitive diagnosis models, the condensation rule describes the logical relationship between the required attributes and the item response, reflecting an explicit assumption about the cognitive processes respondents engage in to solve problems. In practice, multiple condensation rules may apply to an item simultaneously, indicating that respondents should use multiple cognitive processes with different weights to identify the correct response. Coexisting condensation rules reflect the complexity of cognitive processes utilized in problem solving and that the cognitive processes respondents employ in determining item responses may be inconsistent with the expert-designed condensation rule. This study was conducted to evaluate the proposed deterministic input with a noisy mixed (DINMix) model to identify coexisting condensation rules and provide feedback for item revision to increase the validity of the measurement of cognitive processes. Two simulation studies were conducted to evaluate the psychometric properties of the proposed model. The simulation results indicate that the DINMix model can adaptively and accurately identify coexisting condensation rules, existing either simultaneously in an item or separately in multiple items. An empirical example was also analyzed to illustrate the applicability and advantages of the proposed model.

Keywords: cognitive diagnosis; condensation rule; cognitive diagnosis models; DINMix model

1. Introduction

Existing cognitive diagnosis models (CDMs) and diagnostic classification models (e.g., Rupp et al., 2010; von Davier & Lee, 2019) can be classified into three categories based on the condensation rule of how latent attributes (e.g., skills, knowledge, and cognitive processes) influence individuals' observed item responses: conjunctive, disjunctive, and compensatory (Maris, 1995, 1999). Models with the conjunctive condensation rule, such as the deterministic input, noisy 'and' gate (DINA) model (Junker & Sijtsma, 2001), assume that respondents must master all the required attributes of an item to provide the correct response. In contrast, models with the disjunctive condensation rule, such as the deterministic input, noisy 'or' gate (DINO) model (Templin & Henson, 2006), assume that respondents can provide the correct response to an item if they have mastered any of the required attributes. Furthermore, models with the compensatory condensation rule, such as the additive CDM (ACDM) model (de la Torre, 2011), assume that for the correct response probability, the mastery of a particular attribute may compensate for the non-mastery of another attribute.

Applying a CDM should be based on a suitable condensation rule, which is usually designed by experts during the test/item development phase. Condensation rules designed by experts define the theoretical relationship between the measured attributes and the observed item responses, reflecting an explicit assumption about the cognitive processes engaged in by respondents during problem solving. However, in practice, the cognitive processes respondents employ in their item responses may be inconsistent with the expert-designed condensation rule. This phenomenon indicates that the validity of the measurement of cognitive processes is low and has been ignored by most previous studies.

In addition to reduced CDMs with a specific condensation rule, general CDMs involving saturated interaction terms have also been developed, such as the generalized DINA model (GDINA; de la Torre, 2011), log-linear CDM (LCDM; Henson et al., 2009), and general diagnostic model (GDM; von Davier, 2008). General CDMs account for the complexity of cognitive processes from the perspective of statistical analysisⁱ and are therefore not limited to a specific condensation rule, which increases their theoretical application scope. However, this 'rough' approach causes general CDMs to obscure the theoretical relationship between the required attributes and the item response, as it sidesteps the validity of the measurement of cognitive processes. When general CDMs are used, it is difficult to gain a clear understanding of how the required attributes affect correct response probability, which reduces the interpretability of diagnostic results, especially in relation to cognitive processes.

Compared to general CDMs, reduced CDMs may be more appropriate in practice for four reasons. First, reduced CDMs usually require smaller sample sizes for accurate and robust parameter estimation (cf. Jiang & Ma, 2018). Second, the simpler model is preferred if its performance is not significantly worse than that of the more complex model, according to the parsimony principle (e.g., Beck, 1943). Third, item parameters in reduced CDMs typically have more straightforward interpretations, which makes working with them attractive to practitioners (Liu et al., 2009). Fourth, reduced models more clearly reflect the cognitive processes engaged in to solve problems, which increases the interpretability of diagnostic results.

To strike a balance between the general models and reduced models, de la Torre and Lee (2013) and Ma and coworkers (2016) proposed the Wald test approach for selecting a suitable reduced model

from the GDINA model for each item. This approach assumes that each item belongs to only one suitable condensation rule, for example, either the conjunctive or the disjunctive. However, after using this method in practice, one might realize that its results may indicate that the required attributes in an item may satisfy multiple condensation rules simultaneously (i.e., coexisting condensation rules in an item). Specifically, this can be observed in two phenomena. For some items, no reduced model applies to them except the general model. This indicates that the relationship between the required attributes and the item response cannot be simply described by a specific condensation rule. For some other items, there is more than one reduced model with insignificant differences from the general model. Such results indicate that multiple condensation rules may be simultaneously apply to the same item and only differ in degree. These two phenomena can be found when applying the Wald test to the data contained in some previous studies (e.g., Ma et al., 2016; Jang, 2009; Ravand et al., 2013), and they indicate that the correct response to an item may have uncertainty regarding the requirement of cognitive processes. From the perspective of statistical analysis, if such uncertainty can be taken into account in modeling, it may increase the degree of model-data fitting, thus improving diagnostic accuracy. From the perspective of measurement, if we can identify such uncertainty and feed it back into item revision to ensure that the cognitive process used by respondents matches the expert-designed cognitive process, it would help improve the validity of the measurement of cognitive processes.

Unlike general CDMs, which 'roughly' introduce saturated interaction terms to consider the potential for coexisting condensation rules, the current study aims to identify coexisting condensation

rules to provide feedback for item revision. Inspired by but different from mixture/hybrid item response models (e.g., Mislevy & Verhelst, 1990; von Davier, 2008b; Yamamoto, 1989), a new model called the deterministic inputs, noisy mixed (DINMix) model is proposed. The proposed model allows each item to contain multiple condensation rules simultaneously, thus realizing the identification of different condensation rules. Furthermore, the item response function of the proposed model is more concise than that of the general CDMs, which also increases the interpretability of item parameters to some extent, making it more attractive to practitioners.

The remaining sections of this paper are constructed as follows. First, deterministic input, noisy models with three different condensation rules and mixture item response models are briefly reviewed. Next, the proposed DINMix model is presented, followed by two simulation studies that evaluate the psychometric properties of the proposed model. An empirical example is then analyzed to illustrate the applicability and advantages of the proposed model. Finally, the paper concludes with a discussion of the limitations of the proposed model and suggestions for further research.

2. Background

2.1. *Deterministic input, noisy models with typical condensation rules*

In psychometric models, a commonly used item response function of the relationship between observed and latent responses can be expressed as follows:

$$P(y_{ni} = 1 | g_i, s_i, \omega_{ni}) = g_i + (1 - s_i - g_i)\omega_{ni}, \quad (1)$$

where y_{ni} is the observed (dichotomous) response of person n ($n = 1, \dots, N$) to item i ($i = 1, \dots, I$); ω_{ni} is the latent response of person n to item i ; g_i and s_i are the guessing and slip parameters of item i ,

respectively. Typically, a monotonicity restriction, $g_i < 1 - s_i$, can be imposed (Junker & Sijtsma, 2001; Culpepper, 2015). With various choices of ω_{ni} , Equation 1 can describe many psychometric models, such as the four-parameter logistic unidimensional or multidimensional item response models (Reckase, 2009) and the multicomponent latent trait model (Embretson, 1984).

In CDMs, the latent response, ω_{ni} , is the concrete expression of the condensation rule. When assuming that attributes follow the conjunctive condensation rule, the latent response has

$$\omega_{ni} = \prod_{k=1}^K \alpha_{nk}^{q_{ik}}, \quad (2)$$

where α_{nk} is the mastery status of person n of attribute k ($k = 1, \dots, K$), $\alpha_{nk} = 1$ means mastery and $\alpha_{nk} = 0$, otherwise. The Q-matrix (Tatsuoka, 1983) is an I -by- K confirmatory matrix with element q_{ik} indicating whether attribute k is required to answer item i correctly; $q_{ik} = 1$ if the attribute is required, and $q_{ik} = 0$ otherwise. With the conjunctive condensation rule, Equation 1 becomes the DINA model.

In contrast, when assuming that the attributes follow the disjunctive condensation rule, the latent response has

$$\omega_{ni} = 1 - \prod_{k=1}^K (1 - \alpha_{nk})^{q_{ik}}, \quad (3)$$

and Equation 1 becomes the DINO model.

Conjunction and disjunction are two extreme condensation rules, and both divide individuals into two groups: full (non)mastery and partial mastery. Individuals in the partial mastery group are not further differentiated by these rules. To mitigate this issue, the ratio condensation rule can be adopted, where

$$\omega_{ni} = \frac{\sum_{k=1}^K \alpha_{nk} q_{ik}}{\sum_{k=1}^K q_{ik}}. \quad (4)$$

With the ratio condensation rule, Equation 1 becomes the deterministic input, noisy ratio (DINR) model. The ratio condensation rule is a special compensatory condensation rule that assumes increases in the number of mastered attributes are linearly related to increases in correct response probability. In such cases, the DINR model can be considered a special case of the ACDM with the equal main effect of each attribute (i.e., $\frac{1-s_i-g_i}{\sum_{k=1}^K q_{ik}}$). Thus, the DINR model assumes that all required attributes are equally weighted. The DINR model is considered instead of the ACDM to avoid introducing additional item parameters, which keeps the model as simple as possible.

2.2. Mixture item response models

Mixture/hybrid item response models are well-studied and primarily used for detecting and explaining the differential behavior of individuals of multiple latent groups in the population (i.e., population heterogeneity). Typically, in mixture item response models, multiple latent groups of individuals are assumed to be present in the sampled population. These models have been widely applied to diverse research issues, such as identifying problem-solving strategies (Mislevy & Verhest, 1990; Wang & Xu, 2015) and detecting differential item functioning (Cohen & Bolt, 2005). Furthermore, for diagnostic classification purposes, von Davier (2008b) proposed the mixture GDM, which extends the GDM to tests with multiple observed and latent groups. Choi (2010) proposed a diagnostic classification mixture Rasch model, which incorporates the mixture Rasch model (Rost, 1990) and the LCDM. Yamaguchi and Okada (2020) proposed a hybrid CDM, which is defined as a

weighted mixture of the DINA model and the DINO model. This model assumes that there are two latent groups in the population; one group of respondents' responses are suitable for analysis using the DINA model, and the other group' responses are suitable for analysis using the DINO model. However, the distinguishing feature of the proposed model in this study is that mixture modeling is used to accommodate the simultaneous existence of multiple condensation rules rather than multiple latent groups of individuals.

3. Deterministic Input, Noisy Mixed Model

3.1. Model formulation

As stated previously, the purpose of this study is to present a new CDM that can incorporate and identify coexisting condensation rules. In the current study, we interpret the coexisting condensation rules within an item, as the required attributes affect individuals' item responses according to multiple condensation rules with different proportions. Thus, when an item contains coexisting condensation rules, it means that respondents should use multiple cognitive processes with different weights to respond correctly to this item.

Inspired by but different from the mixture item response models, the item response function of the DINMix model can be expressed as

$$P(y_{ni} = 1 | g_i, s_i, \psi_{ni}) = g_i + (1 - s_i - g_i)\psi_{ni}, \quad (5)$$

where ψ_{ni} is the mixed latent response of person n to item i , which can be further defined as

$$\psi_{ni} = \sum_{m=1}^{M_i} \tau_{im} \omega_{nim}, \quad (6)$$

where ω_{nim} is the latent response of person n to item i in condensation rule m ($m = 1, \dots, M_i$); τ_{im} is the

item-level mixing proportion parameter of condensation rule m in item i , which satisfies $\tau_{im} \in [0, 1]$ and $\sum_{m=1}^{M_i} \tau_{im} = 1$ for each item. The item-level mixing proportion parameters can be interpreted as the proportions/weights of different cognitive processes required to correctly solve the problem. M_i is the number of preselected condensation rules for item i . For simplicity, but without losing generality, it can be assumed that $M_i = M$ for all items throughout this study.

In this study, three typical condensation rules are considered for each item: conjunctive (see Equation 2), disjunctive (see Equation 3), and ratio (see Equation 4) condensation rules. Thus, Equation 6 can be further expressed as

$$\Psi_{ni} = \tau_{i1} \prod_{k=1}^K \alpha_{nk}^{q_{ik}} + \tau_{i2} \left(1 - \prod_{k=1}^K (1 - \alpha_{nk})^{q_{ik}} \right) + \tau_{i3} \frac{\sum_{k=1}^K \alpha_{nk} q_{ik}}{\sum_{k=1}^K q_{ik}}. \quad (7)$$

Finally, the DINMix model can be expressed as

$$P(y_{ni} = 1 | g_i, s_i, \tau_i, \alpha_{nk}, q_{ik}) = g_i + (1 - s_i - g_i) \left(\tau_{i1} \prod_{k=1}^K \alpha_{nk}^{q_{ik}} + \tau_{i2} \left(1 - \prod_{k=1}^K (1 - \alpha_{nk})^{q_{ik}} \right) + \tau_{i3} \frac{\sum_{k=1}^K \alpha_{nk} q_{ik}}{\sum_{k=1}^K q_{ik}} \right), \quad (8)$$

where all parameters have been defined above.

Equation 8 is an example of the DINMix model with three typical condensation rules. In practice, by considering uncertainty during the item development phase, experts can assist in determining which condensation rules may be included in the DINMix model for each item in a particular test. For example, when any two of the three condensation rules are pre-selected by experts, Equation 8 can be reduced to another mixed model with only two condensation rules. Furthermore, the DINA, DINO,

and DINR models can all be viewed as special cases of the DINMix model by preselecting a specific condensation rule for all items. If it is difficult for experts to pre-select all the possible condensation rules for a specific item, then Equation 8 can be used to identify potential condensation rules from the data-driven perspective.

Theoretically, the number of item parameters in the DINMix model is $2I + (M - 1)I$, where $M = 3$ in this study. However, when only one attribute is required by an item, there are no differences among all the condensation rules. Thus, an additional constraint can be added to the DINMix model to reduce the number of item parameters, which means that for unidimensional items, the test-level mixing proportion parameters are set as $\boldsymbol{\tau}_i = (\tau_{i1}, \tau_{i2}, \tau_{i3})' \equiv (1, 0, 0)$, indicating that the DINA model is forcibly assigned to unidimensional items.ⁱⁱ

The concept of coexisting condensation rules is more difficult to interpret than that of a single condensation rule. Although coexisting condensation rules describe the complex relationship between the required attributes and the item response, they reduce the interpretability of the model parameters when compared to reduced models. The interpretable and meaningful insights gained from the model are particularly important in educational and psychological applications to meet the need for accountability (Zhan, 2020). Thus, using the proposed model to correctly identify items containing coexisting condensation rules is of considerable practical significance. It may improve the degree of model-data fitting, thus improving diagnostic accuracy, as well as help guide item revision to make revised items contain a single expert-defined condensation rule, thereby improving the validity of the measurement and increasing the interpretability of the model parameters.

3.2. *Bayesian parameter estimation*

The Bayesian Markov chain Monte Carlo (MCMC) method is used to estimate model parameters. In this study, JAGS (version 4.3.0) software was used to automate the estimation process. The corresponding code for the DINMix model and the other models used in this study are available at <https://osf.io/s2yjb/>. More details about how to use JAGS for Bayesian CDM estimation are provided in a tutorial published by Zhan and colleagues (2019).

3.3. *Relationship with existing CDMs*

With three typical and representative condensation rules, the DINMix model is sufficient to cover various relationships between the required attributes and the item response in most cases. Thus, the DINMix model is a more general model than the three aforementioned reduced models. As previously stated, the DINA, DINO, and DINR models can all be viewed as special cases of the DINMix model by preselecting a specific condensation rule.

The differences among the four deterministic inputs, noisy models with the different condensation rules described above are listed in Table 1, along with the item correct response probabilities of eight individuals with different attribute profiles for an item ($\mathbf{q}_i = (1, 1, 1)$, $g_i = s_i = .1$, $\tau_{i1} = .8$, $\tau_{i2} = .1$) based on them. First, compared to the DINA and DINO models, the DINR and DINMix models can better reflect the differences in various attribute profiles. Second, the values of item-level mixing proportions indicate that this item contains three condensation rules simultaneously, and the proportion of the conjunctive condensation rule is the highest. The DINMix model not only differentiates individuals in the partial mastery group but also reflects the feature of the conjunctive condensation rule (i.e.,

individuals in the partial mastery group have low correct-response probabilities).

Additionally, although the DINMix model was originally developed for adaptively identifying coexisting condensation rules, it can still be viewed as a constraint model of the GDINA model after some parameter transformations (see section A1 in the online supplement). It should be noted that for each within-item multidimensional item, the number of item parameters in the GDINA model increases as the number of required attributes increases, while the number of item parameters in the DINMix model is always four (i.e., s_i , g_i , π_{i1} , and π_{i2}). Therefore, similar to the DINA model, the DINMix model has a stable set of item parameters that are easier to interpret than those (e.g., main effects, two-way interactions, three-way interactions, and so on) in the GDINA model. More importantly, the DINMix and GDINA models were developed for different purposes. The former was developed for identifying coexisting condensation rules, and the latter was developed for model generalization.

Table 1. The Correct Response Probabilities of Four Deterministic-Inputs, Noisy Models for an Item.

Attribute Profile	Number of Mastered Attributes	DINA	DINO	DINR	DINMix
(0, 0, 0)	0	0.1	0.1	0.1	0.1
(1, 0, 0)	1	0.1	0.9	0.367	0.207
(0, 1, 0)	1	0.1	0.9	0.367	0.207
(0, 0, 1)	1	0.1	0.9	0.367	0.207
(1, 1, 0)	2	0.1	0.9	0.633	0.233
(1, 0, 1)	2	0.1	0.9	0.633	0.233
(0, 1, 1)	2	0.1	0.9	0.633	0.233
(1, 1, 1)	3	0.9	0.9	0.9	0.9

Note: Item characteristics are $\mathbf{q}_i = (1, 1, 1)$, $g_i = s_i = 0.1$, and $\tau_{i1} = 0.8$, $\tau_{i2} = 0.1$, where $\mathbf{q}_i$ is the required attribute profile of item i , g_i , and s_i is the guessing and slip parameters of item i , respectively; τ_{i1} is the item-level mixing proportion parameter for the conjunctive condensation rule; τ_{i2} is the item-level mixing proportion parameter for the disjunctive condensation rule; DINA = deterministic input, noisy ‘and’ gate model; DINO = deterministic input, noisy ‘or’ gate model; DINR = deterministic input, noisy ratio model; DINMix = deterministic input, noisy mixed model.

3.4. Parameter identifiability

Parameter identifiability is an important issue in CDMs, as it is necessary for the consistent estimation of model parameters and valid statistical inferences (e.g., Chen et al., 2020; Fang et al., 2019; Gu & Xu, 2018, 2019). As mentioned above, the DINMix model can be viewed as a constraint model of the GDINA model. According to Gu and Xu's (2018) classification of CDMs, the DINMix model is a multi-parameter Q-restricted latent class model. Thus, the conditions for generic identifiability (i.e., conditions C5 and C6 in Theorem 7), which were also given by Gu and Xu (2018), are also applicable to the DINMix model.

4. Simulation Studies

Two simulation studies were conducted. The purpose of simulation Study 1 was to determine whether parameters of the DINMix model could be recovered accurately, especially whether the proposed model can correctly identify items that contain coexisting condensation rules, in which the data were simulated from the DINMix model and analyzed. Simulation Study 2 was conducted to compare the performance of the proposed model and that of some other CDMs in six simulated test situations to illustrate the relative advantages and disadvantages of the proposed model.

4.1. Study 1

4.1.1. Design and data generation.

In simulation Study 1, five factors were manipulated, including (a) sample size (N) at two levels, 500 and 1,000, (b) test length (I) at two levels, 15 and 30, and (c) item quality (IQ) at two levels, higher and lower. Referencing Zhan and colleagues(2019), item parameters were generated from a

bivariate normal distribution with a negative correlation coefficient using

$$\begin{pmatrix} \text{logit}(g_i) \\ \text{logit}(s_i) \end{pmatrix} \sim MVN_2 \left(\begin{pmatrix} -2.197 \\ -2.197 \end{pmatrix}, \begin{pmatrix} 1.0 & -0.6 \\ -0.6 & 1.0 \end{pmatrix} \right), \quad (9)$$

for higher quality items and

$$\begin{pmatrix} \text{logit}(g_i) \\ \text{logit}(s_i) \end{pmatrix} \sim MVN_2 \left(\begin{pmatrix} -1.386 \\ -1.386 \end{pmatrix}, \begin{pmatrix} 1.0 & -0.6 \\ -0.6 & 1.0 \end{pmatrix} \right), \quad (10)$$

for lower quality items. These settings led to the guessing and slip probabilities for all items following a positively skewed distribution (mean ≈ 0.1 , minimum ≈ 0.01 , and maximum ≈ 0.6 for higher quality, and mean ≈ 0.2 , minimum ≈ 0.05 , and maximum ≈ 0.7 for lower quality). Assuming that guessing and slip parameters follow a negative correlation is more realistic (Zhan et al., 2019). Additionally, (d) the type of item-level mixing proportion (TM) at two levels of uniform and skew mixing was manipulated. Since the first $I/3$ items in the Q-matrix are unidimensional items, we set $\boldsymbol{\tau}_{1 \sim I/3} = (1, 0, 0)$. Then, for uniform mixing, we set $\boldsymbol{\tau}_{(I/3+1) \sim I} = (1/3, 1/3, 1/3)$. In contrast, for skew mixing, we set $\boldsymbol{\tau}_{(I/3+1) \sim 8I/15} = (0.6, 0.2, 0.2)$, $\boldsymbol{\tau}_{(8I/15+1) \sim 11I/15} = (0.2, 0.6, 0.2)$, and $\boldsymbol{\tau}_{(11I/15+1) \sim I} = (0.2, 0.2, 0.6)$. Furthermore, (e) the latent structural model of attributes (LSM) at two levels of an unstructured and a multivariate normal distribution was manipulated. When an unstructured LSM was used, the true attribute profile of each person was randomly chosen from all possible patterns with equal probability; in these cases, the tetrachoric correlations among attributes were approximately zero. In contrast, when a multivariate normal distribution was used, a continuous latent variable matrix was first generated from a five-dimensional multivariate normal distribution (e.g., Chiu et al., 2009):

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{pmatrix} \sim MVN_5 \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0.6 & 1 & & & \\ 0.6 & 0.6 & 1 & & \\ 0.6 & 0.6 & 0.6 & 1 & \\ 0.6 & 0.6 & 0.6 & 0.6 & 1 \end{pmatrix} \right). \quad (11)$$

Then, the true attribute was determined by

$$\alpha_{nk} = \begin{cases} 0 & \theta_{nk} < 0 \\ 1 & \theta_{nk} \geq 0 \end{cases}. \quad (12)$$

In this case, tetrachoric correlations among the attributes were approximately 0.6.

Five attributes ($K = 5$) were measured, and the simulated Q-matrices are presented in Figure 1. Each Q-matrix contained at least one identity matrix, and each attribute was measured at least three times, which satisfies the conditions for generic identifiability (i.e., conditions C5 and C6 in Theorem 7) described by Gu and Xu (2018). Finally, the observed responses were generated from $y_{ni} \sim \text{Bernoulli}(p_{ni})$, where p_{ni} was given in Equation 8 in the main text.

Figure 1. K -by- I Q' matrix for simulation Study 1.

Note: blank means '0,' and gray means '1'; '*' denotes items used in the $I = 15$ conditions.

4.1.2. Analysis

Thirty replications were implemented in each simulated condition. For each replication, two Markov chains with random starting points were used, and 10,000 iterations were run for each chain. The first 5,000 iterations in each chain were discarded as burn-in. The remaining 10,000 iterations (5,000 in each chain) were retained for model parameter inferences. The potential scale reduction factor (PSRF; Brooks & Gelman, 1998) was computed to assess the convergence of each parameter. In

this study, the PSRFs were generally less than 1.01, suggesting good convergence in the specified settings.

To evaluate parameter recovery, the bias and root mean square error (RMSE) of the item parameter estimates were computed, as $\text{bias}(x) = \sum_{r=1}^R \hat{x}_r - x_r / R$ and $\text{RMSE}(x) = \sqrt{\sum_{r=1}^R (\hat{x}_r - x_r)^2 / R}$, where $\hat{x}$ and x were the estimated value and true value of the model parameters, respectively; R was the total number of replications. The attribute correct classification rate (ACCR) and attribute pattern correct classification rate (PCCR) were computed to evaluate classification accuracy, as $\text{ACCR}_k = \sum_{r=1}^R \sum_{n=1}^N I(\hat{a}_{nr} = a_{nr}) / NR$ and $\text{PCCR} = \sum_{r=1}^R \sum_{n=1}^N I(\hat{a}_{nr} = a_{nr}) / NR$.

4.1.3. Results

Figure 2 summarizes the recovery of attributes (details can be found in Table S1 in the online supplements). Referencing previous studies of CDMs with unstructured LSM (e.g., Ma et al., 2016; Zhan et al., 2019), the classification accuracy of the DINMix model under different conditions is adequate and consistent with expectations. Furthermore, increasing the test length and item quality yielded higher classification accuracy. Higher correlations among attributes (i.e., when attributes were simulated from a multivariate normal distribution) led to higher classification accuracy. The sample size had a limited effect on classification accuracy. The classification accuracy in conditions with the skew type of an item-level mixing proportion seems slightly better than in conditions with the uniform type of item-level mixing proportion.

Figure 3 summarizes the recovery of item parameters (details can be found in Table S2 in the online supplement). It should be noted that for item-level mixing proportion parameters, only the

recovery of them in the multidimensional items (i.e., items 6–15 under $I = 15$ conditions and items 11–30 under $I = 30$ conditions) was computed. The recovery of guessing and slip parameters was better than that of item-level mixing proportion parameters across all conditions. Larger sample sizes, longer test lengths, and higher item quality led to better item parameter recovery. Guessing and slip parameters were well recovered across different conditions. Specifically, the recovery of guessing and slip parameters under conditions with the skew type of item-level mixing proportion seems better than under conditions with the uniform type of item-level mixing proportion. Conversely, the recovery of item-level mixing proportion parameters under conditions with the skew type of item-level mixing proportion was worse than under conditions with the uniform type of item-level mixing proportion, especially for lower quality items. Generally, the recovery of τ_3 was worse than that of τ_1 and τ_2 , mainly because $\tau_{i3} = 1 - (\tau_{i1} + \tau_{i2})$, and thus, τ_{i3} needs to offset both estimation errors of τ_{i1} and τ_{i2} .

Overall, the results of simulation Study 1 indicate that model parameters for the DINMix model can be well recovered, as the proposed model can correctly identify items that contain coexisting condensation rules via the proposed Bayesian MCMC estimation method, especially in conditions with a larger sample, longer test length, and higher item quality.

Figure 2. Summary of the recovery of attributes in simulation Study 1.

Note: LSM = latent structural model; IQ = item quality; N = sample size; I = test length; ACCR = attribute correct classification rate; PCCR = attribute pattern correct classification rate.

Figure 3. Summary of the recovery of item parameters in simulation Study 1.

Note: g = guessing parameter; s = slip parameter; τ_1 = item-level mixing proportion parameter for the conjunctive condensation rule; τ_2 = item-level mixing proportion parameter for the disjunctive condensation rule; τ_3 = item-level mixing proportion parameter for the ratio condensation rule; LSM = latent structural model; IQ = item quality; N = sample size; I = test length; ACCR = attribute correct classification rate; PCCR = attribute pattern correct classification rate.

4.2. Study 2

4.2.1. Design and data generation

In simulation Study 2, six test situations were simulated. Specifically, including the situations in which the relationships between item responses and required attributes (a) simply follow the conjunctive condensation rule (i.e., the DINA model was used to generate data), (b) simply follow the disjunctive condensation rule (i.e., the DINO model was used to generate data), (c) simply follow the ratio condensation rule (i.e., the DINR model was used to generate data), (d) simply follow the compensatory condensation rule (i.e., the ACDM was used to generate data), (e) fuzzily follow some unclear condensation rules (i.e., the GDINA model was used to generate data), and (f) separately follow some clear condensation rules (i.e., five separate CDMs were combined to generate data, including DINA, DINO, DINR, ACDM, and GDINA models). Note that all the above situations are unfair to the DINMix model because it is not used as the data generation model in any of them.

Other factors were set as control variables. More specifically, the number of attributes, sample size, and test length were fixed at 5, 1,000, and 30, respectively. The Q-matrix and corresponding allocation plan of the data generation model are presented in Figure 4. The Q-matrix contained two identity matrices, and each attribute was measured at least three times to make all the mentioned CDMs identifiable (Gu & Xu, 2018, 2019). In all simulated test situations, since the first 10 items were unidimensional (i.e., all CDMs were equivalent), the DINA model was forcibly assigned to them for simplicity. In the simulated test situation (f), for items 11–14, the DINA model was used as the true

model; for items 15–18, the DINO model was used as the true model; for items 19–22, the DINR model was used as the true model; for items 23–26, the ACDM was used as the true model; for items 27–30, the GDINA model was used as the true model. To ensure that data were only determined by the data generation model, items with the same required attributes were assigned to each model, and all item parameters were set to fixed values.

The upper and lower limits of correct response probability were set as 0.9 and 0.1, respectively. Thus, in the DINA, DINO, and DINR models, g_i and s_i were both fixed at 0.1. In the ACDM, the intercept parameters were all fixed at 0.1; for two-dimensional items, two main effects were fixed at 0.5 and 0.3, respectively; for three-dimensional items, three main effects were fixed at 0.35, 0.25, and 0.2, respectively. In the GDINA model, the intercept parameters were all fixed at 0.1; for two-dimensional items, two main effects and one two-way interaction effect were fixed at 0.35, 0.25, and 0.2, respectively; for three-dimensional items, three main effects, three two-way interaction effects, and one three-way interaction effects were fixed at 0.15, 0.1, 0.05, 0.05, 0.1, 0.15, and 0.2, respectively. The unstructured latent structural model was used (i.e., the true attribute profile of each person was randomly chosen from all possible patterns with equal probability). Finally, the observed responses of each item were generated from the corresponding model presented in Figure 1. Thirty datasets were generated in each condition.

Test Simulation	Data Generation Model															
(a) Conjunctive	DINA								DINA							
(b) Disjunctive	DINA								DINO							
(c) Ratio	DINA								DINR							
(d) Compensatory	DINA								ACDM							
(e) Fuzzily	DINA								GDINA							
(f) Separately	DINA								DINA				DINO			
Q-matrix																

Figure 4. *K*-by-*I* Q' matrix for simulation Study 2 and the data generation model for each item.

Note: blank means '0' and gray means '1'; DINA = deterministic input, noisy 'and' gate model; DINO = deterministic input, noisy 'or' gate model; DINR = deterministic input, noisy ratio model; ACDM = additive cognitive diagnosis model; GDINA = generalized DINA model; DINMix = deterministic input, noisy mixed model.

4.2.2. Analysis

In the simulated test situations (a) to (e), only the true model and the DINMix model were used to fit the generated data; while in the simulated test situation (f), five models were used to fit the generated data: DINA, DINO, DINR, DINMix, and GDINA models. All models were estimated using the Bayesian MCMC method. Analysis processes were identical to those used in simulation Study 1 (see section S1 in the online supplement). In addition to bias, the RMSE, the PCCR, the deviance information criterion (DIC; Spiegelhalter et al., 2002), and the log conditional predictive ordinate (Kim & Bolt, 2007; Levy & Mislevy, 2016, p. 247) multiplied by -2 (i.e., $-2LCPO$) were computed for model selection. Both the test- and item-level $-2LCPO$ were reported. A smaller value of the DIC and $-2LCPO$ indicates a better model-data fit.

4.2.3. Results

Table 2 presents the overall performance of six models in simulation Study 2, including model-data fitting, the recovery of item parameters, and the recovery of attributes (details can be found

in Table S3 in the online supplements). For the first five test situations, it was unsurprising that the overall performance of the DINMix model was worse than that of the data generation model itself, but the relative disadvantage of the former was very small, even in these extremely unfair test situations. Additionally, both model-data fitting indicators can successfully identify the true model, indicating that the degree of model-data fitting can be used as evidence of the validity of the measurement of cognitive processes. For the last test situation, the overall performance of the six models was evaluated in the following relative order: $GDINA \geq DINMix \gg ACDM \geq DINR > DINA > DINO$. More specifically, first, the GDINA model performs best in complex test situations, as expected, because it is a saturated model; second, the performance of the DINMix model was slightly worse than that of the GDINA model; third, the performance of the first two models was much better than that of the last four reduced models; fourth, the ACDM performed only slightly better than the DINR model.

Figure 5 presents the estimates of item-level mixing proportion parameters of the DINMix model in six test situations. First, by using the estimate of $\tau_{i1} > 0.9$ and $\tau_{i2} > 0.9$ as a judgment condition, τ_{i1} and τ_{i2} can accurately identify the conjunctive and disjunctive condensation rules for each item across all test situations, respectively. For τ_{i3} , we cannot make a judgement directly by using a certain cut-point (e.g., 0.9), as for τ_{i1} and τ_{i2} ; however, by judging whether τ_{i3} is simultaneously larger than τ_{i1} and τ_{i2} , the items that follow the ratio/compensatory condensation rule can still be identified. Second, when the relationships between item responses and required attributes fuzzily followed some condensation rules (i.e., the test situation (e)), τ_{i1} , τ_{i2} , and τ_{i3} seemed to show the following pattern: the proportion of τ_{i1} and τ_{i3} was much higher than that of τ_{i2} . Of course, this pattern may change depending

on the simulated values of different item parameters in the GDINA model. Third, for test situation (f), even in such a complex test situation, the DINMix model can identify the condensation rules followed by each item by adaptively adjusting the estimates of τ_{i1} , τ_{i2} , and τ_{i3} . If the identified condensation rules for some items were found to be inconsistent with those predefined by experts, then the experts may consider revising these items.

To better illustrate the comparative advantages and disadvantages of the proposed model, the following discussion focuses on test situation (f). Figure 6 summarizes the recovery of the item parameters of six models (details can be found in Table S4 in the online supplement). Only the recovery of the upper and lower limits of the correct response probability (i.e., the guessing and slip parameters) were computed. The DINMix and GDINA models performed well in recovering the parameters in all items and were unaffected by the true condensation rule followed by each item. Additionally, when the true condensation rule followed by an item did not match the condensation rule adopted by the analysis model, the DINA model substantially overestimated the guessing parameter, while the DINO model substantially overestimated slip parameters, and the DINR model's results were a bit more complicated. Specifically, when the true condensation rule was conjunctive, the DINR model substantially overestimated slip parameters and slightly underestimated guessing parameters; however, when the true condensation rule was disjunctive, the DINR model substantially overestimated guessing parameters and slightly underestimated slip parameters. Furthermore, the ACDM still performed similar to the DINR model.

Figure 7 displays the item-level $-2LCPO$ of the six models (details can be found in Table S5 in the

online supplement). The GDINA and the DINMix models were more effective than the other four reduced models in fitting the data on most, if not all, items, regardless of the true model for each item. The DINA, DINO, DINR, and ACDM models had a relative high degree of item-level fit, but only when their adopted condensation rule conformed to the true condensation rule followed by an item; in contrast, the GDINA and DINMix models seemed to be unaffected by the true condensation rule followed by the item.

Overall, despite these unfair test situations, the DINMix model performed well. The results of simulation Study 2 indicate that (a) the DINMix model can adaptively identify different condensation rules existing separately in multiple items, and (b) the overall performance of the DINMix model is similar to that of the GDINA model, both of which are suitable for complex test situations. However, it should be noted that the DINMix and GDINA models deal with complex test situations in different ways. The former identifies condensation rules followed by different items by adjusting the item-level mixing proportion parameters, while the latter accounts for all possible condensation rules through saturated interaction effects. Thus, the analysis results of the former is more helpful for item revision from the perspective of improving validity.

Table 2. Summary of the Results of Simulation Study 2.

Test Situation	Analysis Model	DIC	Test_2LCPO	RMSE_g	RMSE_s	PCCR
(a) Conjunctive	DINA	20748.69	21201.02	0.012	0.021	0.805
	DINMix	20852.34	21256.41	0.024	0.022	0.804
(b) Disjunctive	DINO	20746.62	21200.64	0.022	0.013	0.806
	DINMix	20827.94	21251.88	0.022	0.024	0.803
(c) Ratio	DINR	29963.85	30625.57	0.020	0.020	0.853
	DINMix	30020.02	30676.92	0.021	0.021	0.852
(d) Compensatory	ACDM	29613.86	30390.21	0.020	0.020	0.853

	DINMix	30084.68	30713.51	0.022	0.021	0.844
(e) Fuzzily	GDINA	28541.86	29260.53	0.020	0.022	0.845
	DINMix	28743.04	29372.46	0.023	0.024	0.840
(f) Separately	DINA	29348.12	29971.11	0.195	0.035	0.746
	DINO	30078.30	30616.86	0.039	0.227	0.726
	DINR	28386.28	28972.58	0.079	0.093	0.827
	ACDM	28266.83	28911.75	0.079	0.092	0.828
	GDINA	25595.61	26341.86	0.022	0.022	0.900
	DINMix	25744.26	26397.83	0.021	0.022	0.899

Note: The value is the mean value of 30 replications; DIC = deviance information criterion; Test₋₂LCPO = test-level -2 log conditional predictive ordinate; RMSE = root mean square error; PCCR = attribute pattern correct classification rate; DINA = deterministic input, noisy ‘and’ gate model; DINO = deterministic input, noisy ‘or’ gate model; DINR = deterministic-inputs, noisy ratio model; ACDM = additive cognitive diagnosis model; GDINA = generalized DINA model; DINMix = deterministic-inputs, noisy mixed model.

Figure 5. Estimates of the item-level mixing proportion parameter in simulation Study 2 from the DINMix model.

Note: The value is the mean value of 30 replications; τ_1 = item-level mixing proportion parameter for the conjunctive condensation rule; τ_2 = item-level mixing proportion parameter for the disjunctive

condensation rule; τ_3 = item-level mixing proportion parameter for the ratio condensation rule; DINMix = deterministic-inputs, noisy mixed model; GDINA = generalized DINA model.

Figure 6. Summary of the recovery of item parameters in simulation Study 2.

Note, RMSE = root mean square error; DINA = deterministic input, noisy ‘and’ gate model; DINO = deterministic input, noisy ‘or’ gate model; DINR = deterministic-inputs, noisy ratio model; ACDM = additive cognitive diagnosis model; GDINA = generalized DINA model; DINMix = deterministic-inputs, noisy mixed model.

Figure 7. Item-level -2LCPO of six models in simulation Study 2.

Note: The value is the mean value of 30 replications; $-2\text{LCPO} = -2 \log$ conditional predictive ordinate;

DINA = deterministic input, noisy ‘and’ gate model; DINO = deterministic input, noisy ‘or’ gate model; DINR = deterministic-inputs, noisy ratio model; ACDM = additive cognitive diagnosis model; GDINA = generalized DINA model; DINMix = deterministic-inputs, noisy mixed model.

5. An Empirical Example

A commonly used empirical dataset was used as an example to illustrate the applicability and advantages of the proposed model: the fraction subtraction (FS) dataset (Tatsuoka, 2002). Since this dataset has been widely used in many previous studies, it does not need to be described in detail here. In addition to the DINA, DINO, DINR, ACDM, GDINA, and DINMix models, a selected mixing model via the Wald test (Ma et al., 2016) was also used to fit the data. For the first six models, the analysis processes were identical to those used in simulation studies. To make the results comparable, for the selected mixing model, the Wald test was first used to select a suitable reduced CDM for each item using the GDINA package (Ma & de la Torre, 2020) in R software. The candidate reduced models included the DINA, DINO, and ACDM. Then, according to the selected modelsⁱⁱⁱ, Bayesian estimation was used. The DIC and $-2LCPO$ were computed for model selection.

The FS dataset consists of responses given by 536 individuals to 20 items measuring 8 attributes. The Q-matrix was published by Tatsuoka (2002)^{iv}. Previous studies have shown that the DINA model can fit this data well (e.g., DeCarlo, 2011; de la Torre & Douglas, 2004). Table 3 presents the DIC and test-level $-2LCPO$ of seven models. The DINMix model was preferred based on the DIC, and the GDINA model was preferred based on the test-level $-2LCPO$. Additionally, Figure 5 displays the item-level $-2LCPO$ of the seven models. The DINA, DINMix, GDINA, and selected mixing models have approximate fittings, mainly because the latter three also agree that the conjunctive condensation rule is more suitable for most items.

Figure 7 presents the estimated item parameters of the DINMix model for the FS data (the results of other models are provided in Tables S6 and S7 in the online supplement). The estimates of τ_{ims} indicate that the conjunctive condensation rule accounts for the largest proportion in most items. When using the judgement rules obtained in simulation Study 2 (e.g., $\tau_{i1} > 0.9$ or $\tau_{i2} > 0.9$), it was difficult to determine which specific condensation rule that items 1, 4, 5, 12, 14, 16, and 18 followed, primarily because they were judged to contain coexisting condensation rules. Similarly, the results of the Wald test also suggest that no particular condensation rule applied to items 1, 4, 12, 14, and 16. This consistency also indicates that the proposed model can further explain why the Wald test cannot find a specific condensation rule for some items. Additionally, take item 14, $3\frac{4}{5} - 3\frac{2}{5}$,^v as an example. Attributes α_2 (separate a whole number from a fraction) and α_7 (subtract numerators) were required to give a correct response according to the Q-matrix. However, respondents who mastered α_7 but not α_2 could still identify that the correct answer was 2/5. The first reason is that respondents may ignore the integer part and only focus on the difference of the fraction part. The second reason is that alternative attributes that are unspecified by the Q-matrix can probably be used to answer this item, such as *convert mixed number to fraction* (Mislevy, 1996). Apparently, for whatever reason, the expert-defined conjunctive condensation rule does not fully apply to this item, which is also what the DINMix model indicated.

Table 3. The DIC and –2LCPO of Seven Models for the Fraction Subtraction Data.

Analysis Model	DIC	Test-level –2LCPO
DINA	8347.341	7882.304
DINO	9038.505	8665.394
DINR	8911.684	8293.461
ACDM	10528.705	7826.567

GDINA	11096.800	7690.248
DINMix	8330.826	7803.378
Wald-selected	8785.672	7783.750

Note: DIC = deviance information criterion; $-2\text{LCPO} = -2 \log$ conditional predictive ordinate; DINA = deterministic input, noisy ‘and’ gate model; DINO = deterministic input, noisy ‘or’ gate model; DINR = deterministic-inputs, noisy ratio model; ACDM = additive cognitive diagnosis model; GDINA = generalized DINA model; DINMix = deterministic-inputs, noisy mixed model; Wald-selected = selected mixing model via Wald test.

Figure 6. Item-level -2LCPO of seven models for the fraction subtraction data.

Note: DINA = deterministic input, noisy ‘and’ gate model; DINO = deterministic input, noisy ‘or’ gate model; DINR = deterministic-inputs, noisy ratio model; ACDM = additive cognitive diagnosis model; GDINA = generalized DINA model; DINMix = deterministic-inputs, noisy mixed model; Wald-selected = selected mixing model via Wald test.

Figure 7. The estimated item parameters for the fraction subtraction data from the DINMix models.

Note: g = guessing parameter; s = slip parameter; τ_1 = item-level mixing proportion parameter for the conjunctive condensation rule; τ_2 = item-level mixing proportion parameter for the disjunctive condensation rule; τ_3 = item-level mixing proportion parameter for the ratio condensation rule; DINMix = deterministic-inputs, noisy mixed model.

6. Summary and Discussion

The condensation rule describes the logical relationship between the required attributes and the

item response, which reflects the explicit assumption about the cognitive processes respondents engage in to solve problems. When an item contains coexisting condensation rules, it means that the contribution of required attributes to the correct item response probability follows multiple condensation rules with different proportions. Coexisting condensation rules reflect the complexity of cognitive processes in problem solving, as well as reflect that the cognitive processes respondents employ in their item responses are inconsistent with the expert-designed condensation rule. This study proposed the DINMix model to identify coexisting condensation rules for the purpose of providing feedback for item revision. Two simulation studies were conducted to evaluate the psychometric properties of the proposed model. The simulation results indicate that (a) the model parameters for the DINMix model can be well recovered; (b) the DINMix model can adaptively and accurately identify coexisting condensation rules, either existing simultaneously in an item or existing separately in multiple items; (c) the overall performance of the DINMix model is similar to that of the GDINA model, both of which are suitable for complex test situations. An empirical example was also analyzed to illustrate the applicability and advantages of the proposed model.

The work represented in this article is an initial attempt to simultaneously consider multiple condensation rules in a single CDM. Despite the promising results, some limitations remain. First, the utilized model framework (see Equations 1 and 5) models aberrant responses at the item level. However, in practice, such aberrant responses may occur at the attribute rather than the item level, such as noisy input, deterministic, and gate model (Junker & Sijtsma, 2001). Exploring ways to incorporate attribute-level aberrant responses into the proposed model is worthy of further research, as Equation 11 (de la Torre, 2011) seems to give us a reference. Second, within-item characteristic dependency (Zhan et al., 2019), which means that dependency exists between the guessing and slip parameters within an item, was not considered in the proposed model. It can be incorporated into the

proposed model to increase the estimation accuracy of item parameters in a future study. Third, only the dichotomous scoring item and binary attribute were modeled in the proposed model. It would be meaningful and practical to extend the current model to consider polytomous scoring items (e.g., Ma & de la Torre, 2016) and polytomous attributes (e.g., Zhan et al., 2020). Fourth, in recent years, some studies have focused on Q-matrix validation or estimation (Chen, Culpepper, Chen, & Douglas, 2018; de la Torre & Chiu, 2016) and multiple strategies for problem solving (Ma & Guo, 2019), which are not covered in the current study. Fifth, model identifiability is very important for valid statistical inferences. Although the conditions for generic identifiability for the multi-parameter Q-restricted latent class model are also applicable to the DINMix model, it remains to be explored whether there are specific identifiable conditions for the DINMix model. Sixth, inspired by the modeling logic of mixture item response models, this study proposed the idea of the coexistence of multiple factors (e.g., condensation rules) at the item level. In fact, this item-level mixed-modeling idea may have some extended uses worth exploring, such as coexisting problem-solving strategies and coexisting Q-matrices (i.e., q-vectors, to be exact) at the item level. Seventh, notably, the generalizability of the findings of this study is dependent upon the limitations of the design of the simulation studies, such as using a fixed number of attributes and assuming the Q-matrix is correct. To further generalize these findings, a wider range of simulated conditions should be considered in future studies.

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ⁱ Specifically, it follows the data-first philosophy (i.e., by developing more complex models to realize the analysis of data in complex situations). In contrast, for the perspective of measurement, considering the viewpoint of garbage-in, garbage-out for predetermined measurement objects (e.g., attributes and cognitive processes) to ensure the reliability and validity of measurement, factors unrelated to the measurement should be excluded (i.e., be revised or be deleted) as much

as possible.

ii Any one element in τ_i can be set to 1 because all CDMs (for binary attributes) are identical to each other for unidimensional items.

iii According to the p -values of the Wald test, items 1, 4, 12, 14, and 16 are applicable to the GDINA model, while the remaining items are applicable to the DINA model.

iv The author does, however, note that this Q-matrix has been suggested to be revised by some previous studies (e.g., Chen et al., 2018). But it is still fair to compare several models with the same Q-matrix, especially when there is no definite conclusion about the revision of this Q-matrix.

v The specific content of each item can be found in Table 1 in DeCarlo (2011).