A Hierarchical Gaussian Signal Detection Model with Unequal Variance for Binary Responses

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Abstract Gaussian signal detection models with equal variance are commonly used in simple yes-no detection and discrimination experiments whereas more flexible models with unequal variance require additional information. Here, a hierarchical Bayesian model with equal-variance is extended to an unequal-variance model so that it is applicable to binary responses from a random sample of participants. This unequal-variance model is investigated analytically, in simulations and three applications using existing data sets. The results suggest that parameters of this model are accurate and reliable, providing a promising alternative to the ubiquitous equal-variance model.

Keywords Bayesian inference · MCMC sampling · bivariate normal distribution · latent parameter estimation

1 Introduction

Signal detection theory has a long tradition in psychophysics and provides a general framework for studying responses under uncertainty (Fechner, 1860; Tanner & Swets, 1954; Green & Swets, 1966; Falmagne, 1985; Macmillan & Creelman, 2005). In a standard signal detection paradigm each participant gives binary responses in a series of randomised signal and noise trials. Responses are usually aggregated for signal and noise trials and summarised in a confusion matrix, by reporting the number of hits, misses, false alarms and correct rejections. The main parameters of interest are typically sensitivity and response bias in a detection task or discriminability together with a decision criterion in a discrimination task. Both parameters are derived from a signal

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Fig. 1 Illustration of signal detection models with equal variance $\sigma_n^2 = \sigma_s^2$ (left panel) and unequal variance $\sigma_n^2 < \sigma_s^2$ (right panel) for a single participant i . The distance between the mean of the signal and noise distribution defines sensitivity or discriminability d_i . The response criterion c_i separates the two response categories and is described as a deviation from the midpoint $d_i/2 = 0$ between the means of the signal and noise distribution. Hit rate θ_i^h and false alarm rate θ_i^f correspond to the hatched area under the signal distribution and the grey area under the noise distribution, respectively.

detection model with overlapping signal and noise distribution defined on a common decision axis (e.g., Wickens, 2002).

Gaussian signal detection models are widely used to estimate discriminability and a response criterion. They fall into two main categories – models with equal and models with unequal variance (Green & Swets, 1966). Equal-variance models assume that signal and noise distributions have different means, yet the same variance, whereas signal detection models with unequal variance are more flexible because they introduce signal variance as an additional free parameter (see Fig. 1).

However, estimating an unequal variance model is not always possible. For a single participant additional information is required because three unknown parameters cannot be estimated from two variables, such as a hit and a false-alarm rate. As a consequence, researchers either settle for an equal-variance model or collect additional data to establish an unequal-variance model.

An unequal-variance model is more flexible than an equal-variance model because it can accommodate not only symmetric but also an asymmetric receiver operating characteristics (ROCs). Asymmetric ROCs have been observed in a number of domains leading to parameter estimates that are less biased than their equal-variance counterparts. (e.g., Rotello, Masson & Verde, 2008; Jang, Wixted & Huber, 2009; Starns & Ratcliff, 2014; Starns, Rotello & Hautus, 2014)

In a typical detection or discrimination experiment each participant performs in several conditions so that an ROC of the participant can be established from observed hit and false-alarm rates across conditions. For example,

if rewards, penalties or proportions of signal and noise trials are systematically manipulated within participants then an equal or unequal variance model can be fit to the hit and false-alarm rates.

In other paradigms, researchers have used magnitude estimation (Stevens, 1957), confidence ratings (Dorfman & Alf, 1969) as well as choices and response times (Starns & Ratcliff, 2014; Weidemann & Kahana, 2016) to approximate individual ROCs and fit signal detection models with unequal variance. Although confidence ratings and continuous response scales are popular in behavioural research their properties as interval scales are debated (e.g., Stevens, 1957; MacKay, 1963; Narens, 1996; Luce, 2002).

Nonetheless, researchers are usually not just interested in individual performances but also in the overall performance of a sample of participants. While aggregating data across participants, items, and other factors can lead to biased parameter estimates, these factors can be exploited in hierarchical models (Gelman et al., 2014). In general, this improves statistical inference because shrinkage towards the population mean makes parameter estimates more robust and less susceptible to noise and outliers (Efron & Morris, 1977). For rich data sets, for example with confidence ratings from many trials, items and participants, various hierarchical signal detection models with unequal variance have been developed. These models are superior to equal-variance models because they estimate discriminability, response criterion and signal variance at the individual and population level, typically providing better data fits and more reliable parameter estimates (e.g., Morey, Pratte & Rouder, 2008; DeCarlo, 2011; Pratte, Morey & Rouder, 2010; Vuurde, 2017; Paulewicz & Blaut, 2020).¹

To the best of my knowledge, however, no hierarchical signal detection model with unequal variance has been developed for binary responses. Rather than observing hits and false-alarm rates from a single participant in multiple conditions, the distribution and variability of observed hit and false-alarm rates across participants is used to estimate signal variance at the population level.

This appears to be at odds with conventional wisdom whereby parameters of an unequal-variance model are not identifiable if only binary responses in a single condition are observed (e.g., Wickens, 2002; p.50). Although this is true for non-hierarchical models, a hierarchical Bayesian model can take advantage of distribution properties at the sampling and population level. More specifically, if individual signal and noise distributions have equal variance then random and independent sampling of discriminability and criterion values across participants should perturb the transformed hit and false-alarm rates in the same way. If, however, individual signal and noise distributions have unequal variance, that is approximately constant across participants, then on average the ratio of the sample variances of the transformed hit and false-alarm rates reveals the scaling of the signal distributions. As a consequence, a hierarchical

¹ Please note that in some publications 'individual' or 'participant' level is labeled 'group' level whereas in other publications 'group' level refers to the 'population' level.

signal detection model with unequal variance can be applied to binary response data if the sample of participants is sufficiently large. This is explained in detail in the following sections.

A signal detection model that can accommodate equal as well as unequal variances is more flexible and therefore produces more accurate parameter estimates than a model that assumes equal variance from the start. In addition, a hierarchical model with unequal variance makes it possible to conduct more detailed analyses of the data because discriminability, response criterion and signal variance can be estimated for each experimental condition. At the same time, this reduces the burden of collecting data across multiple conditions in order to establish individual ROCs and signal variances. It is also easier to interpret the results from binary decisions with a single response criterion than from confidence ratings with several thresholds.

In the following it is shown how signal variance together with discriminability and criterion can be estimated in a hierarchical Bayesian signal detection model with unequal variance. Possible applications of this hierarchical unequal-variance signal detection (hUVSD) model concern binary response data collected from a random sample of participants, animals, subjects or other independent units using the same stimuli and detection or discrimination task. Therefore, the hUVSD model should be of interest to psychologists, behavioural scientists, neuroscientists and biomedical researchers working in different domains.

First, the hierarchical equal-variance signal detection hEVSD model as suggested by Lee and Wagenmakers (2013) is outlined. This model is extended to a hUVSD model by introducing signal variance as an additional scaling parameter that is approximately constant for all participants.

2 Hierarchical Equal-Variance Model

The hierarchical equal-variance signal detection model (hEVSD) is based on binary responses in signal and noise trials from participant i in a sample of $i = 1, \dots, n$ participants (Lee & Wagenmakers, 2013; Chap. 11). Observed number of hits h_i and false alarms f_i are divided by the number of signal s_i and noise n_i trials, respectively, and the resulting rates feature as individual parameter estimates θ_i^h and θ_i^f of two binomial distributions. Next, the rate parameters θ_i^h and θ_i^f are re-parameterised into discriminability d_i and criterion c_i of a Gaussian signal detection model that is based on standard Gaussian cumulative distributions for signal and noise with equal variance (left panel of Fig. 1). Discriminability d_i is defined as the distance between signal and noise distribution for each participant i . The threshold at which participant i selects a Yes or No response is denoted by c_i .

In a linear parameter expansion of the hEVSD model, individual discriminability d_i and criterion c_i are defined as

$$\begin{aligned} d_i &= \mu_d + \xi_d \delta_i^d \\ c_i &= \mu_c + \xi_c \delta_i^c \end{aligned} \quad (1)$$

where δ_i^d has a normal distribution with mean 0 and precision λ_d , the reciprocal of variance λ_d^{-1} . The precision λ_d of the normal distribution has Gamma(0.1, 0.1) as prior distribution.

$$\begin{aligned} \delta_i^d &\sim \mathcal{N}(0, \lambda_d^{-1}) \\ \delta_i^c &\sim \mathcal{N}(0, \lambda_c^{-1}) \end{aligned} \quad (2)$$

Similarly, δ_i^c is normally distributed with mean 0 and precision λ_c . The precision λ_c of the normal distribution has a Gamma(0.1, 0.1) distribution as prior.

In this expanded hEVSD model two auxiliary random variables $\xi_d, \xi_c \sim \text{Beta}(1, 1)$ can be introduced to avoid convergence issues of the equal-variance model with four population parameters (Lee & Wagenmakers, 2013). Introducing independent random variables in a linear expansion may improve convergence because an auxiliary random variable makes the model more flexible and can prevent MCMC sampling from getting stuck in a local extremum (Liu, Rubin & Wu, 1998). However, even if sampling of the critical population parameters converges, ξ_d and ξ_c have no meaning on their own. The main parameters can be estimated but the two auxiliary parameters ξ_d and ξ_c are not sufficiently identified. If, however, the hierarchical equal-variance model is extended to an unequal-variance model, as shown below, then parameters converge and are straightforward to interpret.

3 Hierarchical Unequal-Variance Model

The hUVSD model (right panel of Fig. 1 and Fig. 2) assumes that each hit rate θ_i^h corresponds to an area under a signal distribution with variance σ_s^2 and each false alarm rate θ_i^f to an area under a noise distribution with variance σ_n^2 . Similar to the hierarchical equal-variance model, both cumulative probability estimates are expressed by distance d_i between the mean of signal and noise distribution and the position of criterion c_i for each participant i . In addition, signal variance σ_s^2 may be different from noise variance σ_n^2 . By introducing standard deviation σ_s of the signal distribution and $\sigma_n = 1$ of the noise distribution, that are both approximately constant across participants, the hierarchical equal-variance model with four parameters is extended to a hUVSD model with five parameters at the population level.

Binomial hit and false-alarm rates of each participant i are re-parameterised in a Gaussian signal detection model with unequal variance. The definition of c_i as a deviation from 0, the midpoint between the mean of the noise distribution $-d_i/2$ and the mean of the signal distribution $+d_i/2$, is crucial because

Fig. 2 Graphical representation of the hierarchical unequal-variance signal detection (hU-VSD) model. The model is a straightforward extension of the hierarchical equal-variance model proposed by Lee and Wagenmakers (2013). Square and circular nodes describe discrete and continuous variables whereas gray and white nodes refer to observed (or known) and latent variables, respectively. Nodes with double circles are continuous variables that can be derived from other variables. Arrows indicate parent-child dependencies in a directed acyclic graph. The rectangular plate surrounds nodes that are indexed by $i = 1 \dots n$ participants.

then each pair of individual hit and false-alarm rates depends on the two parameters d_i and c_i , scaled by the same parameter σ_s and $\sigma_n = 1$, respectively.

$$\begin{aligned}\theta_i^h &= \Phi[(+0.5d_i - c_i)/\sigma_s] \\ \theta_i^f &= \Phi[(-0.5d_i - c_i)/\sigma_n]\end{aligned}\quad (3)$$

$\Phi[\cdot]$ denotes the standard Gaussian cumulative distribution function (cdf). The inverse of the Gaussian cdf $\Phi^{-1}[\cdot]$ for each transformed hit and false-alarm rate, written as $z(\theta_i^h)$ and $z(\theta_i^f)$, so that Eq. (3) can be expressed as

$$\begin{aligned}\sigma_s z(\theta_i^h) &= +0.5d_i - c_i \\ \sigma_n z(\theta_i^f) &= -0.5d_i - c_i\end{aligned}\quad (4)$$

Each transformed hit rate $z(\theta_i^h)$ is scaled by the same σ_s whereas each transformed false alarm rate $z(\theta_i^f)$ is multiplied by $\sigma_n = 1$.

Re-arranging the two equation systems and solving for d_i and c_i gives

$$\begin{aligned}d_i &= \sigma_s z(\theta_i^h) - \sigma_n z(\theta_i^f) \\ c_i &= -0.5[\sigma_s z(\theta_i^h) + \sigma_n z(\theta_i^f)]\end{aligned}\quad (5)$$

so that discriminability and criterion values are described by a system of $i = 1 \dots n$ pairs of linear functions where each $z(\theta_i^h)$ is scaled by σ_s and each $z(\theta_i^f)$ is multiplied by $\sigma_n = 1$.

Individual parameters d_i , and c_i can be derived from Eq. (5). They have population mean of μ_d and μ_c and variance of σ_d^2 and σ_c^2 , respectively. Importantly, the ratio of signal and noise variance σ_s^2/σ_n^2 of the hUVSD model corresponds to the ratio of the sample variances $\text{var}[z(\theta^f)]/\text{var}[z(\theta^h)]$ (see Proposition 1 in Appendix A).

3.1 Bivariate Normal Distributions

Discriminability d_i and response criterion c_i are modelled as random draws from independent normal distributions with unknown population means μ_d, μ_c and variances σ_d^2, σ_c^2 . Implementing these assumptions into a generative hUVSD model gives

$$\begin{aligned} d_i &\sim \mathcal{N}(\mu_d, \lambda_d^{-1}) \\ c_i &\sim \mathcal{N}(\mu_c, \lambda_c^{-1}) \end{aligned} \quad (6)$$

where d_i is a draw from a normal distribution with population mean μ_d and variance λ_d^{-1} ($= \sigma_d^2$). Similarly, c_i is a draw from a normal distribution with population mean μ_c and variance λ_c^{-1} ($= \sigma_c^2$).

In many (hierarchical) signal detection models the variance-covariance matrix of the bivariate sampling distribution remains only implicit or is ignored altogether (see however Vuorre, 2017; Suero, Botella & Privado, 2018). Here, independence between discriminability and criterion is stated explicitly

$$\begin{pmatrix} d_i \\ c_i \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \text{ with } \boldsymbol{\mu} = \begin{pmatrix} \mu_d \\ \mu_c \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_d^2 & \rho \sigma_d \sigma_c \\ \rho \sigma_c \sigma_d & \sigma_c^2 \end{pmatrix} \quad (7)$$

so that discriminability and criterion of the bivariate normal distribution approximate a correlation $\rho = 0$ and a covariance $\sigma_{d,c} = \rho \sigma_d \sigma_c = 0$ (see also Fig. 2 and right panel of Fig. 5).

The transformed hit and false alarm rate $z(\theta_i^h)$ and $z(\theta_i^f)$ as illustrated in the left panel of Fig. 5 may be correlated whereas re-parameterised discriminability and criterion estimates are approximately independent. The bivariate sampling distribution of transformed rates is normal and has five observable parameters. After dropping index i , the vector of transformed hit rates $z(\theta^h)$ has a mean of $z(\theta^h)$ and sample variance of $\text{var}[z(\theta^h)]$. Similarly, the vector of transformed false-alarm rates $z(\theta^f)$ has a mean of $z(\theta^f)$ and sample variance $\text{var}[z(\theta^f)]$. In addition, there is the covariance $\text{cov}[z(\theta^h), z(\theta^f)]$. These five observable parameters match up with the five population parameters of the hUVSD model (see Proposition 2 in Appendix A). The correspondence between observable and estimated parameters ensures identifiability of the model

as $\lim(n) \rightarrow \infty$.

$$\begin{pmatrix} z(\theta_i^h) \\ z(\theta_i^f) \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z), \text{ with } \boldsymbol{\mu}_z = \begin{pmatrix} (\mu_d/2 - \mu_c)/\sigma_s \\ -\mu_d/2 - \mu_c \end{pmatrix}, \quad (8)$$

$$\boldsymbol{\Sigma}_z = \begin{pmatrix} (\sigma_c^2 + \sigma_d^2/4)/\sigma_s^2 & (\sigma_c^2 - \sigma_d^2/4)/\sigma_s \\ (\sigma_c^2 - \sigma_d^2/4)/\sigma_s & \sigma_c^2 + \sigma_d^2/4 \end{pmatrix}$$

In the present approach all parameters are estimated by MCMC sampling because there is no known analytical solution of the estimation problem. More specifically, the observed hit θ_i^h and false-alarm rates θ_i^f are modelled as binomial distributions, transformed and re-parameterised in a hierarchical signal detection model with unequal variance with samples d_i and c_i from independent normal distributions.

3.2 Prior Distributions

Following the recommendations for estimating unknown variances in hierarchical models (Gelman, 2006), a conjugate prior for unknown precision of normal-distributed discriminability and criterion is the gamma distribution, written as $\text{Gamma}(\alpha, \beta)$, with shape parameter $\alpha = \nu/2$ and inverse scale or rate parameter $\beta = \nu/2$. If degrees of freedom for the prior are set to $\nu = 1$ then $\text{Gamma}(0.5, 0.5)$ which is equivalent to a chi-square distribution $\chi_1^2(1)$ with scale $s_\alpha^2 = \alpha/\beta = 1$ and degrees of freedom $\nu_\alpha = 2\alpha = 1$. Accordingly, both unknown parameters for precision, λ_d and λ_c , have $\text{Gamma}(0.5, 0.5)$ as their prior.

The truncated normal distribution $\mathcal{N}(1, 2)_{I(0, \infty)}$ is suggested as a prior for signal variance σ_s^2 . The prior is truncated at zero because signal variance cannot be negative. Introducing this ‘informative prior’ may appear contentious. However, this prior favours an equal over an unequal variance model because the distribution is centred on 1.0, leading to a more conservative estimation of unequal variances. Any departure of $\hat{\sigma}_s = 1.0$ is the result of empirical evidence in the data overwhelming the prior. The strength of this evidence depends on the sample of hit and false-alarm rates. If the evidence is strong then the prior can be changed from an ‘informative’ normal to a ‘weakly informative’ gamma or ‘uninformative’ uniform distribution without affecting parameter estimation.

The normal distributed priors for the population means μ_d and μ_c are centred on 0 with large variance, e.g., $\mathcal{N}(0, 1000)$. Both of these ‘weakly-informative’ priors have negligible effects on parameter estimation.

3.3 Marginal Posterior Distributions

A family of t -distributions, $t_\nu(\mu, \sigma^2)$, is characterised by three parameters: location μ , scale σ^2 , and degree of freedom ν which determines the shape of the distribution. The densities of t -distributions are symmetric, and ν must fall in the range $(0, \infty)$. For $\nu = 1$, the t -distribution is equivalent to the Cauchy

distribution with infinite mean and variance. As $\nu \rightarrow \infty$ the t -distribution approximates the normal distribution. Generalised Student's t -distributions have heavier tails than the standard normal distribution and feature frequently in hierarchical Bayesian models because they can accommodate outliers or extreme parameter estimates (e.g., Gelman et al., 2013). and because of their relation to the normal distribution: When the precision (variance) of a normally distributed random variable is unknown and a conjugate prior is placed on the precision (variance) that follows an (inverse-)gamma distribution then the marginal posterior has a Student's t -distribution (e.g., Johnson & Kotz, 1972; Bishop, 2006; Gelman, 2006; Gelman et al., 2013). Similarly, the marginal posterior of the precision (variance) follows a (Inverse-)Gamma distribution.

In the context of the hUVSD model, compounding a normal distribution with a gamma-distributed variance, the marginal posterior distributions for population mean μ_d and μ_c are also t -distributed whereas for variance σ_d^2 and σ_c^2 marginal posterior distributions follow Gamma distributions. However, these (marginal) posterior distributions are difficult to derive analytically. This is the case because population estimates of μ_d and σ_d as well as μ_c and σ_c rely on samples of individual estimates d_i and c_i . These estimates are not directly observable but are linear functions of transformed estimate $z(\theta_i^h)$, scaled by σ_s , and false-alarm rate estimate $z(\theta_i^f)$, multiplied by σ_n . Estimates are therefore conditional on the (transformed) binomial hit and false alarm rates, their variances and covariance.

In summary, the hUVSD model assumes that the five observable parameters of the bivariate hit and false-alarm rate distribution (two means, two variances and a covariance) can be re-parameterised in terms of the five population parameters of the hUVSD model. Unlike other hierarchical models the hUVSD model estimates signal variance from variability across participants rather than from variability across response categories or conditions within participants.

4 Simulations

In order to benchmark the performance of the hUVSD model a series of simulation studies were conducted in R 3.6.3 (R Core Team, 2020) using different population parameter values to generate data for a sample of synthetic participants. The hUVSD model was implemented in JAGS (Plummer, 2006; a clone of BUGS; Lunn et al., 2000) and sampled from 10,000 iterations in 4 chains with random initialisations of λ_c and λ_d using the package `rjags` and `runjags` in R. The first 4,000 iterations (burn-ins) of each chain were removed. Each parameter of the model converged with $\hat{R} < 1.01$ (Gelman & Rubin, 1992).

4.1 Parameter Recovery

First, parameter recovery of arbitrarily chosen population parameter values is investigated. Recovery of parameter values $\mu_d = 2.0$, $\mu_c = -0.5$, $\sigma_s = 1.5$,

Fig. 3 Parameter recovery of the hUVSD model. Discriminability $\hat{d}_i$ (left panel) and criterion $\hat{c}_i$ (right panel) were estimated from simulated data generated from (arbitrarily chosen) population parameter values of $\mu_d = 2.0$, $\mu_c = -0.5$, $\sigma_d = 0.6$, $\sigma_c = 0.4$, and $\sigma_s = 1.5$ randomly sampled from $n=500$ synthetic participants in $k=200$ signal and $k=200$ noise trials.

($\sigma_n = 1.0$), $\sigma_d = 0.6$ and $\sigma_c = 0.4$ is reported for a sample of $n = 500$ synthetic participants with binary responses in $k = 200$ signal and $k = 200$ noise trials. The population parameters define independent normal distributions for discriminability and for response criterion from which random samples are drawn. Each pair of random values in this sample was transformed into a corresponding hit and false-alarm rate using binomial approximations of Gaussian cumulative distribution functions with unequal variance.

In the next step, the hUVSD model was fit to the simulated hit and false alarm rates of a sample of synthetic participants. In Fig. 3 mean estimated posterior values of individual discriminability $\hat{d}_i$ (left panel) and criterion $\hat{c}_i$ (right panel) are compared to the simulated values. The sample of $n=500$ values remains close to the main diagonal. Pearson correlations between estimated and simulated parameter values of $r_d = 0.975$ and $r_c = 0.987$ indicate very good parameter recovery.

4.2 Parameter Bias

Posterior means estimated from simulated data for criterion values μ_c ranging from -0.75 to $+0.75$ are listed in Table 1, for discriminability values μ_d ranging from 0.5 to 2.5 in Table 2 and for signal variances σ_s ranging from 0.5 to 2.5 in Table 3. The simulations illustrate that population parameters are reliably recovered for a wide range of values without systematically affecting other parameter estimates.

If the hEVSD model rather than the hUVSD model is fit to simulated binary data generated by an unequal variance model with $\sigma_s^2 > \sigma_n^2$ then estimates are systematically biased, underestimating discriminability and overestimating criterion as shown in the left and right panel of Fig. 4. In contrast,

Fig. 4 Parameter estimates of the hEVSD model are systematically biased when fit to data generated by the same population parameters as in Fig 3.

if simulated hit and false-alarm rates are derived from population parameters with equal variance $\sigma_s^2 = \sigma_n^2 = 1$ and are fit to the hUVSD model then estimates closely approximate the parameter values, showing no systematic bias.

This illustrates the key advantage of the hUVSD model over the hEVSD model: If the hUVSD model is fit to binary data generated by an equal-variance model then parameter estimates are not biased and signal variance approximates 1. In contrast, if the hEVSD model is fit to binary data based on an unequal-variance model then discriminability and criterion parameters systematically deviate from the true values (e.g., Rotello, Verde & Masson, 2008). Corresponding parameter shifts are expected when applying the hUVSD and hEVSD model to empirical data.

In Fig. 5 the left panel shows transformed hit and false-alarm rates for $n=500$ synthetic participants from $k=200$ signal and $k=200$ noise trials. Each data point denotes a synthetic participant of the hUVSD model. The scatterplot of superimposed estimated (open) compared to simulated (filled) data points illustrates shrinkage towards the mean of the bivariate sampling distribution of transformed hit and false alarm rates. The gray ellipse describes a 95% fit of the bivariate normal distribution and the gray line denotes the least-squares regression. The solid black line corresponds to the estimated and the dotted line to the true z -ROC. The operating characteristic was established by inserting estimated and true population parameter values into equation

$$z(\bar{\theta}^h) = \frac{1}{\sigma_s} z(\bar{\theta}^f) + \frac{\mu_d}{\sigma_s} \quad (9)$$

where $z(\bar{\theta}^h)$ and $z(\bar{\theta}^f)$ are the sample mean of simulated and transformed hit and false alarm rates, $1/\sigma_s$ defines the slope, and μ_d/σ_s the intercept of this linear function. Data points that fall onto the iso-sensitivity line describe synthetic participants who have the same discriminability as the population

Fig. 5 The graph on the left shows the receiver operating characteristic (z -ROC) for transformed hit and false alarm rates. Filled circles denote simulated data points of $n=500$ synthetic participants and superimposed open circles denote the corresponding data points estimated by the hUVSD model. The solid and dotted lines represent the estimated and true operating characteristic (iso-sensitivity), respectively. The graph on the right shows the corresponding estimated and simulated data points for criterion and discriminability. Gray ellipses fit 95% of the normally distributed data and gray lines represent least-squares regression lines.

mean but different criterion values. Fig. 5 shows that the linear combination of individual transformed hit and false alarm rates in the left panel approximate independent normal distributions of discriminability and criterion estimates in the right panel. In this simulation the sample covariance between $z(\theta^h)$ and $z(\theta^f) = 0.048$ which is close to the value derived from the population parameters $(0.4^2 - 0.6^2/4)/1.5 = 0.047$ (see Proposition 2). The sample covariance between d and c $\text{cov}[d, c] = -0.0045$ is also close to the expected value of 0.

4.3 Parameter Robustness

In order to create more realistic data, signal variance was varied across synthetic participants. A normal rather than a gamma distribution was used to generate individual signal variances $\sigma_{s,i}^2$. This simplifies introducing a correlation between individual discriminability d_i and signal standard deviation $\sigma_{s,i}$. In six simulations the number of participants together with the number of trials were halved from $n=512$ to 16 participants and from $k=256$ to 8 trials (see Table 4). For each sample size n the variability of signal standard deviations $\sigma_{s,i}$ was matched to the standard error of the sample variance ($\sigma_{s,i} \sim \mathcal{N}(\sigma_s, 2\sigma_s^4/n)$) with $\sigma_s = 1.5$). However, if variability of $\sigma_{s,i}$ is larger than the standard error of the sample variance then this can lead to extreme values and unreliable parameter estimates. These simulations explore parameter recovery if signal variance is no longer constant but varies across participants together with individual discriminability and response criterion. For each simulated data set the

generated number of hit and false alarms were fit to both hierarchical models (hEVSD, hUVSD) without any 'edge-correction' of values.

Although credibility intervals increase for smaller sample sizes and lower trial numbers the mean posterior estimates suggest that parameter recovery remains robust, irrespective of introducing a correlation of 0.0 or 0.5 between discriminability d_i and individual signal standard deviation $\sigma_{s,i}$ (see Table 4). In contrast, if the hEVSD model is fit to simulated data from the hUVSD model (with the same seed and mean signal variance) main parameter estimates are systematically biased with the true values frequently outside the 95% credibility intervals (see Table 5).

5 Applications

In the following the hUVSD and hEVSD model is applied to three existing data sets. These data sets are from three domains with different sample sizes, tasks and trial numbers.

In the first application parameter estimates of more sophisticated hierarchical models with unequal variance (Pratte, Rouder & Morey, 2010; Selker et al., 2019) based on confidence ratings served as a benchmark for the hUVSD model using only dichotomised confidence ratings. In more challenging applications, observed hit and false-alarm rates were estimated from binary responses based on experiments on reasoning with a medium sample size and only a few trials per participant (Heit & Rotello, 2005). In the third application responses came from an auditory discrimination task with five conditions and a small sample size (Tanner, Haller & Atkinson, 1967).

5.1 Word Recognition

Pratte, Rouder, and Morey (2010) collected confidence ratings from $n=97$ participants in a word recognition task. In a study phase each participant encountered 240 words. In each trial a word was shown for 1,850 ms followed by a 250 ms inter-trial interval. The words were randomly selected from a total sample of 480 words. In the subsequent memory test participants had to indicate how confident they were that a word was part of the previously seen study list on a six-point Likert scale using the ratings 'sure new', 'believe new', 'guess new', 'guess studied', 'believe studied', and 'sure studied' for each of the 480 words. In this experiment, 240 words from the study list were used as targets in signal trials whereas 240 words that were not from the study list were used as foils in noise trials. In order to apply the hUVSD model the confidence ratings, ranging between 1 and 6, were dichotomised into a *yes* response if ratings were ≤ 3 and into a *no* response, otherwise. The dichotomised responses were aggregated across signal and noise trials in order to establish hit and false-alarm rates for each participant.

The hUVSD model was applied to data set `prm09` from package `hbm` using `JAGS` (package `rjags`). Parameters were sampled for 10,000 iterations in

4 chains with random initialisations for λ_d and λ_c . The first 1,000 iterations (burn-ins) of each chain were removed. Each parameter of the model converged with $\hat{R} < 1.01$. The model fit passed various convergence diagnostics. Changing the prior for signal variance to $\text{Gamma}(0.5, 0.5)$ or to a uniform distribution $\text{dunif}(0, 5)$ did not affect mean parameter estimates. Bivariate distributions of estimated population parameters in Fig. 10 show a positive correlation between μ_d and σ_s and a negative correlation between σ_s and μ_c .

The receiver-operating characteristic (ROC) and area under the curve (AUC) of the hierarchical threshold model (Selker et al., 2019) are based on the confidence ratings (right panel in Fig. 6) whereas ROC and AUC of the hUVSD model are derived from aggregated hit and false alarms using the dichotomised confidence ratings (left panel in Fig. 6). At the population level the discriminability estimate of the hUVSD model was $\hat{\mu}_d = 1.31$, with 95% credible interval $\text{CrI}[1.17, 1.48]$ and the estimate of the standard deviation of signal distribution was $\hat{\sigma}_s = 1.45$, $\text{CrI}[1.20, 1.75]$ (see Table 6). Despite using dichotomised rating data the estimates closely match the estimates $\hat{\mu}_d = 1.38$ and $\hat{\sigma}_s = 1.36$ of the best-fitting hierarchical unequal-variance model with item-, lag- and participant-specific random effects (Model 1) as reported by Pratte et al. (2010). Selker et al. (2019) reported slightly higher values for $\hat{\mu}^{(d)} = 1.41$ and $\hat{\sigma}^{(s)} = 1.54$ when applying a parsimonious hierarchical threshold model to the original confidence data.

In comparison to the hierarchical threshold model (Selker, et al., 2019) the ROC (and AUC) of the hUVSD model has smaller credible intervals (see Fig. 6). The hUVSD model underestimates variability of the ROC because responses were aggregated across response categories, items and lags (Pratte & Rouder, 2011).

The hUVSD model was also compared to the equal variance (hEVSD) model by setting signal variance σ_s^2 equal to noise variance $\sigma_n^2 = 1$. As expected population discriminability $\hat{\mu}_d$ is reduced and population criterion $\hat{\mu}_c$ increased compared to the hUVSD model (see Table 6) and as illustrated in simulations (e.g., Fig. 4).

The deviance information criterion (**dic**; Spiegelhalter et al., 2002; Plummer, 2008) was employed to compare model performance. The small difference between **dic**=1,469 for the hUVSD model and **dic**=1,470 for the hEVSD model suggests equivalent model performance. The model fit for hUVSD and hEVSD model was replicated using a Metropolis-Hamiltonian sampler **STAN** (**rstan**). This gave almost identical parameter estimates (Table 6) with a negligible difference between the widely applicable information criteria (**waic**; Watanabe, 2010).

In the z -ROC graph of Fig. 7 individual responses are dichotomised, aggregated over trials in hit and false-alarm rates and then transformed into z -values. Each data point denotes a different participant. The scatterplot of observed (filled) with estimated (open) data points superimposed illustrates shrinkage in the hUVSD and hEVSD model. The solid, dotted and dashed line describe iso-sensitivity (discriminability) as predicted by the hUVSD model, hierarchical UVSD Model 1 (Pratte et al., 2010) and hierarchical UVSD thresh-

Fig. 6 Receiver operating characteristic (ROC) with 95% credible interval in grey and area under the curve (AUC) density distribution using the data from Pratte, Rouder, and Morey (2010). Left panel shows results for the hUVSD model based on dichotomised confidence ratings and the right panel results for the hierarchical threshold model (Selker et al., 2019) based on original confidence ratings.

Fig. 7 The graph on the left shows the receiver operating characteristic (z-ROC) for dichotomised and transformed response rates from Pratte, Rouder, and Morey (2010). Black circles denote observed data of individual participants $(z(\theta_i^f), z(\theta_i^h))$ and superimposed white circles denote values estimated by the hUVSD model. The solid, dotted and dashed line represents iso-sensitivity (discriminability) of the hUVSD model, the hierarchical UVSD Model 1 (Pratte et al., 2010), and the hierarchical UVSD threshold model (Selker et al., 2019), respectively. The graph on the right plots criterion values c_i against discriminability values d_i . The gray lines represent least-squares regressions and the gray ellipses surround 95% of the bivariate normal distribution.

old model (Selker et al., 2019), respectively. Each of the linear functions was established by inserting population parameter estimates of the corresponding hierarchical model into Eq. 9.

In the present example (Fig. 7), the solid line of the hUVSD model has a slope of $1/1.45 = 0.69$, crosses the x -axis at $x_0 = -1.31$ and intersects the y -axis at $y_0 = 0.90$ with a slope of $1/1.45 = 0.69$. The hUVSD function is close to the lines of the hierarchical unequal variance model as reported by Pratte et al. (2010) and by Selker et al. (2019). In contrast to the latter models the hUVSD model uses only dichotomised confidence ratings and is based on a single population estimate of signal variance.

Finally, the posterior mean of parameter estimate $\hat{\mu}_d=1.31$, $\hat{\sigma}_d = 0.49$, $\hat{\mu}_c=0.05$, $\hat{\sigma}_c=0.36$ and $\hat{\sigma}_s=1.45$ were used to create normal distributions for discriminability and for criterion from which pairs of random probabilities were independently drawn to simulate the performance of $n=97$ synthetic participants according to the hUVSD model. Each pair of random values was then approximated by a corresponding hit and false alarm rate. As in the previous simulations the hUVSD model was then fit to the simulated hit and false alarm rates, drawing samples of posterior values at the individual and population level. In Fig. 11 mean estimated posterior values of individual discriminability d_i (left panel) and criterion c_i (right panel) were compared to the generated posterior values. Pearson correlations of $r_d = 0.977$ and $r_c = 0.987$ between estimated and generated parameter values again suggest very good parameter recovery for the simulated data with $k = 240$ trials and $n = 97$ participants.

In Fig. 12 predicted hit and false alarm rates, generated from the population parameter values above, are plotted against observed hit and false alarm rates as reported by Pratte et al. (2010). If predicted and observed hit rates (left panel) and false alarm rates (right panel) are independently rank-ordered then correlations reach $r_h=.994$ and $r_f=.984$, respectively.

5.2 Inductive and Deductive Reasoning

The data sets are from an empirical study by Heit and Rotello (2005) who investigated a conjecture by Rips (2001). They proposed that inductive and deductive reasoning may be described in a signal detection framework where the strength of arguments is expressed on a single decision axis. According to this approach, deductive reasoning can be viewed as a more stringent form of inductive reasoning. More specifically, if discriminability and criterion are independent of each other, then deductive reasoning should shift the decision criterion of inductive reasoning without affecting discriminability. This would have implications for the understanding of reasoning processes (Oaksford & Chater, 2007).

Heit and Rotello (2005) collected inductive (deductive) judgments on four strong (valid) and four weak (invalid) arguments. Despite a medium sample size the low number of signal and noise trials gave rise to very coarse hit and false-alarm rates, making model fitting challenging. They employed a between-

Fig. 8 Receiver operating characteristic (ROC) with 95% credible interval in grey and density of area under the curve (AUC) for data from Heit and Rotello (2005). The hUVSD model was applied to binary responses of the inductive (left panel) and deductive reasoning task (right panel).

subjects design so that $n=40$ participants judged whether the conclusion of each argument was *strong* or *not strong* in the inductive reasoning task while a new sample of $n=40$ participants judged whether the conclusion of each argument was *valid* or *not valid* in the deductive reasoning task. Each data set was fit separately by the hUVSD as well as the hEVSD model. The results are summarised in Table 7.

As before, the hUVSD and hEVSD model was implemented in JAGS (package `rjags`). MCMC sampling was conducted with 10,000 iterations in each of four chains and random initialisation of λ_d and λ_c . The first 1,000 iterations (burn-in) of each chain were removed. Although all parameters of the model converged with $\hat{R} < 1.01$ sampling was less efficient and accurate as in the previous application.

The bivariate posterior distributions of population parameter estimates in Fig. 13 and 14 in the Appendix shows a positive correlation between μ_d and σ_s and negative correlations between μ_d and μ_c as well as σ_s and μ_c . Although the bivariate posterior distributions for inductive reasoning in Fig. 13 and deductive reasoning in Fig. 14 look similar the parameter estimates are quite different.

ROC and AUC of the hUVSD model are based on hits and false alarms for inductive and deductive reasoning (see Fig. 8). For inductive reasoning mean posterior discriminability of the hUVSD model is $\hat{\mu}_d=2.64$ with 95% credible interval $\text{CrI}[1.46, 3.98]$. For deductive reasoning discriminability is estimated as $\hat{\mu}_d=4.19$ with credible interval $\text{CrI}[2.76, 6.12]$. For inductive as well as deductive reasoning the standard deviation of the signal distribution is $\hat{\sigma}_s=1.57$ with $\text{CrI}[0.94, 2.18]$ and $\hat{\sigma}_s=1.63$ with $\text{CrI}[1.05, 2.21]$, respectively. Signal variance is

larger than 1.0 and close to 1.6 in the inductive as well as deductive reasoning task.

The hUVSD model has increased discriminability estimates compared to the hierarchical EVSD model in Lee and Wagenmakers (2013) with $\hat{\mu}_d=1.92$ for inductive reasoning and $\hat{\mu}_d=3.28$ for deductive reasoning (Table 7).

A model comparison in terms of `dic` and `waic` again suggests that the hUVSD model performed as good as the hEVSD model for inductive ($\Delta\text{dic}=2.6$; $\Delta\text{waic}=-1.0$) and deductive ($\Delta\text{dic}=0.5$; $\Delta\text{waic}=-0.9$) reasoning. Since predicted as well as observed hit and false-alarm rates were unreliable and resulted in a number of extreme values, z -ROC graphs and predictive checks were omitted.

In line with previous results the hUVSD model indicates clear shifts of response criterion estimates as well as discriminability estimates between inductive and deductive reasoning. The increased difference between discriminability estimates of inductive and deductive reasoning speaks against the conjecture that a shift in response criterion can explain performance (Rips, 2001); instead it promotes a ‘mirror effect’ where increased discriminability leads to a more conservative decision criterion.

It is noteworthy that the estimates of signal variance for inductive and deductive reasoning are clearly larger than 1.0. This suggests that hEVSD estimates of discriminability and criterion are biased and that both ‘strong’ and ‘valid’ arguments are better represented by a wider distribution than ‘weak’ and ‘invalid’ arguments. Interestingly, the signal variance estimates for inductive and deductive reasoning are very similar, suggesting that noise and signal variance are not affected by the increase of discriminability between conditions. Avoiding the intricate debate on how to model inductive and deductive reasoning (e.g., Rotello, Heit & Kelly, 2019), results for the hUVSD model confirm previous effects on discriminability and criterion while providing new evidence about signal variance and a potential bias of estimates in equal-variance models.

5.3 Auditory Discrimination

In this classic data set from Tanner et al. (1967) a small sample of $n = 12$ naive participants with normal hearing engaged in an auditory discrimination task. In each trial a participant had to decide which of two auditory amplitudes had occurred. The two tones were 1,000 Hz sinusoids presented for 100 ms. The amplitude of the ‘loud’ tone had 70 dB whereas the ‘soft’ tone was adjusted individually for each participant so that they were 70% correct after practice trials (mean 67.5 dB, range 66.8 dB to 68.0 dB). In successive five-day blocks of trials each participant performed for one day on one of the five conditions which had different proportions of signal and noise stimuli, ranging from $S/N=0.1$ to 0.9 in steps of 0.2 .

The participants were given no information concerning the presentation schedules and no trial-to-trial feedback during the experiment (Tanner et

al. 1967). The probability data were expressed as number of hit and false alarms adjusted for the number of signal and noise trials in the five S/N conditions. The data sets of each condition was analysed using the hEVSD model with four ($\sigma_s = 1$) and hUVSD model with five population parameters (see Fig. 9 and Table 8).

A default prediction for the five experimental conditions is that with increasing proportion of signal trials only the criterion should shift from negative to positive values, describing step changes from more 'liberal' to more 'conservative' response thresholds. Discriminability, on the other hand, is expected to remain constant, showing approximately the same value across all five conditions.

Again, model performance in terms of `dic` were very similar for both models. The hEVSD and hUVSD model both illustrate the predicted increase of criterion estimates across the five conditions. In addition, the hEVSD model not only has systematically higher criterion estimates and lower discriminability estimates but shows a modulation in discriminability values across the five conditions.

This was confirmed in two analyses using a linear mixed-effect model with subject-specific random intercepts (R-package `lme4`; Bates et al., 2015) on participants' discriminability $\hat{d}_i$ and criterion $\hat{c}_i$ estimates across the five conditions and both hierarchical signal detection models. For the criterion estimates a statistically significant fixed effect (Satterthwaite's method) of model ($t(1)=-3.80, p < 0.0003$) and condition ($t(1)=8.50, p < 0.0001$) was found. For the discriminability estimates a statistically significant fixed effect for model ($t(1)=5.15, p < 0.0001$) as well as an interaction between model and condition ($t(1)=-2.04, p = 0.044$) emerged. The significant interaction suggests that the hUVSD and hEVSD model produce different discriminability estimates across conditions. The hUVSD model resulted in nearly constant discriminability values whereas the hEVSD model suggests a peak at S/N=0.5.

This is illustrated in the left panel of Figure 9 where posterior discriminability estimates for the hEVSD model are lower compared to the hUVSD model with a peak at S/N=0.5. The hUVSD model, on the other hand, has higher and relatively constant posterior discriminability estimates. The results demonstrate that even for a small sample size of $n = 12$ the hUVSD model provides a more detailed picture than the hEVSD model. If signal variance is allowed to vary across conditions (right panel of Figure 9) then discriminability and criterion estimates are systematically affected across conditions.

6 Discussion

The hUVSD model assumes that individual discriminability d_i and response criterion c_i are draws from independent normal distributions and that signal variance is approximately constant across participants. The resulting hierarchical Bayesian model takes advantage of information in the bivariate sampling

Fig. 9 Discriminability (left panel), criterion (middle panel) and signal standard deviation (right panel) estimates of the hEVSD (gray squares) and hUVSD model (white circles) for data from Tanner et al. (1967). MCMC estimates are shown for five proportions of signal trials ranging from S/N= 0.1 to 0.9. Error bars denote ± 1 standard deviation from the posterior mean. (See text for explanation.)

distribution of (transformed) hit and false-alarm rates to estimate signal variance σ_s^2 . This systematically affects the other population estimates.

Comparing population estimates between the hUVSD and hEVSD model across applications confirms that an increase in discriminability μ_d is accompanied by a reduction of the criterion μ_c in the hUVSD model. The criterion is moved to the left by half the amount of the increase in discriminability as can be expected from the underlying signal detection model centered on 0. Performance measures `dic` and `waic` suggest equivalent performance because both models fit the observed hit and false-alarm rates equally well whereas characteristics at the population level, such as independence between d and c , may differ between the models.

For a sufficiently large sample of randomly selected participants with approximately constant signal variance the hUVSD model can estimate signal variance from the variability of hit and false alarm rates. However, this population estimate is an approximation and may not be as accurate as estimates of hierarchical models where signal variances are first established at the individual level (e.g., Pratte, Rouder & Morey, 2010). Nevertheless, if individual signal variances are not directly observable, as for binary responses, then the hUVSD model provides a preferable alternative to the ubiquitous equal-variance model because bias of discriminability and criterion estimates is reduced when signal variance is different from noise variance.

The hUVSD model bridges the gap between hierarchical equal-variance models for yes-no responses (Lee & Wagenmakers, 2013) and hierarchical unequal-variance models based on confidence ratings with observable signal variance at the individual level (Pratte, Morey & Rouder, 2010; Selker et al., 2019). It allows researchers to compare hEVSD and hUVSD parameter estimates in applications based on binary responses. Performance may be evaluated for yes-no responses, dichotomised confidence ratings, as well as other

measures and tasks. For certain tasks and paradigms it may be necessary to modify the hUVSD model so that it can address more specific hypotheses in an experimental design but the present hierarchical Gaussian model provides a good starting point.

Conditional and marginal posterior distributions have been derived in closed form for hierarchical models with unknown Gaussian mean and variance (e.g., Bernardo & Smith, 1994; Gelman et al., 2013) but the hUVSD model is more complex and has no known analytical (Bayesian or maximum likelihood) solution. Individual discriminability and criterion values cannot be observed directly and rely on parameters that are themselves conditional on binomial parameters derived from a sample of observed hit and false alarm rates.

As shown by Wolpert and Macready (1997) there is no “free lunch” in optimisation and this applies to fitting signal detection models. The hUVSD model is no exception as it makes a number of critical assumptions: (1) Hit and false alarms are modelled by binomial distributions requiring a sufficient number of independent trials. (2) Signal and noise variance may be unequal but are approximately the same across participants. (3) Discriminability and criterion values vary independently and normally in a sufficiently large random sample of participants.

Each of these assumptions is plausible but has obvious limitations: Ad (1): Any sequential dependency or auto-correlation between trials challenges the assumption of binomially distributed responses. Ad (2): It is well known (e.g., Egan, 1975) that the signal-to-noise ratio depends on the underlying distributions of the decision model. For example, log-normal, gamma, or double-exponential signal and noise distributions may have the same operational characteristic but different signal-to-noise ratios (Rouder, Pratte & Morey, 2010). In addition, it seems likely that not only signal variance but also noise variance varies across participants. Ad (3): The assumption of independent normal distributed discriminability and criterion values is plausible but difficult to verify. However, assuming independence of parameters among participants from a random sample may be less contentious than assuming constant signal variance across conditions in a within-subjects design with multiple conditions. The hUVSD model is more flexible than the hEVSD model but requires that hit and false alarm rates can be estimated from a sufficient number of trials and participants to estimate signal variance together with the other population parameters. It is usually easier to increase the sample size of participants than to gather more data from the same participant in multiple conditions or sessions where learning and fatigue may confound the data.

Although the constraints of the hUVSD model are plausible, they introduce dependencies between parameter estimates. This does not invalidate the model but can affect MCMC sampling and therefore estimation. If parameters of the hUVSD model are strongly correlated then a Metropolis-Hamiltonian (**STAN**) rather than a Gibbs sampler (**JAGS**) may improve convergence of parameter estimates. Therefore, the hUVSD and hEVSD model were also implemented in **STAN** and applied to the memory and reasoning data using package **rstan** in R. Running the hUVSD and hEVSD model for **JAGS** and **STAN** gave almost

identical parameter estimates for the memory data (Table 6) and similar estimates for the reasoning data (Table 7). Convergence diagnostics in terms of $\hat{R}$, efficiency, and auto-correlation of posterior estimates were generally more reliable for STAN compared to JAGS.

It is important to acknowledge that not only sampling of participants but also sampling of items and other factors can contribute to variability in observed data (Baayen et al. 2008; Rouder & Lu, 2005; DeCarlo, 2011). Similarly, introducing additional conditions, response categories and different tasks are likely to have an effect on performance and therefore on some or all of the parameters. A possible influence that is not modelled here are sequential dependencies between trials that may affect successive responses (e.g., Lages, 2002; Lages & Treisman, 2010; Treisman & Lages, 2010; Lages & Jaworska, 2012). These stimulus, task and design-specific issues warrant careful inspection when applying the hUVSD model. Nevertheless, the simulations and three applications above illustrate that the hUVSD model offers an alternative that can reveal detailed and novel information compared to the ubiquitous equal-variance model.

7 Conclusion

Hierarchical signal detection models with equal variance can be extended to a model with unequal variance by imposing plausible constraints on the distribution of discriminability and response criterion for a sufficiently large random sample of participants. The hierarchical signal detection model with unequal variance was investigated analytically, explored in simulations and applied to dichotomised confidence ratings of a word recognition, binary responses of inductive and deductive reasoning as well as auditory discrimination across five conditions.

The results indicate that the hUVSD model performs well as long as trial numbers and sample sizes are sufficiently large. More importantly, unlike the hEVSD model, the hUVSD model does not introduce bias to discriminability and criterion estimates if the signal variance deviates from the noise variance. In the three applications parameters were systematically different between the two models and for auditory discrimination only the hUVSD model produced approximately constant discriminability values across conditions. The empirical results suggest that the hierarchical unequal-variance model is more flexible and less biased when describing signal detection performance in binary tasks.

It is concluded that the hUVSD model is a promising alternative to the ubiquitous (hierarchical) models with equal variance. In general however, parameter estimation based on MCMC sampling needs to be carefully examined and evaluated in each application.

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Appendix A: Analytical Results

In Proposition 1 it is shown that independence between normal distributed discriminability and criterion is necessary and sufficient to estimate parameter σ_s in the hUVSD model. Proposition 2 establishes the correspondence between the five parameters of the sampling distribution and the five re-parameterised population parameters.

Let $z(\theta^h) = (z(\theta_1^h), \dots, z(\theta_n^h))$ denote a random sample of transformed hit rates and $z(\theta^f) = (z(\theta_1^f), \dots, z(\theta_n^f))$ denote the corresponding random sample of transformed false-alarm rates. If the vectors $d = (d_1, \dots, d_n)$ and $c = (c_1, \dots, c_n)$ contain independent samples from normal distributions then for $n \rightarrow \infty$ the variances are additive with covariance $\text{cov}[d, c] = \sigma_{d,c} = 0$.

Proposition 1 *Assume a hUVSD model with Gaussian signal and noise distributions then for $\lim(n) \rightarrow \infty$ the ratio of sample variances $\text{var}[z(\theta^f)]/\text{var}[z(\theta^h)]$ equals signal variance σ_s^2 if and only if $d_i \sim \mathcal{N}(\mu_d, \sigma_d)$ and $c_i \sim \mathcal{N}(\mu_c, \sigma_c)$ are sampled from independent normal distributions.*

Proof Since $\sigma_s z(\theta_h) = +0.5 d - c$ and $\sigma_n z(\theta_f) = -0.5 d - c$, with pairwise independent d_i and c_i the two variances $\text{var}[+0.5 d - c]$ and $\text{var}[-0.5 d - c]$ are equal. It follows that

$$\begin{aligned} \text{var}[\sigma_s z(\theta^h)] &= \text{var}[\sigma_n z(\theta^f)] \\ \sigma_s^2 \text{var}[z(\theta^h)] &= \sigma_n^2 \text{var}[z(\theta^f)] \\ \sigma_s^2 / \sigma_n^2 &= \text{var}[z(\theta^f)] / \text{var}[z(\theta^h)] \end{aligned}$$

If discriminability and criterion values are sampled from independent normal distributions and $\sigma_n = 1$ then σ_s can be approximated by $\text{sd}[z(\theta^f)]/\text{sd}[z(\theta^h)]$.

Conversely, it is shown that $\sigma_s^2 = \text{var}[z(\theta^f)]/\text{var}[z(\theta^h)]$ implies $\text{cov}[d, c] = 0$: According to the bilinear property of the covariance

$$\begin{aligned} \text{cov}[d, c] &= \text{cov}[\sigma_s z(\theta^h) - z(\theta^f), -0.5 \sigma_s z(\theta^h) - 0.5 z(\theta^f)] \\ &= -0.5 \sigma_s^2 \text{var}[z(\theta^h)] - 0.5 \sigma_s \text{cov}[z(\theta^f), z(\theta^h)] + \dots \\ &\quad \dots + 0.5 \sigma_s \text{cov}[z(\theta^f), z(\theta^h)] + 0.5 \text{var}[z(\theta^f)] \\ &= -0.5 \sigma_s^2 \text{var}[z(\theta^h)] + 0.5 \text{var}[z(\theta^f)] \end{aligned}$$

Inserting $\text{var}[z(\theta^f)]/\text{var}[z(\theta^h)]$ for σ_s^2 into the expression above gives $\text{cov}[d, c] = 0$.

□

Proposition 2 *Assume a hUVSD model with Gaussian signal and noise distributions, then for $\lim(n) \rightarrow \infty$*

$$\begin{aligned} E[z(\theta^h)] &= (\mu_d/2 - \mu_c)/\sigma_s \\ \text{var}[z(\theta^h)] &= (\sigma_c^2 + \sigma_d^2/4)/\sigma_s^2 \\ E[z(\theta^f)] &= -\mu_d/2 - \mu_c \\ \text{var}[z(\theta^f)] &= \sigma_c^2 + \sigma_d^2/4 \\ \text{cov}[z(\theta^h), z(\theta^f)] &= (\sigma_c^2 - \sigma_d^2/4)/\sigma_s \end{aligned}$$

Proof The expected values of the transformed hit rate and false alarm rate follow from Eq. (4), assuming independence between d and c .

$$\begin{aligned} \text{var}[z(\theta^h)] &= \text{var}[(0.5d - c)/\sigma_s] \\ &= (\text{var}[0.5d] + \text{var}[c] - 2\text{cov}[0.5d, c])1/\sigma_s^2 \\ &= (\sigma_d^2/4 + \sigma_c^2)/\sigma_s^2 \\ \text{var}[z(\theta^f)] &= \text{var}[-(0.5d + c)] \\ &= \text{var}[-0.5d] + \text{var}[c] - 2\text{cov}[-0.5d, c] \\ &= \sigma_d^2/4 + \sigma_c^2 \end{aligned}$$

From the bilinear property of covariances and Eq. (4) follows that

$$\begin{aligned} \text{cov}[z(\theta^h), z(\theta^f)] &= \text{cov}[(0.5d - c)/\sigma_s, -(0.5d + c)] \\ &= -0.25/\sigma_s \text{cov}[d, d] - 0.5/\sigma_s \text{cov}[d, c] + \dots \\ &\quad \dots + 0.5/\sigma_s \text{cov}[c, d] + 1/\sigma_s \text{cov}[c, c] \\ &= -1/(4\sigma_s) \text{var}[d] + 1/\sigma_s \text{var}[c] \\ &= (\sigma_c^2 - \sigma_d^2/4)/\sigma_s. \end{aligned}$$

□

Appendix B: Simulation Results

Parameter Recovery of hUVSD Model for Different Criterion μ_c						
μ_c	$\hat{\mu}_d$ (1.5)	$\hat{\mu}_c$ (-0.75 to +0.75)	$\hat{\sigma}_d$ (0.5)	$\hat{\sigma}_c$ (0.25)	$\hat{\sigma}_s$ (1.25)	
-0.75	1.51 [1.31,1.77]	-0.75 [-0.87,-0.65]	0.53 [0.47,0.59]	0.27 [0.24,0.30]	1.26 [1.10,1.48]	
-0.50	1.55 [1.36,1.75]	-0.56 [-0.66,-0.47]	0.54 [0.48,0.61]	0.25 [0.23,0.28]	1.33 [1.15,1.51]	
-0.25	1.56 [1.41,1.72]	-0.31 [-0.39,-0.24]	0.48 [0.43,0.55]	0.25 [0.22,0.28]	1.38 [1.21,1.58]	
± 0.0	1.53 [1.41,1.67]	-0.05 [-0.12,0.01]	0.53 [0.47,0.59]	0.26 [0.23,0.29]	1.32 [1.17,1.51]	
+0.25	1.63 [1.53,1.74]	0.23 [0.17,0.28]	0.51 [0.46,0.58]	0.30 [0.26,0.33]	1.48 [1.29,1.69]	
+0.50	1.47 [1.40,1.55]	0.48 [0.44,0.52]	0.52 [0.46,0.58]	0.27 [0.24,0.31]	1.31 [1.15,1.50]	
+0.75	1.50 [1.44,1.57]	0.75 [0.71,0.78]	0.50 [0.45,0.57]	0.27 [0.24,0.31]	1.31 [1.15,1.50]	

Table 1 Posterior means and 95% credible intervals in brackets estimated from simulated data with sample size $n=256$, signal and noise trial number $k=128$. Hit and false alarm rates were generated from population parameter values as shown in parentheses in the top row with μ_c ranging from -0.75 to +0.75 in steps of 0.25.

Parameter Recovery of hUVSD Model for Different Discriminability μ_d						
μ_d	$\hat{\mu}_d$ (0.5 to 2.5)	$\hat{\mu}_c$ (-0.25)	$\hat{\sigma}_d$ (1.25)	$\hat{\sigma}_c$ (0.75)	$\hat{\sigma}_s$ (1.5)	
0.5	0.62 [0.50,0.74]	-0.24 [-0.31,-0.17]	1.25 [1.15,1.35]	0.75 [0.70,0.81]	1.48 [1.36,1.63]	
1.0	1.09 [0.97,1.23]	-0.24 [-0.32,-0.17]	1.26 [1.17,1.37]	0.75 [0.70,0.81]	1.50 [1.38,1.63]	
1.5	1.60 [1.46,1.75]	-0.24 [-0.32,-0.16]	1.25 [1.16,1.35]	0.75 [0.69,0.81]	1.50 [1.37,1.63]	
2.0	2.10 [1.95,2.26]	-0.25 [-0.33,-0.17]	1.23 [1.14,1.33]	0.74 [0.69,0.80]	1.48 [1.37,1.61]	
2.5	2.58 [2.41-2.76]	-0.24 [-0.33,-0.15]	1.24 [1.14,1.34]	0.74 [0.69,0.80]	1.48 [1.35,1.61]	

Table 2 Posterior means and 95% credible intervals in brackets estimated from simulated data with sample size $n=500$, signal and noise trial number $k=100$. Hit and false alarm rates were generated from population parameter values as shown in parentheses in the top row with μ_d ranging from 0.5 to 2.5 in steps of 0.5.

Parameter Recovery of hUVSD Model for Different Signal Variance σ_s						
σ_s	$\hat{\mu}_d$ (2.0)	$\hat{\mu}_c$ (-0.5)	$\hat{\sigma}_d$ (0.75)	$\hat{\sigma}_c$ (0.5)	$\hat{\sigma}_s$ (0.5 to 2.5)	
0.5	2.18 [2.02-2.37]	-0.60 [-0.70,-0.51]	0.76 [0.70,0.83]	0.52 [0.48,0.56]	0.57 [0.50,0.64]	
1.0	2.08 [1.92,2.26]	-0.54 [-0.63,-0.45]	0.74 [0.68,0.80]	0.52 [0.48,0.56]	1.05 [0.95,1.15]	
1.5	2.09 [1.95,2.26]	-0.56 [-0.65,-0.49]	0.74 [0.68,0.80]	0.52 [0.48,0.56]	1.59 [1.47,1.74]	
2.0	2.04 [1.90,2.20]	-0.53 [-0.61,-0.45]	0.74 [0.68,0.80]	0.51 [0.47,0.55]	2.05 [1.88,2.23]	
2.5	1.95 [1.81,2.09]	-0.47 [-0.55,-0.40]	0.71 [0.66,0.77]	0.50 [0.47,0.54]	2.38 [2.19,2.59]	

Table 3 Posterior means and 95% credible intervals in brackets estimated from simulated data with sample size $n=500$, signal and noise trial number $k=100$. Hit and false alarm rates were generated from population parameter values as shown in parentheses in the top row with σ_s ranging from 0.5 to 2.5 in steps of 0.5.

Parameter Recovery of hUVSD Model for Different Participants n and Trials k								
n	k	$\hat{\mu}_d$ (2.0)	$\hat{\mu}_c$ (-0.5)	$\hat{\sigma}_d$ (0.6)	$\hat{\sigma}_c$ (0.4)	$\hat{\sigma}_s$ (1.5)		
512	256	1.98 [1.85,2.10]	-0.46 [-0.53,-0.39]	0.55 [0.51,0.59]	0.41 [0.38,0.44]	1.48 [1.37,1.59]		
256	128	1.99 [1.85,2.16]	-0.44 [-0.53,-0.36]	0.54 [0.49,0.61]	0.42 [0.38,0.47]	1.46 [1.34,1.63]		
128	64	1.93 [1.68,2.23]	-0.44 [-0.60,-0.30]	0.53 [0.44,0.63]	0.43 [0.37,0.51]	1.37 [1.14,1.65]		
64	32	2.04 [1.67,2.45]	-0.45 [-0.68,-0.25]	0.61 [0.48,0.79]	0.52 [0.42,0.64]	1.49 [1.16,1.88]		
32	16	2.05 [1.54,2.61]	-0.52 [-0.85,-0.23]	0.57 [0.39,0.76]	0.55 [0.41,0.76]	1.58 [1.10,2.10]		
16	8	2.13 [1.32,3.09]	-0.57 [-1.06,-0.16]	0.85 [0.49,1.53]	0.48 [0.31,0.78]	1.46 [0.84,2.09]		

Table 4 Estimated posterior means and 95% credible intervals in brackets for simulated data with decreasing sample size n and trial number k generated from population parameter values as shown in the top row with variable $\sigma_{s,i} \sim \mathcal{N}(\sigma_s, 2\sigma_s^4/n)$ and a correlation of 0.5 between d_i and $\sigma_{s,i}$.

Parameter Recovery of hEVSD Model for Different Participants n and Trials k								
n	k	$\hat{\mu}_d$ (2.0)	$\hat{\mu}_c$ (-0.5)	$\hat{\sigma}_d$ (0.6)	$\hat{\sigma}_c$ (0.4)	σ_s (1.5)		
512	256	1.51 [1.46,1.55]	-0.27 [-0.26,-0.20]	0.48 [0.45,0.51]	0.35 [0.32,0.37]	(1.0)		
256	128	1.54 [1.48,1.60]	-0.21 [-0.26,-0.17]	0.47 [0.43,0.52]	0.36 [0.33,0.39]	(1.0)		
128	64	1.54 [1.45,1.63]	-0.19 [-0.25,-0.11]	0.46 [0.40,0.53]	0.39 [0.34,0.45]	(1.0)		
64	32	1.56 [1.43,1.69]	-0.24 [-0.35,-0.14]	0.47 [0.38,0.59]	0.40 [0.33,0.49]	(1.0)		
32	16	1.60 [1.37,1.84]	-0.30 [-0.47,-0.14]	0.53 [0.37,0.75]	0.41 [0.31,0.56]	(1.0)		
16	8	1.66 [1.18,2.20]	-0.25 [-0.62, -0.08]	0.78 [0.46,1.30]	0.45 [0.30,0.71]	(1.0)		

Table 5 Estimated posterior means and 95% credible intervals in brackets for simulated data with decreasing sample size n and trial number k generated from population parameter values as shown in the top row with variable $\sigma_{s,i} \sim \mathcal{N}(\sigma_s, 2\sigma_s^4/n)$ and a correlation of 0.5 between d_i and $\sigma_{s,i}$.

Appendix C: Application Results

Parameter Estimates for Word Recognition Task								
Model	sampler	dic/waic	A_z	$\hat{\mu}_d$	$\hat{\mu}_c$	$\hat{\sigma}_d$	$\hat{\sigma}_c$	$\hat{\sigma}_s$
Pratte et al.	n/a	n/a	0.793	1.38	n/a	n/a	n/a	1.36
Selker et al.	JAGS	n/a	0.778	1.41	n/a	n/a	n/a	1.54
hEVSD	JAGS	1,470	0.785	1.12	0.15	0.42	0.31	(1.0)
hUVSD	JAGS	1,469	0.771	1.31	0.05	0.49	0.36	1.45
hEVSD	STAN	1,422	0.788	1.13	0.15	0.42	0.31	(1.0)
hUVSD	STAN	1,422	0.772	1.32	0.05	0.49	0.36	1.46

Table 6 Population parameter estimates of hierarchical models for word recognition task (Pratte et al., 2010). Models from Pratte et al. (2010) and Selker et al. (2019) have different parameters (n/a) because they are based on confidence ratings. Parameter estimates are the result of Gibbs (JAGS) and Metropolis-Hamiltonian (STAN) MCMC sampling reporting **dic** and **waic**, respectively.

Parameter Estimates for Reasoning Task								
Inductive Reasoning								
Model	sampler	dic/waic	A_z	$\hat{\mu}_d$	$\hat{\mu}_c$	$\hat{\sigma}_d$	$\hat{\sigma}_c$	$\hat{\sigma}_s$
hEVSD	JAGS	169.6	0.913	1.92	-1.18	1.87	0.75	(1.0)
hUVSD	JAGS	167.0	0.922	2.64	-1.54	2.08	0.72	1.57
hEVSD	STAN	154.2	0.916	1.95	-1.19	1.88	0.76	(1.0)
hUVSD	STAN	155.2	0.924	2.93	-1.70	2.06	0.82	1.78
Deductive Reasoning								
Model	sampler	dic/waic	A_z	$\hat{\mu}_d$	$\hat{\mu}_c$	$\hat{\sigma}_d$	$\hat{\sigma}_c$	$\hat{\sigma}_s$
hEVSD	JAGS	156.1	0.990	3.28	-0.30	2.23	0.77	(1.0)
hUVSD	JAGS	155.6	0.986	4.19	-0.62	2.74	0.83	1.63
hEVSD	STAN	134.9	0.991	3.37	-0.31	2.30	0.80	(1.0)
hUVSD	STAN	135.8	0.984	4.40	-0.71	2.74	0.88	1.80

Table 7 Population parameter estimates for inductive and deductive reasoning task by Heit and Rotello (2005). Parameter estimates are the result of Gibbs (JAGS) and Metropolis-Hamiltonian (STAN) MCMC sampling using **dic** and **waic**, respectively.

Parameter Estimates for Auditory Discrimination Task								
Model	S/N	dic	A_z	$\hat{\mu}_d$	$\hat{\mu}_c$	$\hat{\sigma}_d$	$\hat{\sigma}_c$	$\hat{\sigma}_s$
hEVSD	0.1	180.9	0.804	1.21	-0.57	0.50	0.59	(1.0)
hUVSD		181.7	0.821	1.43	-0.68	0.48	0.57	1.19
hEVSD	0.3	194.9	0.825	1.32	-0.38	0.47	0.52	(1.0)
hUVSD		195.2	0.833	1.49	-0.47	0.44	0.50	1.17
hEVSD	0.5	197.5	0.835	1.38	-0.22	0.54	0.55	(1.0)
hUVSD		197.9	0.838	1.46	-0.26	0.52	0.51	1.09
hEVSD	0.7	197.0	0.825	1.32	0.09	0.54	0.51	(1.0)
hUVSD		197.1	0.821	1.44	0.03	0.54	0.49	1.21
hEVSD	0.9	186.1	0.808	1.23	0.22	0.64	0.54	(1.0)
hUVSD		186.2	0.789	1.42	0.13	0.69	0.57	1.46

Table 8 Population parameter estimates of hEVSD and hUVSD model for auditory discrimination data from Tanner et al. (1967). In five conditions the proportion of signal trials was increased from S/N=0.1 to 0.9 in steps of 0.2. Parameter estimates are the result of Gibbs (JAGS) MCMC sampling.

Fig. 10 MCMC parameter estimates (JAGS) of the hUVSD model for recognition data by Pratte et al. (2010). Bivariate hexagon plots for all pairs of parameters at the population level. Light to dark hexagons indicate increasing frequencies. Pearson correlations between posterior samples are shown on top of each panel.

Fig. 11 Parameter recovery for discriminability d_i (left panel) and criterion c_i (right panel) from synthetic data generated with population parameters, $k=240$ trials and $n=97$ participants as in Pratte et al. (2010).

Fig. 12 Posterior predictive check for hit and false alarm rates using observed data and hUVSD population estimates from Pratte et al. (2010). Predicted and observed hit rates (left panel) as well as predicted and observed false alarm rates (right panel) were independently rank-ordered for simulated and real participants.

Fig. 13 MCMC parameter estimates (JAGS) of the hUVSD model for inductive reasoning data by Heit and Rotello (2005). Bivariate hex plots for parameters pairs at the population level. Light to dark hexagons indicate increasing frequencies. Pearson correlations between posterior samples are shown on top of each panel.

Fig. 14 MCMC parameter estimates (JAGS) of the hUVSD model for deductive reasoning data by Heit and Rotello (2005). Bivariate hex plots for all pairs of parameters at the population level. Light to dark hexagons indicate increasing frequencies. Pearson correlations between posterior samples are shown on top of each panel.

References

1. Baayen, R., Davidson, D., Bates, D.: Mixed-effects modeling with crossed random effects for subjects and items. *Journal of Memory and Language* **59**, 390–412 (2008)
2. Bates, D., Maechler, M., Bolker, B., Walker, S.: Fitting linear mixed-effects models using lme4. *Journal of Statistical Software* **67**(1), 1–48 (2015). URL doi.org/10.18637/jss.v067.i01
3. Bernardo, J., Smith, A.: *Bayesian Theory*. New York: Wiley (1994)
4. Bishop, C.: *Pattern recognition and machine learning*. New York: Springer (2006)
5. DeCarlo, L.: Signal detection theory with item effects. *Journal of Mathematical Psychology* **55**, 229–239 (2011)
6. Dorfman, D., Alf Jr., E.: Maximum likelihood estimation of parameters of signal detection theory and determination of confidence intervals – rating-method data. *Journal of Mathematical Psychology* **6**, 487–496 (1969)
7. Efron, B., Morris, C.: Stein’s paradox in statistics. *Scientific American* **236**, 119–127 (1977)
8. Falmagne, J.C.: *Elements of psychophysical theory*. Oxford psychology series; no. 6. New York: Oxford University Press (1985)
9. Fechner, G.: *Elemente der Psychophysik* (Band 1 und 2). Leipzig: Breitkopf und Härtel (1860)
10. Gelman, A.: Prior distributions for variance parameters in hierarchical models. *Bayesian Analysis* **1**(3), 515–533 (2006)
11. Gelman, A., Carlin, J., Stern, H., Dunson, D., Vehtari, A.: *Bayesian data analysis* (3rd Ed.). New York: Chapman & Hall/CRC (2013)
12. Gelman, A., Rubin, D.: Inference from iterative simulation using multiple sequences (with discussion). *Statistical Science* **7**(4), 457–511 (1992)
13. Green, D., Swets, J.: *Signal detection theory and psychophysics*. Wiley: New York (1966)
14. Heit, E., Rotello, C.: Are there two kinds of reasoning? In: *Proceedings of the 27th Annual meeting of the Cognitive Science Society*, pp. 923–928 (2005)
15. Jang, Y., Wixted, J., Huber, D.: Testing signal-detection models of yes/no and two-alternative forced-choice recognition memory. *Journal of Experimental Psychology: General* **138**(2), 291–306 (2009). DOI [10.1037/a0015525](https://doi.org/10.1037/a0015525)
16. Johnson, N., Kotz, S.: *Distributions in statistics*. 4 vols. New York: Wiley (1972)
17. Lages, M.: Ear decomposition for pair comparison data. *Journal of Mathematical Psychology* **46**(1), 19–39 (2002). DOI [10.1016/0003-4916\(63\)90068-X](https://doi.org/10.1016/0003-4916(63)90068-X)
18. Lages, M., Jaworska, K.: How predictable are ‘spontaneous decisions’ and ‘hidden intentions’? comparing classification results based on previous responses with multivariate pattern analysis of fmri bold signals. *Frontiers in Psychology* **3**(56), 1–8 (2012). DOI [10.3389/fpsyg.2012.00056](https://doi.org/10.3389/fpsyg.2012.00056)
19. Lages, M., Treisman, M.: A criterion setting theory of discrimination learning that accounts for anisotropies and context effects. *Seeing and Perceiving* **23**(5-6), 401–434 (2010). DOI [10.1163/187847510X541117](https://doi.org/10.1163/187847510X541117)
20. Lee, M., Wagenmakers, E.J.: *Bayesian modeling for cognitive science: a practical course*. Cambridge: Cambridge University Press (2013)
21. Liu, C., Rubin, D., Wu, Y.: Parameter expansion to accelerate em: The px-em algorithm. *Biometrika* **85**, 755–770 (1998)
22. Luce, R.: A psychophysical theory of intensity proportions, joint presentations, and matches. *Psychological Review* **109**(3), 520–532 (2002). DOI [10.1037/0033-295X.109.3.520](https://doi.org/10.1037/0033-295X.109.3.520)
23. Lunn, D., Thomas, A., Best, N., Spiegelhalter, D.: Winbugs — a bayesian modelling framework: concepts, structure, and extensibility. *Statistics and Computing* **10**, 325–337 (2000)
24. MacKay, D.: Psychophysics of perceived intensity: A theoretical basis for Fechner’s and Stevens’ laws. *Science* **139**, 1213–1216 (1963)
25. Macmillan, N., Creelman, C.: *Detection theory: A user’s guide*. Cambridge: Cambridge University Press (2005)

26. Morey, R., Pratte, M., Rouder, J.: Problematic effects of aggregation in zROC analysis and a hierarchical modeling solution. *Journal of Mathematical Psychology* **52**, 376–388 (2008)
27. Narens, L.: A theory of ratio magnitude estimation. *Journal of Mathematical Psychology* **40**(2), 109–129 (1996). DOI 10.1006/jmps.1996.0011
28. Oaksford, M., Chater, N.: *Bayesian rationality: The probabilistic approach to human reasoning*. Oxford: Oxford University Press (2007)
29. Paulewicz, B., Blaut, A.: The bhsdtr package: a general-purpose method of bayesian inference for signal detection theory models. *Behavior Research Methods* **52**, 2122–2141 (2020). DOI 10.3758/s13428-020-01370-y
30. Plummer, M.: Penalized loss functions for Bayesian model comparison. *Biostatistics* **9**(3), 523–539 (2008). DOI 10.1093/biostatistics/kxm049
31. Plummer, M., Best, N., Cowles, K., Vines, K.: Coda: Convergence diagnosis and output analysis for mcmc. *R News* **6**(3), 7–11 (2006)
32. Plummer, M., Stukalov, A., Denwood, M.: *R JAGS: Bayesian Graphical Models using MCMC* (2019). URL <http://mcmc-jags.sourceforge.net>
33. Pratte, M., Rouder, J.: Hierarchical single- and dual-process models of recognition memory. *Journal of Mathematical Psychology* **55**, 36–46 (2011). DOI 10.1016/j.jmp.2010.08.007
34. Pratte, M., Rouder, J., Morey, R.: Separating mnemonic process from participant and item effects in the assessment of ROC asymmetries. *Journal of Experimental Psychology: Learning, Memory and Cognition* **36**, 224–232 (2010). DOI 10.3758/s13428-019-01231-3
35. R Core Team: *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria (2020). URL <https://www.R-project.org/>
36. Rips, L.: Two kinds of reasoning. *Psychological Science* **12**, 129–134 (2001)
37. Rouder, J., Lu, J.: An introduction to bayesian hierarchical models with an application in the theory of signal detection. *Psychonomic Bulletin & Review* **12**, 573–604 (2005). DOI 10.3758/s13428-019-01231-3
38. Selker, R., van den Bergh, D., Criss, A., Wagenmakers, E.J.: Parsimonious estimation of signal detection models from confidence ratings. *Behavior Research Methods* **51**, 1953–1967 (2019). DOI 10.3758/s13428-019-01231-3
39. Spiegelhalter, D., Best, N., Carlin, B., van der Linde, A.: Bayesian measures of model complexity and fit. *Journal of the Royal Statistical Society, Series B (Stat. Methodology)* **64**, 583–639 (2002). DOI 10.1111/1467-9868.00353
40. Stan Development Team: *RStan: the R interface to Stan* (2019). URL <https://cran.r-project.org>
41. Stan Development Team: *Stan: Modeling Language User’s Guide and Reference Manual* (2019). URL <http://mc-stan.org>
42. Starns, J., Ratcliff, R.: Validating the unequal- variance assumption in recognition memory using response time distributions instead of roc functions: A diffusion model analysis. *Journal of Memory and Language* **70**, 36–52 (2014). DOI 10.3758/s13428-019-01231-3
43. Starns, J., Rotello, C., Hautus, M.: Recognition memory zroc slopes for items with correct versus incorrect source decisions discriminate the dual process and unequal variance signal detection models. *Journal of Experimental Psychology: Learning, Memory and Cognition* **40**, 1205–1225 (2014). DOI 10.1037/a0036846
44. Stevens, S.: On the psychophysical law. *Psychological Review* **64**(3), 153–181 (1957)
45. Suero, M., Botella, J., Privado, J.: Estimating the expected value and variance of sdt indexes with heterogeneous individuals. *Journal of Mathematical Psychology* **83**, 12–23 (2018). URL doi.org/10.1016/j.jmp.2018.02.001
46. Tanner Jr., T., Haller, R., Atkinson, R.: A decision-making theory of visual detection. *Psychological Review* **61**, 401–409 (1954)
47. Tanner Jr., W., Swets, J.: A decision-making theory of visual detection. *Psychological Review* **61**, 401–409 (1954)
48. Treisman, M., Lages, M.: Sensory integration across modalities: How kinaesthesia integrates with vision in visual orientation discrimination. *Seeing and Perceiving* **23**, 435–462 (2010). DOI 10.1163/187847510X541126

-
49. Vuorre, M.: Bayesian estimation of signal detection models (2017). URL <https://sometimesir.com/posts/2017-10-09-bayesian-estimation-of-signal-detection-theory-models>
 50. Watanabe, S.: Asynptotic equivalence of bayes cross validation and widely applicable information criterion in singular learning theory. *Journal of Machine Learning Research* **11**, 3571–3594 (2010)
 51. Weidemann, C., Kahana, M.: Assessing recognition memory using confidence ratings and response times. *R. Soc. Open Sci.* **3**, 150670 (2016). DOI 10.1098/rsos.150670
 52. Wickens, T.: *Elementary signal detection theory*. Oxford University Press: New York (2002)
 53. Wolpert, D., Macready, W.: No free lunch theorems for optimization. *IEEE Transactions on Evolutionary Computation* **1**, 67–82 (1997)