Parts and Wholes: A Note on Interpretation of Partial Covariance Matrices and Latent Variable Models

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Abstract

In a discussion of partial covariance matrices, Gaussian graphical models (GGMs), and unidimensional latent variable models (ULVMs), Waldorp and Marsman (2021) make a claim that “for the ULVM, observed partial correlations would all be positive... [proving] that the GGM applied to data coming from a ULVM will be fully-connected and not empty.” In a note, we show that although this is technically true, it is misleading, as with a ULVM parts of the partial covariance matrix corresponding to less informative indicators will in fact become approximately empty as other indicator variables become increasingly informative about the latent variable. We note that in this way, even though in their entirety covariance and concentration matrices are statistically equivalent, interpretations of their elements are not. We discuss interpretation of partial covariance matrices under random and incomplete samplings of observed variables, which is the norm in the behavioral sciences.

Keywords: network analysis, latent variable model, factor analysis, precision matrix

Waldorp and Marsman (2021) recently published a discussion of relationships between partial covariance matrices, Gaussian graphical (i.e., network) models (GGMs), and unidimensional latent variable models (ULVMs). Written at least in part in response to an article of ours in this journal (Forbes et al., 2019), their article addresses the general topic of how to interpret shared variance in structural models such as GGMs based on partial covariance matrices.

One central claim in their paper is that “for the ULVM, observed partial correlations would all be positive... [proving] that the GGM applied to data coming from a ULVM will be fully-connected and not empty.” They make this claim to support a more general position that, even with unmodeled latent variables, network models do not include edges reflecting residual covariance.

Here, we show that although this claim is true in a strict sense, it becomes approximately false as measurement error decreases among a subset of variables. For many realistic scenarios, if there is differential measurement error among a set of variables, with some subset of them being informative (as in a bifactor model, for instance) it may not be true in practice. Very small, near-zero partial correlations can emerge as a result of the informative subset increasingly approximating the latent variable in an information sense.

More broadly speaking, we make a distinction between interpretation of a partial covariance matrix as a whole and its individual components which are often the focus of interpretation. Although Waldorp and Marsman (2021) are correct that covariance and partial covariance matrices are invertible and therefore exchangeable in an important sense, interpretation of any given partial covariance or set of partial covariances critically depends on other elements of the matrix. This extends to broader design considerations, such as the random nature of subsets of variables constituting the matrix, in that random changes in one set of variables due to secondary factors of concern can change partial covariances between other sets of variables. As such, researchers interpreting individual paths or sets of paths might overlook the “big picture” of the broader matrix, especially if it is fragmented through imperfect estimates, and “miss the forest for the trees.”

Concentrations as Measurement Information Increases

Assume a concentration matrix Θ can be subsetting into blocks, corresponding to subsets of variables $\mathbf{X}$,

$\mathbf{Y}$, and $\mathbf{Z}$, with i , j , and k variables in each subset, respectively. Assume further that the variables conform to an orthogonal latent variable model. For example, the model for the variables $\mathbf{X}$ is given by

$$\mathbf{X} = \mathbf{\Lambda}_x \boldsymbol{\eta}_x + \boldsymbol{\epsilon}_x \quad (1)$$

and the covariance matrix is given by

$$\boldsymbol{\Sigma}_x = \mathbf{\Lambda}_x \mathbf{\Lambda}_x' + \boldsymbol{\Psi}_x \quad (2)$$

where $\mathbf{\Lambda}_x$ is the loading matrix for subset $\mathbf{X}$, $\boldsymbol{\epsilon}_x$ is an error or residual variable vector, those residuals are uncorrelated, and $\boldsymbol{\Psi}_x$ is the error covariance matrix (in general we denote a matrix corresponding to a variable subset with subscripts).

In general, the partial covariances between $\mathbf{X}$ and $\mathbf{Y}$ given $\mathbf{Z}$ are given by (e.g., Anderson, 2003)

$$\boldsymbol{\Theta}(\mathbf{X}, \mathbf{Y} \mid \mathbf{Z}) = \boldsymbol{\Theta}(\mathbf{X}, \mathbf{Y}) - \boldsymbol{\Theta}(\mathbf{X}, \mathbf{Z}) \boldsymbol{\Theta}(\mathbf{Z}, \mathbf{Z})^{-1} \boldsymbol{\Theta}(\mathbf{Z}, \mathbf{Y}) \quad (3)$$

where $\boldsymbol{\Theta}(\mathbf{X}, \mathbf{Y} \mid \mathbf{Z})$ is the $(i+j)$ by $(i+j)$ symmetric subblock of the covariance matrix involving $\mathbf{X}$ and $\mathbf{Y}$, partialing $\mathbf{Z}$, $\boldsymbol{\Theta}(\mathbf{X}, \mathbf{Y})$ is the $(i+j)$ by $(i+j)$ symmetric subblock of the covariance matrix involving $\mathbf{X}$ and $\mathbf{Y}$ (that is, the covariance matrix of the variables of interest not partialing $\mathbf{Z}$), $\boldsymbol{\Theta}(\mathbf{Z}, \mathbf{Z})$ is the k by k symmetric subblock of the covariance matrix involving $\mathbf{Z}$ (the covariances between the variables to be partialled), and $\boldsymbol{\Theta}(\mathbf{X}, \mathbf{Z})$ is the i by k rectangular subblock of the covariance matrix involving $\mathbf{X}$ and $\mathbf{Z}$.

To see how the concentration matrix behaves as a function of measurement properties, first assume that $\mathbf{Z}$ consists of a set of orthogonal standardized latent variables. Then, given Equation 2, Equation 3 can be rewritten as

$$\Theta(\mathbf{X}, \mathbf{Y} \mid \mathbf{Z}) = (\Lambda_{\mathbf{X}} \Lambda_{\mathbf{Y}}' + \Psi_{\mathbf{XY}}) - (\Lambda_{\mathbf{X}} \Lambda_{\mathbf{Y}}') = \Psi_{\mathbf{XY}} \quad (4)$$

That is, if $\mathbf{Z}$ is a set of the latent variables, the partial covariance matrix is equal to the error or uniqueness covariance matrix. This is consistent with the definitional assumption that observed variables are independent given the set of latent variables, and also with observations by many that the uniquenesses in a factor model are provided by the inverse covariance matrix (e.g., Guttman, 1956; Krijnen, 2006).

Note, however, that Equation 4 also holds approximately even if $\mathbf{Z}$ is observed, if $\mathbf{Z}$ is made arbitrarily close to the set of latent variables, such as in the case where the $\epsilon_{\mathbf{Z}}$ approach 0. In this case you have

$$\Theta(\mathbf{X}, \mathbf{Y} \mid \mathbf{Z}) = (\Lambda_{\mathbf{X}} \Lambda_{\mathbf{Y}}' + \Psi_{\mathbf{XY}}) - (\Lambda_{\mathbf{X}} \Lambda_{\mathbf{Z}}' + \Psi_{\mathbf{XZ}})_{\mathbf{XZ}} (\Lambda_{\mathbf{Z}} \Lambda_{\mathbf{Z}}' + \Psi_{\mathbf{Z}})^{-1} (\Lambda_{\mathbf{Z}} \Lambda_{\mathbf{Y}}' + \Psi_{\mathbf{YZ}})_{\mathbf{YZ}} \approx \Psi_{\mathbf{XY}} \quad (5)$$

where the subscripts on parentheses denote the rectangular subblocks containing cross-variable covariances only. As the $\epsilon_{\mathbf{Z}}$ go to 0, relevant portions of $\Psi_{\mathbf{Z}}$, $\Psi_{\mathbf{XZ}}$, and $\Psi_{\mathbf{YZ}}$ go to 0 matrices, and the $\Lambda_{\mathbf{Z}}$ cancel (as $[\Lambda_{\mathbf{Z}} \Lambda_{\mathbf{Z}}']^{-1} = [\Lambda_{\mathbf{Z}}']^{-1} [\Lambda_{\mathbf{Z}}]^{-1}$). Then one is again left with the residual covariance matrix, $\Psi_{\mathbf{XY}}$, approximately. As measurement precision increases among a subset of variables, because the errors are uncorrelated (per the standard LVM, as in Waldorp and Marsman's paper), the coconcentrations involving less precise variables will go to zero.

This phenomenon is illustrated with examples in Table 1. In the table, the variables are assumed to conform to the standard factor model (Equations 1-2), with the first set of three variables having loadings of 0.50, the second set of three variables having loadings of either 0.70 or 0.90, and the observed variables in a standardized metric (i.e., the observed matrix is a correlation matrix). As can be seen, as the second set of variables' loadings increase, and the error variances decrease, the elements of the concentration matrix corresponding to the first set of variables decrease toward zero.

Figure 1 illustrates this for different values of large loadings, and for different numbers of variables having larger versus smaller loadings. A couple of trends are evident in the figure. First, all minimum absolute

concentrations approach 0 with increases in the most informative loadings. Second, the rate at which the minimum absolute concentrations approach 0 is greater with larger numbers of informative items: having more informative items pushes the expected minimum concentration lower, and seems to slightly increase the rate at which the minimum concentrations approach 0. Waldorp and Marsman note related phenomena as the number of variables goes to infinity, citing other authors.

In practice, this implies that if the ULVM (or classical linear orthogonal LVM more broadly) holds, and errors of variables $\mathbf{Z}$ decrease (or standardized loadings increase) as those of other sets of variables $\mathbf{X}$ and $\mathbf{Y}$ remain constant, the partial covariances corresponding to $\mathbf{X}$ and $\mathbf{Y}$ will also go to zero. This is not uncommon with many forms of data, such as those behaving as a bifactor model, where the observed variables may be highly correlated, with a subset having especially high loadings. In this scenario, the variables with smaller standardized loadings will have concentration matrix elements relatively close to zero under the ULVM. The informative variables $\mathbf{Z}$ act collectively as a statistical proxy for the latent variable, rendering the other variables conditionally independent.

Technically, this result does not contradict what Waldorp and Marsman claim, as the partial covariances are technically nonzero even as the residuals decrease and standardized loadings of a subset increase. However, in practice, the partial covariances can go to zero in an approximate sense, which can be of relevance in practice when estimating model parameters. This is especially true given that the covariances in question will generally not be population covariances, but sample covariances, leading to further complications in estimation.

Practically speaking, these values will be relatively close to zero. Methods designed to accurately represent the concentration matrix may represent this accurately. However, in the case of the ULVM, even if the elements should be zero in some sense, this increases the number of approximately-zero elements being modeled, increasing the chance of false nonzero estimates. Moreover, in a very real sense, the sense of the observed population concentration matrix, these elements are not actually nonzero, even though they do reflect elements of the measurement error matrix. More precisely, they reflect unreliability of the anchor set of indicator variables, and the distance of those anchor variables from perfect estimates of the latent variables.

It is important to note that in their treatment, Waldorp and Marsman make the assumption that the errors of the ULVM have means of 0 and variances of 1 (that is, $\Psi = \mathbf{I}$; Equations 1-4 of their paper). That is, not only are the measures quasi-parallel, in the sense that they have unequal loadings but equal error variances, but they furthermore all have *unit* error variances. Examining Equation 5 one can see that assuming $\Psi = \mathbf{I}$ for all $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ has a number of important consequences. First, it prevents the error covariances for the different subsets of variables from differing. Second, and more importantly, it artificially disallows Ψ from ever approaching a zero matrix, which is critical to the appearance of zero coconcentrations. In a critical sense, Waldorp and Marsman artificially disallow the ULVM from having zero or near-zero off-diagonal coconcentrations by defining the latent variable model in such a way that zero or near-zero values cannot be observed.

Much of their discussion focuses on their Equation 9 (see also their Appendix), where they define the concentration matrix values as

$$\mathbf{I} = \frac{\mathbf{\Lambda}\mathbf{\Lambda}'}{\mathbf{\Lambda}'\mathbf{\Lambda} + 1} \quad (6)$$

where $\mathbf{I}$ is the identity matrix. The appearance of $\mathbf{I}$ in their formulae depends on the assumption of unit error variances, $\Psi = \mathbf{I}$. If the error variances are allowed to be unrestricted, application of the Sherman-Morrison theorem gives instead

$$\Psi^{-1} = \frac{\Psi^{-1}\mathbf{\Lambda}\mathbf{\Lambda}'\Psi^{-1}}{\mathbf{\Lambda}'\Psi^{-1}\mathbf{\Lambda} + 1} \quad (7)$$

which has different implications for interpretation of concentration matrices.

The Random Nature of Concentration Matrices

This phenomenon, whereby the measurement error of one set of variables can affect partial covariances between other variables, is part of a broader phenomenon whereby one set of variables in a concentration matrix affects the coconcentrations of another set of variables. In general, coconcentrations between variables are not invariant to changes in the overall set of variables: the coconcentration between two variables can change, even at the population level, solely because other variables in the matrix change (e.g., Guttman, 1953).

This is different from covariances: at the population level, the covariance between x and y remains the same regardless of what values are observed in $\mathbf{Z}$. The coconcentration between x and y , in contrast, can and often does change depending on the values observed in $\mathbf{Z}$. Changes in one set of measures used in analysis can change the population values of coconcentrations between other variables.

This noninvariance to changes in the variable set is reasonable if one assumes that the entire set of causally-relevant variables has been observed, and partial covariances are used to isolate, in some sense, the shared relationship between two variables accounting for the rest of the variables in the set. In network analysis, for instance it is often assumed that there is a dynamic system of variables, and that the variables in the system have been observed such that one can model that entire system, either as a closed set, or as an open set where relevant variables inside and outside the system have been exhaustively sampled. In other words, the noninvariance of partial covariances to changes in the set of variables being modeled is often implicitly addressed by assuming that the variable set is invariant across the universe of samples (e.g., that the components of major depression cannot be defined or measured differently in a different study).

The assumption that the variable set is invariant is problematic because the set of observed variables one might be interested in in the psychological sciences is almost necessarily random, in the sense that they almost certainly will be variable and not fixed. This occurs for a number of reasons. First, measures are often the target of development for various reasons, usually because of changes in theories of what a measure should comprise, or relatedly, because of concerns about lack of generalizability of measurements from those

measures. These measure changes, even in secondary variables within a system, can change partial covariances between primary variables of interest. Second, even when dealing with perfect measures, the way a psychological variable space can be measured is often fluid, in the sense that boundaries between constructs are often fuzzy and perhaps arbitrary. Idiographic theories of personality and psychopathology, for example, essentially argue that the structure of variables might be intensively individual-specific; the random nature of variables in these frameworks is fairly explicit and foundational. Finally, and relatedly, one might argue that in psychological systems, the relevant set of variables is fluid and changeable, in the sense that it is impossible to define a closed set of causally relevant variables when discussing human behavior, even if the measures are perfect representations of their corresponding measurands.

For example, one can consider the symptoms of PTSD on their own as a system, but the components of that PTSD system are undoubtedly related to other variables described by other diagnostic criteria, as well as extradiagnostic variables such as sociological and biological factors. Unlike covariances, the coconcentrations between two PTSD symptoms, however defined, will change at the population level depending on whether or not variables such as substance use, trauma history, social support, and so forth are also included in the model. Each of those variables will change, moreover, as measures of the variables change, whether due to design constraints, improvements in measurements, changes in theory, or changes in the individuals being studied. Perhaps most significantly, however, the coconcentrations between two variables, even if they themselves are measured the same, will change if *other* variables in the matrix are measured differently.

This property of concentration matrices might be desirable from one perspective, but from another perspective, it obscures the precise mechanisms by which modeled relationships are changing as other observed variables are changing. In this sense, the phenomenon whereby coconcentrations can go to zero as other measures in a variable set become more reliable (Equations 3-5) points to the importance of considering not only what is observed in a sample, or what would be observed in a population, but what would be observed under a theoretically ideal design.

Methods designed to model measurement error within networks (Epskamp, et al., 2018) provide one

way to address this problem, but at some level require some explicit representation of a theoretical process by which variability in the observed variables arise. This might be a measurement process involving a latent variable or some other process, but it would not be agnostic to the scope of the specific variables being modeled. In other words, the model explicitly represents a difference between the variables that happened to have been sampled in the data, and those that might exist in some limit. In an important sense, the introduction of latent variables into a model can provide a risky test of competing accounts, in that they delineate, even if implicitly, a range of expectable population scenarios under different assumptions.

Conclusions

Waldorp and Marsman are correct that all the information contained in the concentration matrix is contained in the covariance matrix and vice versa. However, although this is true in a statistical sense, it is not necessarily true in an interpretive sense. Researchers tend to focus on specific elements of the concentration matrix, as the individual paths are often part of the appeal of graph models as used in psychometric network analysis. However, as shown here, if this is not done carefully, misleading interpretations can arise, as interpretation of one element of a partial covariance matrix depends on the rest of the matrix.

Relationships between latent variable and network models have long been noted (e.g., Snijders & Nowicki, 1997; for a history and general discussion see Smith, Asta, and Calder, 2019), and we have commented on this ourselves (Markon & Jonas, 2016). As some of us have noted, the equivalence may extend beyond statistical concerns into theoretical equivalences, in that higher-order causal explanations might be required to account for structural features of networks (in the sense of communities; Markon & Jonas, 2016). Even still, if either causal interpretation can be applied, due to lack of causally informative information or strong equivalence, a focus on individual paths or subsets of paths might be misleading without awareness of alternate possible interpretations.

Waldorp and Marsman claim to have addressed our concerns about instability and reliability by showing that all variance in the correlation matrix is captured in symptom networks of partial correlations, but our central concerns remain. While a given variance component may technically be part of the model

regardless of the framework used, it may not be considered in the detailed features of symptom networks that are the focus of their interpretation, such as the presence, absence, sign, and strength of individual edges. These detailed features are based on individual elements of the concentration matrix, which are vulnerable to random sampling of persons, measures, and other sources of variability affecting concentration matrices as discussed above.

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Table 1

$\Lambda_z = 0.70$					
1.242	-0.091	-0.091	-0.187	-0.187	-0.187
-0.091	1.242	-0.091	-0.187	-0.187	-0.187
-0.091	-0.091	1.242	-0.187	-0.187	-0.187
-0.187	-0.187	-0.187	1.575	-0.386	-0.386
-0.187	-0.187	-0.187	-0.386	1.575	-0.386
-0.187	-0.187	-0.187	-0.386	-0.386	1.575
$\Lambda_z = 0.90$					
1.303	-0.030	-0.030	-0.214	-0.214	-0.214
-0.030	1.303	-0.030	-0.214	-0.214	-0.214
-0.030	-0.030	1.303	-0.214	-0.214	-0.214
-0.214	-0.214	-0.214	3.746	-1.517	-1.517
-0.214	-0.214	-0.214	-1.517	3.746	-1.517
-0.214	-0.214	-0.214	-1.517	-1.517	3.746

Note. Values in table are concentration matrices corresponding to unidimensional latent variable models with 6 items, 3 of which have loadings of 0.50, and the other 3 of which have loadings of either 0.70 or 0.90.

Figure 1

Note. Values are minimum absolute (co)concentrations as a function of the large loading, assuming a factor model as described in the text. Minimum absolute concentrations were calculated for loading values from 0.50 to 0.99 (“large loadings”), assuming the remaining loadings (“small loadings”) were 0.50. Different lines represent different numbers of items with larger and smaller loadings, respectively, as labeled in the key (that is, “6, 3” represents the case of 6 items all having the large loading value, and 3 items with a loading of 0.50).