Taking the Relational Structure of Fractions Seriously:
Relational Reasoning Predicts Fraction Knowledge in Elementary School Children

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Abstract

Understanding and using symbolic fractions in mathematics is critical for access to advanced STEM concepts. However, children and adults consistently struggle with fractions. Here, we take a novel perspective on symbolic fractions, considering them within the framework of relational structures in cognitive psychology, such as those studied in analogy research. We tested the hypothesis that relational reasoning ability is important for reasoning about fractions by examining the relation between scores on a domain-general test of relational reasoning (TORR Jr.) and a test of fraction knowledge consisting of various types of fraction problems in 201 second grade and 150 fifth grade students. We found that relational reasoning was a significant predictor of fractions knowledge, even when controlling for non-verbal IQ and fractions magnitude processing for both grades. The effects of relational reasoning also remained significant when controlling for overall math knowledge and skill for second graders, but was attenuated for fifth graders. These findings suggest that this important subdomain of mathematical cognition is integrally tied to relational reasoning and opens the possibility that instruction targeting relational reasoning may prove to be a viable avenue for improving children's fractions skills.

Keywords: fractions, relational reasoning, numerical cognition

Introduction

Proficiency with symbolic fractions is critical for advanced learning in STEM disciplines (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 1989, 2000; National Mathematics Advisory Panel, 2008; Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; Siegler, Fazio, Bailey, & Zhou, 2013). As Beckmann and Izsák (2014) assert “The concept of ratio is key in mathematics, providing access to such central topics as slope of a line, similarity, trigonometric ratios, and probability as well as applications of these ideas to the physical and social sciences.” However, decades of work demonstrate that both children and adults struggle when reasoning about symbolic fractions (e.g., Bright, Behr, Post, & Wachsmuth, 1988; Dufour-Janvier, Bednarz, & Belanger, 1987; Kerslake, 1986; Lesh, Behr, & Post, 1987; Mack, 1995; Ni, 2001; Novillis, 1979). Common errors in fraction reasoning are often attributed to a “whole-number bias,” the inclination to consider the components of a fraction separately rather than considering the fraction as determined by the holistic relations between its two components (Braithwaite & Siegler, 2017; Ni & Zhou, 2005). Several domain-specific explanations for these difficulties exist, ranging from interference from prior knowledge (e.g., Hartnett & Gelman, 1998; Streefland, 1978; Vamvakoussi & Vosniadou, 2004; see Pitkethly & Hunting, 1996, for a review) to the possible absence of necessary neurocognitive startup tools that would support intuitions of rational number constructs (Feigenson, Dehaene, & Spelke, 2004; Gelman, 2015; Gelman & Meck, 1983; Bonato, Fabbri, Umiltà, & Zorzi, 2007). In this study, we explore another possible, more domain general factor: this tendency to focus on fraction components may reflect difficulties with relational reasoning more broadly. There is a strong *a priori* case for the role of relational reasoning in fractions understanding, as the bipartite structure of a symbolic fraction is inherently relational. Indeed, a fraction cannot be meaningfully understood without relating the two components (numerator and denominator) to

each other. If the tendency to think about the components of a symbolic fraction separately reflects a general weakness in relational reasoning ability, then measures of general relational reasoning should be predictive of competence with symbolic fractions. Moreover, this should be true over and above other domain specific measures of numerical competence with whole numbers. The current study sought to test this hypothesis in two groups of elementary school children who either have, or have not yet, had extensive formal instruction with fractions.

Fractions as a special case of relational reasoning

The structure of symbolic fractions can be compared to the structure of general propositional relations frequently studied in cognitive psychological research on analogy. Viewed from this research perspective, relational reasoning allows us to make inferences and to generalize based on the *roles* that entities play, rather than the *properties* of the entities (Doumas, Hummel, & Sandhofer, 2008; Hummel & Holyoak, 2003). In an analogy, the relation or relational structure is the key quality shared by the source and the target (Alexander et al., 2016; Gentner & Smith, 2012). For example, in the verbal analogy “cat: mouse:: frog: fly” (cat is to mouse as frog is to fly), the implied relation is “preys on.” The entities cat and frog take the role “predator”, and the entities “mouse” and “fly” take the role “prey.” Cat and frog cannot be aligned based on surface features or properties – nor can mouse and fly. However, by recognizing the common relation in both sets, we can make inferences from one to the other. For example, if we know that bringing cats into an ecosystem can reduce the rodent population, it is reasonable to infer that introducing frogs (the cat analog) could reduce the fly population (rodent analog).

Symbolic fractions share several formal properties with propositional relations, and respecting those relations similarly helps promote inferences and understanding of the fractions domain. In a common symbolic fraction of the form “a/b” (such as $\frac{3}{4}$), the roles are numerator

and denominator. As with many of the relations studied in analogy and relational reasoning research, fractions are not commutative (“cat preys on mouse” $\neq$ “mouse preys on cat” and $5/7 \neq 7/5$). Also in parallel with general propositional relations, the numerator-denominator relation is represented abstractly when those roles are represented independently of any particular component numerals (entities) or their properties; such abstract representations facilitate transfer (Gick & Holyoak 1980, 1983).

Although some previous research has highlighted the relational nature of symbolic fractions, a domain general object-role relational reasoning approach has not been widely applied to fractions. Math education researchers have pointed out for some time that symbolic fractions have many possible interpretations (such as the part-whole, part-part, measure, or operator interpretations/subconstructs), many of which hinge on the relational nature of the fraction in some way (Behr, Lesh, Post, & Silver, 1983; Empson et al., 2010; Ohlsson, 1988). Despite this complexity, considerable recent work in psychology has emphasized the importance of a single interpretation: the unidimensional magnitude of a fraction (for example, its position on number line) (Siegler, 2016). These studies demonstrate that understanding of fraction magnitudes (particularly with fraction number lines) is an important predictor of fraction knowledge and overall mathematical competence (Fazio, Bailey, Thompson, & Siegler, 2014a; Hansen et al., 2017; Hurst & Cordes, 2018a, 2018b; Yoshida & Shinmachi, 1999) and suggest that this subconstruct should be emphasized in instruction (Fazio, Kennedy, & Siegler, 2016; Siegler, 2016). While fraction magnitude understanding is clearly important, additional relation-based approaches to fractions could complement the work done on fraction magnitudes.

There are in fact two senses in which we are referring to fractions as relations. The first is a sort of formal relation, based only on the assignment of roles, and potentially applies to any

fraction interpretation/subconstruct. For example, saying that “ $2/3$ means 2 parts out of 3” is analogous to “ $4/7$ means 4 parts out of 7” in the sense that each has a denominator, which can be interpreted as a whole or total, and each has a numerator, which can be interpreted as a subset of the whole. Understanding the entailments of $2/3$ —for example, that it has a complement which can be found by subtracting from the total in the form $3/3$ ($3/3 - 2/3 = 1/3$)—allows us to make inferences about $4/7$ ($7/7 - 4/7 = 3/7$). Although we have used the part-whole subconstruct here, any interpretation of numerator and denominator fits this sense of relational. An understanding of the qualitative relation shared by all fractions is necessary for transfer to examples or problems using numerators and denominators different from those used for training/teaching.

However, there is also a more specific type of *quantitative relation* that is shared across multiple exemplars from the equivalence class of fractions of a given magnitude. For example, the fractions $3/4$ and $75/100$ are not merely analogous, but are in fact equivalent (members of the same equivalence class). They represent the same quantitative relation (or relative quantity). In some math education literature, this is referred to as a multiplicative relation, and may be particularly salient for fraction subconstructs/interpretations such as operator/scaling factor. Behr et al. (1983) describe this subconstruct of a fraction as follows: “[in this subconstruct] p/q is thought of as a function that transforms geometric figures to similar geometric figures p/q times as big, or as a function that transforms a set into another set with p/q times as many elements.” They also refer to it as a “stretcher-shrinker” function or machine (p. 6). Although the components may change ($3/4$, $75/100$, $8/12$), the multiplicative relation is constant.

Relational reasoning and common fractions errors

Reframing common errors studied in numerical cognition research as failures of relational reasoning may help to link these errors with broader domain general accounts of errors

in relational reasoning. For example, errors that are often attributed to a “whole-number bias” can be seen as a tendency to focus on the components of the fraction separately, rather than considering the holistic, relational meaning of the fraction (Braithwaite & Siegler, 2017; Ni & Zhou, 2005). Two of the most common sources of error are attention to the numerator or denominator separately rather than attending to the holistic magnitude (for example, choosing $4/9$ as larger than $3/4$ because $4 > 3$ and/or $9 > 4$) and treating equivalent fractions with larger components as being larger (treating $16/20$ as larger than $4/5$ even though they are equivalent) (Braithwaite & Siegler, 2017; DeWolf & Vosniadou, 2015; Fazio et al., 2014a; Ganor-Stern, Karasik-Rivkin, & Tzelgov, 2011; Meert, Gregoire, & Noel, 2009; Zhang, Fang, Gabriel, & Szucs, 2014). This whole-number bias has previously been considered in terms of issues specific to numerical cognition, such as difficulties in mapping number symbols to number concepts (Mix et al., 1999; Sophian et al., 1997), differences in the representations of discrete vs. continuous quantities (Hartnett & Gelman, 1998; Gelman, 1991), or differences in the possibility of experiential or embodied grounding (Confrey, 1994; Streefland, 1991).

We suggest that these errors might be better characterized as failures to think relationally. If students treat the components as separate entities to compare, rather than as role-fillers in a relation, the student will make exactly these types of errors: for example selecting $2/3 > 3/4$ because $2 > 3$ or $3 > 4$ could be a failure to recognize and engage with the relation between 2 (numerator) and 3 (denominator) and between 3 (numerator) and 4 (denominator). That error could also be characterized as attention to the surface features (magnitude/identity) of the components rather than the *roles* they are assigned to. Turning to arithmetic operations with fractions, additional difficulties may stem from failures to consider and maintain these roles when carrying out operations. On the other hand, identifying equivalent fractions requires the

more specific, “quantitative” relational understanding of equivalence classes. Given this relational framing of fractions and fraction errors, it might be the case that reasoning about fractions depends on the ability to reason about relations more generally.

Previous studies of fractions and relational reasoning

Previous researchers have highlighted links between relational reasoning and human higher cognition in general (Halford, Wilson, & Phillips, 2010) and mathematics in particular (Alexander, 2017), even specifically implicating fractions. For example, Richland, Stigler, and Holyoak (2012) argued that conceptual understanding in mathematics depends on relational structure, and specifically point to previous research on learning by analogy. They illustrated their argument for the importance of relational concepts in mathematics with examples of poor reasoning about fractions among college students, such as their difficulty placing rational numbers (fractions and decimals) on a number line, or their failures to correctly answer the question “Which is greater? $a/5$ or $a/8$.”

These theoretical arguments have been bolstered by empirical results. Domain-general relational reasoning measures have been found to predict early math performance in preschoolers (Collins & Laski, 2018) and calculation skill in 5-13 year olds (Fyfe, Evans, Matz, Hunt, & Alibali, 2017), as well as intervention effectiveness in engineering students (Dumas & Schmidt, 2015). Moreover, there is some precedent in cognitive psychology for connecting relational reasoning and symbolic fractions in particular. In general, decimals are thought to be easier to process than fractions. For example, DeWolf and colleagues (DeWolf, Grounds, Bassok, & Holyoak, 2014) observed greater reaction times for fractions than other types of compositional numbers (multidigit integers, decimals) in a magnitude comparison task. They attributed this difference to greater processing demands created by the greater relational complexity of

symbolic fractions' bipartite structure. However, the affordances of symbolic fractions can also facilitate certain aspects of numerical reasoning. DeWolf, Bassok, and Holyoak (2015a) find that symbolic fractions facilitate relational problem solving compared with solving the same types of problems with decimals (a representation that privileges magnitude information over relational structure). In another study, DeWolf and colleagues demonstrated relational priming from multiplication with fractions to other arithmetic operations within the same multiplicative schema (DeWolf, Son, Bassok, & Holyoak, 2017).

However, most psychology research on fractions and relational reasoning to date has focused on fractions as a bridge to algebraic reasoning (Booth & Newton, 2012; Empson et al., 2010) and has used math domain-specific measures, (such as fraction arithmetic performance or relational fraction knowledge) rather than domain-general measures of relational reasoning as predictors. For example, Hurst and Cordes (2018a, 2018b) found that fraction arithmetic performance predicted algebraic reasoning above and beyond rational number magnitude knowledge and decimal arithmetic. DeWolf, Bassok, and Holyoak (2015b) also introduced the construct of *relational fraction knowledge* and tasks that tap relational fraction knowledge, such as non-calculation questions focused on part-part or part-whole relationships. They found that this relational fraction knowledge explained unique variance in middle-school students' early algebraic reasoning. They also found that priming with a relational fraction task facilitates algebraic reasoning more than priming with a procedural fraction task in both pre-algebra students and adults (DeWolf, Bassok, & Holyoak, 2016).

The Current Study

However, there is currently no research on whether and to what extent domain-general relational reasoning ability, like that seen in analogy research, is related to fraction knowledge in

itself. Given the structural similarity between common fractions and propositional relations, we hypothesized that a common cognitive process might support both domain-general relational reasoning and reasoning about fractions. To test this hypothesis, we used a domain-general assessment of relational reasoning with known psychometric properties, the Test of Relational Reasoning Jr. (TORR Jr.), and a paper-and-pencil test of fraction knowledge, the Fraction Knowledge Assessment (FKA), in two groups of school-age children: 2nd graders, who have little or no formal instruction with symbolic fractions, and 5th graders, who have received or are receiving formal instruction with symbolic fractions. Because we expected general math ability, fraction magnitude understanding, and general non-verbal reasoning ability to be important predictors of fraction knowledge, we included measures of these abilities as controls to examine whether relational reasoning predicts unique variance in fraction knowledge above and beyond what is accounted for by these measures.

Methods

Participants

We recruited participants from multiple local school districts, including a mix of urban and suburban districts with varying demographics, as part of a longitudinal study of children's acquisitions of fractions. In total, 351 children participated. Of these, 201 were second graders when they entered the study (mean age = 8.04 years, 109 male) and 150 were fifth graders (mean age = 10.92 years, 82 male). Two participants were excluded from the current analysis because the parents disclosed a learning disability diagnosis (1 ADHD, 1 dyslexia) after initial participation in the study. Not all scores were available for all participants due either to discontinuation due to participant fatigue/non-compliance, or experimenter error (see Tables 4 & 5 for details). Participants received \$10 per hour of participation and a small toy per study visit. All study procedures were approved by the IRB (IRB #2016-0665).

Procedure

Participants completed assessments in two lab visits on separate days. At the first visit they completed standardized measures of cognitive capacities (WISC and WJ subtests, see below), and at the second visit they completed fraction magnitude comparison and fraction number line tasks, a Fraction Knowledge Assessment, and the TORR Jr. The duration of each visit was about 60 minutes.

Materials

Dependent variable—fraction knowledge.

Fraction Knowledge Assessment (FKA). The FKA is a pencil-and-paper assessment of both procedural and conceptual fraction knowledge. We constructed this instrument using items taken from key national and international assessments, including the National Assessment of Educational Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS), and from instruments developed by psychology and math-education researchers (Carpenter, 1981; Hallett, Nunes, Bryant, & Thorpe, 2012). A separate FKA was constructed for each grade, featuring grade-appropriate items (i.e., there was a second grade version of the FKA and a fifth grade version of the FKA). Each version of the FKA contained 41 items, and each had strong internal consistency (second grade FKA Cronbach's $\alpha = .90$; fifth grade FKA Cronbach's $\alpha = .92$).

Relational reasoning measure.

Test of Relational Reasoning Jr. (TORR Jr.). The TORR Jr. was developed by Alexander et al. (Alexander, Dumas, Grossnickle, List, & Firetto, 2015; Dumas & Alexander, 2016; Jablansky, Alexander, Dumas, & Compton, 2015) specifically to test relational reasoning among children. All items in the TORR are figural (i.e. non-verbal) and do not require any

outside information to solve. The entire assessment consists of four subtests of eight items each (total 32 items) designed to test different forms of relational reasoning: analogy, anomaly, antinomy, and antithesis. For the present study, we used only the analogy subtest of the TORR Jr. for several reasons: First, as this study was part of a larger longitudinal study, time constraints prevented administration of the entire TORR Jr. Second, we initially piloted testing with the analogy and antithesis subtests based on their reported psychometric properties (Alexander et al., 2015; Jablansky et al., 2015), and for construct face-validity and consistency with previous research (analogical reasoning tasks are the most widely used measure of relational reasoning). We observed a problematic floor effect for the antithesis subtest in our pilot sample, so we proceeded with only the analogy subtest.

Domain-general predictors.

WJ-III subtests: Spatial Relations and Auditory Working Memory. We used two subtests of the Woodcock-Johnson Tests of Abilities, 3rd Edition (WJ-III, Woodcock et al., 2001) as domain-general predictors. In the Spatial Relations subtest, participants are shown a whole shape, then asked to select the component shapes necessary to construct the whole from a set of options. Spatial Relations items probe visual-spatial abilities such as visual segmentation and rotation. Scores on this subtest are understood to reflect non-verbal reasoning ability. During the Auditory Working Memory subtest, participants listen to a mixed series of words and digits and then must rearrange them by first saying the words in order and then the numbers. Working memory, and in particular verbal working memory, is known to be an important predictor of math skills (e.g., Formoso et al., 2018).

WISC-IV subtests: Matrix Reasoning and Block Design. We also used two subtests of the Wechsler Intelligence Scale for Children, 4th edition, (WISC-IV, Wechsler, 2004) to assess

general cognitive abilities. In the Matrix Reasoning subtest, children are shown an array of pictures with one missing square and instructed to select the picture that fits the array from five options. Some late items in the Matrix Reasoning task could be solved by analogical reasoning, but many require only pattern recognition, such as recognizing sequences in object size, color, orientation, or identity. In the Block Design subtest, children put together red-and-white blocks in a pattern according to a displayed model. Block Design is timed, and some of the more difficult puzzles award bonuses for speed. These two subtests were chosen because they reflect non-verbal reasoning and spatial abilities, which have been previously found to predict math abilities, and because they contribute to the Performance IQ (PIQ) sub-score of the WISC.

Non-verbal Reasoning Composite. Given the similarity in task demands and hypothesized construct for Matrix Reasoning, Block Design, and Spatial Relations, we hypothesized that they would load on a single factor. Therefore, we performed a 1-factor confirmatory factor analysis (see supplement for details) and used the resulting factor scores as a predictor variable in our regression models.

Math domain-specific predictors.

WJ-III Subtests: Math Fluency and Calculation. The Math Fluency subtest is a timed assessment in which participants must correctly answer as many simple (single digit, two operand) written addition, subtraction and multiplication problems as possible within 3 minutes. In contrast, the Calculation subtest is untimed and therefore may provide more information about participants' numerical reasoning skills than recall of arithmetic facts (which are reflected more in the Math Fluency test). Items on this subtest range from number identification through arithmetic operations (addition, subtraction, multiplication, division), to geometric, trigonometric, and even calculus operations at the high end. Some items on the Calculation

subtests may involve fraction arithmetic or other fraction operations (for example, simplifying algebraic expressions).

Fraction magnitude understanding predictors.

Fraction magnitude understanding is known to be an important component of overall fraction competence (Fazio, Bailey, Thompson, & Siegler, 2014b; Fazio et al., 2016; Hamdan & Gunderson, 2017; Liu, 2018). In order to assess whether relational reasoning predicts fraction knowledge above and beyond what is predicted by fraction magnitude awareness, we included two measures of fraction magnitude awareness as predictors.

Fraction number line estimation task. The number-line estimation task was modeled on Siegler, Thompson, and Schneider (2011). Participants used a mouse to indicate the position of fraction stimuli on a number line with the end points 0 and 1 in the 0-1 condition, and between 0-2 and 0-5 in two other conditions. Performance was defined as each participants' mean percentage of absolute error (PAE), where $PAE = (|answer - correct\ answer| / \text{numerical range})$, as is typical for these tasks. A smaller PAE indicates greater accuracy.

Fraction magnitude comparison task. This task was adapted from the task used for adults in a previous study (Binzak et al., 2019). In each trial, participants viewed a screen on which two fractions appeared and were instructed to choose the larger of the two. The fractions included all the non-unit proper fractions with denominators of 2-9. Reaction time and accuracy were recorded on each trial, and a mean RT and accuracy rate were calculated for each participant.

Results

Overall performance

Table 1 displays the means and standard deviations for standardized test scores for both second and fifth graders. Mean performance was well above chance on all experimental tasks. As

expected for raw scores, fifth graders scored significantly higher than second graders on all assessments except the FKA. Importantly, the second grade FKA and fifth-grade FKA are different instruments, each with age-appropriate items as well as some overlapping (anchor) items. The current results suggest that for our sample, the second grade FKA was easier for the second graders than the fifth grade FKA was for the fifth graders.

Although we observed a wide range of scores on the standardized assessments, the average scores for both second graders and fifth graders on all subtests were above the expected averages for their respective grades. This likely reflects the self-selection aspect of recruitment (i.e., families with higher income, parental education and familiarity with academic settings were more likely to respond to our recruitment methods).

Table 1. Performance on all measures by grade, with independent sample-t-test by grade

	Possible Raw Score Range	<u>2nd</u> <i>M (SD)</i> <i>n = 201</i>	<u>5th</u> <i>M (SD)</i> <i>n = 150</i>	<i>t</i> <i>df = 349</i>	<i>p</i>
Spatial Relations	0-81	64.33 (6.85)	69.26 (5.99)	7.03	<.001
Block Design	0-68	32.58 (12.04)	42.79 (11.81)	7.92	<.001
Matrix Reasoning	3-35	20.51 (4.87)	24.75 (4.29)	8.48	<.001
Spatial/Block/Matrix composite	NA	-0.34(0.80)	.45 (.72)	9.50	<.001
Auditory working memory	0-45	19.9 (5.28)	25.27 (5.26)	9.44	<.001
Math Fluency	0-160	44.51 (15.91)	75.32 (21.62)	15.38	<.001
Calculation	0-45	13.43 (3.83)	23.15 (4.93)	20.80	<.001
Number Line PAE	0-1	0.27 (0.09)	0.11 (0.08)	15.50	<.001
Fraction Comparison Accuracy	0-1	0.84 (0.13)	0.93 (0.08)	7.00	<.001
FKA Score	0-1	0.63 (0.21)	0.56 (0.26)	2.58	0.01
TORR Jr. Analogy	0-8	3.7 (1.96)	5.37 (1.63)	7.82	<.001

Correlations

Pairwise correlations among the behavioral measures are presented in Tables 2 and 3. For both grades, all predictors were significantly positively correlated as might be expected due to positive manifold effects and the design of the WJ-III and WISC-IV. Visual inspection of the

first column of each table confirms that the TORR Jr. Analogy (and all the other predictors) were significantly correlated with our outcome of interest, the Fraction Knowledge Assessment.

Furthermore, as seen in bivariate scatterplots of Figure 2, there appears to be a positive linear relationship between TORR Jr. Analogy score and FKA score in both grades. However, this observed relationship could be due to other shared factors between the TORR Jr. and FKA. For example, if both scores are related to common underlying factors, including more general cognitive abilities, then those common factors would better explain the observed relationship. In the following regression analyses, we controlled for some of the most likely possible factors.

Table 2. Zero-order correlations between measures for 2nd Graders

	FKA	Analogy	WJ-III WM	WJ-III SR	WISC BD	WISC Matrix	Non-verbal Factor	WJ-III Math	WJ-III Calc	NLE PAE	Mag. Comp.
FKA	1										
Analogy	0.49***	1									
WJ-III WM	0.53***	0.36***	1								
WJ-III SR	0.33***	0.41***	0.38***	1							
WISC BD	0.49***	0.45***	0.4***	0.57***	1						
WISC Matrix	0.48***	0.47***	0.45***	0.48***	0.52***	1					
Non-verbal Factor	0.54***	0.54***	0.50***	0.81***	0.88***	0.79***	1				
WJ-III Math	0.46***	0.31***	0.34***	0.33***	0.37***	0.28***	0.41***	1			
WJ-III Calc	0.58***	0.38***	0.44***	0.42***	0.44***	0.44***	0.53***	0.65***	1		
NLE PAE	-0.54***	-0.29***	-0.33***	-0.10***	-0.28***	-0.24***	-0.26**	-0.31***	-0.47***	1	
Mag. Comparison	0.49***	0.41***	0.26***	0.32***	0.33***	0.40***	0.43***	0.47***	0.42***	-0.20***	1

*p<.05, **p<.01, ***p<.001

Table 3. Zero-order correlations between measures for 5th graders

	FKA	Analogy	WJ-III WM	WJ-III SR	WISC BD	WISC Matrix	Non-verbal Factor	WJ-III Math	WJ-III Calc	NLE PAE	Mag. Comp.
FKA	1										
Analogy	0.41***	1									
WJ-III WM	0.46***	0.4***	1								
WJ-III SR	0.29***	0.45***	0.27***	1							
WISC BD	0.39***	0.43***	0.50***	0.48***	1						
WISC Matrix	0.21***	0.44***	0.41***	0.33***	0.56***	1					

Non-verbal Factor	0.38***	0.54***	0.51**	0.73**	0.90***	0.77***	1				
WJ-III Math	0.32***	0.27**	0.47***	0.14***	0.29***	0.2***	0.27***	1			
WJ-III Calc	0.46***	0.5***	0.53***	0.38***	0.53***	0.4***	0.55***	0.53***	1		
NLE PAE	-0.45***	-0.45***	-0.50***	-0.27*	-0.51***	-0.35***	-0.49***	-0.38***	-0.5***	1	
Mag. Comparison	0.3***	0.22*	0.30***	0.17**	0.28**	0.19*	0.28**	0.44***	0.41***	-0.3***	1

*p<.05, **p<.01, ***p<.001

Figure 2. Scatterplot of FKA proportion correct against TORR Jr. Analogy score for second graders (left) and fifth graders (right) with OLS fitted regression lines. Shaded areas represent 95% confidence intervals.

Multiple regression analyses

We conducted a series of two-stage linear regressions to test whether individual differences in relational reasoning predicted fraction knowledge (FKA score) above and beyond individual differences in domain-general cognitive capacities and math-specific facilities. In the first stage of each analysis, we entered the control variables (either domain-general or math-specific). In the second stage, we added the TORR Jr. analogy score. Regression results are reported in Tables 4 -8. We report standardized coefficients because they facilitate interpretation of effect sizes. We conducted these regressions separately for 2nd graders and 5th graders because 2nd graders scored systematically lower on the TORR and the raw scores of the standardized assessments, and therefore combining 2nd and 5th graders in a regression model would introduce a confound based on grade. We also included age in months because developmental differences within a grade could be a potential confounding variable. Because of the potential overlap

between general non-verbal reasoning ability and relational reasoning ability, we retained the non-verbal reasoning factor as a control predictor in all subsequent models.

Controlling for domain-general cognitive abilities.

One plausible hypothesis explaining the observed relationship between TORR Jr. analogy scores and FKA scores is that TORR Jr. is measuring some domain-general ability such as general non-verbal reasoning ability. To test this hypothesis, we examined the relationship between TORR Jr. analogy and FKA scores while statistically controlling for domain-general predictors, specifically the non-verbal reasoning factor and a measure of auditory working memory. These results are shown in Model 1 and Model 2 of Tables 4 and 5. For both second graders and fifth graders, TORR analogy remained a significant predictor of FKA scores, contributing unique variance. Age in months was also an important predictor of FKA score for second graders but not for fifth graders.

Table 4. Second grade regression models for domain-general variables.

	Model 1	Model 2
Spatial/Block/Matrix composite	0.364***	0.242**
Age in months	0.222***	0.217***
Auditory Working Memory	0.343***	0.314***
TORR Jr. Analogy		0.248***
N	158	158
R-sq	0.452	0.494
adj. R-sq	0.441	0.48
ΔR^2		0.039

Table 5. Fifth grade regression models for domain-general variables.

	Model 1	Model 2
Spatial/Block/Matrix composite	0.381***	0.275**
Age in months	-0.074	-0.07
Auditory Working Memory	0.364***	0.332***
TORR Jr. Analogy		0.227**
N	127	127
R-sq	0.44	0.476
adj. R-sq	0.427	0.459
ΔR^2		0.032

Controlling for math ability and knowledge.

Next, we considered the hypothesis that TORR Jr. analogy score might be accounted for by general mathematical ability. Overall mathematics ability has been demonstrated to be an important predictor of fraction knowledge (Hansen et al., 2017; Ye et al., 2016). It might be the case that TORR predicts fraction knowledge because mathematics ability mediates the relationship between them. For second graders, but not fifth graders, TORR Jr. analogy remained a significant predictor of FKA scores when controlling for WJ-III Math Fluency and Calculation scores, and the additional variance explained by including TORR Jr. analogy made the fit of Model 3 better than that of Model 4 (Tables 6 & 7).

Table 6. Second grade regression models for math predictors.

Second Grade	Model 3	Model 4
Spatial/Block/Matrix composite	0.347***	0.233**
Age in months	0.140*	0.142*
Math Fluency	0.133	0.121
Calculation	0.297***	0.273***
TORR Jr. Analogy		0.235***
<i>N</i>	158	158
<i>R-sq</i>	0.475	0.513
<i>adj. R-sq</i>	0.461	0.496
ΔR^2		0.038

Table 7. Fifth grade regression models for math predictors.

Fifth Grade	Model 3	Model 4
Spatial/Block/Matrix composite	0.340***	0.281***
Age in months	-0.109	-0.103
Math Fluency	0.182*	0.180*
Calculation	0.346***	0.302***
TORR Jr. Analogy		0.155
<i>N</i>	127	127
<i>R-sq</i>	0.511	0.526
<i>adj. R-sq</i>	0.495	0.507
ΔR^2		0.015

Controlling for fraction magnitude understanding.

Finally, we considered whether TORR Jr. analogy could explain performance on the FKA above and beyond fraction magnitude understanding (as measured by fraction number line estimation and fraction magnitude comparison). For both second and fifth graders, TORR Jr. analogy remained a significant predictor. Indeed, for 5th graders, it had an effect as large as fraction comparison accuracy (see Models 5 and 6, Tables 8 and 9).

Table 8. Second grade regression models for fraction magnitude knowledge.

Second Grade	Model 5	Model 6
Spatial/Block/Matrix composite	0.300***	0.242**
Age in months	0.106	0.112
Fraction Comparison Accuracy	0.272***	0.245***
Number Line PAE	-0.377***	-0.354***
TORR Jr. Analogy		0.144*
<i>N</i>	129	129
<i>R-sq</i>	0.547	0.561
<i>adj. R-sq</i>	0.533	0.543
ΔR^2		0.014

Table 9. Fifth grade regression models for fraction magnitude knowledge.

Fifth Grade	Model 5	Model 6
Spatial/Block/Matrix composite	0.306***	0.239**
Age in months	-0.104	-0.11
Fraction Comparison Accuracy	0.153*	0.160*
Number Line PAE	-0.407***	-0.363***
TORR Jr. Analogy		0.171*
<i>N</i>	110	110
<i>R-sq</i>	0.492	0.512
<i>adj. R-sq</i>	0.473	0.489
ΔR^2		0.020

Discussion

We have demonstrated that domain-general relational reasoning ability is related to children's competence with symbolic fractions, suggesting that understanding the relational structure of fractions depends on—or at least can be addressed with—the ability to reason about relations more generally. Specifically, we found an association between a standardized measure of domain-general relational reasoning ability and performance on a paper-and-pencil test of fraction knowledge, in both second graders and fifth graders. For both age groups, the effect size of relational reasoning (as seen in the standardized beta coefficients) as a predictor of fraction knowledge was modest but not trivial by Cohen's guidelines (Cohen, 1988, 1992). The association between relational reasoning ability and fraction performance could not be explained by general non-verbal reasoning, demonstrating that relational reasoning in particular is important for fraction understanding, above and beyond other measures of other types of fluid reasoning. Relational reasoning ability remained a significant predictor even when we controlled for participants' understanding of fraction magnitudes, suggesting that magnitude and relational conceptions of fractions may be complementary, both necessary for fraction mastery. In second graders, but not fifth graders, relational reasoning also explained variance in fraction knowledge above and beyond calculation ability and arithmetic fact knowledge. In contrast, arithmetic fact knowledge (math fluency) contributed to fraction knowledge above and beyond relational reasoning and calculation ability for fifth graders, but not for second graders.

One compelling explanation for this difference between age groups is based on the difference in formal instruction with fractions between the groups. Within our sample, as in the U.S. more generally, fifth graders have usually received at least 2 years of instruction with symbolic fractions, whereas second graders have usually received little or no formal instruction

with symbolic fractions. It may be the case that fifth graders are therefore more able to leverage their math fact knowledge to help them solve fraction arithmetic problems because they recognize symbolic fractions as a subset of the number system they are already familiar with. It could similarly be the case that they have memorized certain arithmetic facts about symbolic fractions, particularly highly frequent/familiar fractions. A related possibility is that fifth graders have learned and are able to use procedures to solve fraction problems, as instruction in such procedures is common in 3rd-5th grade fraction instruction in the U.S. Once they learn these procedural strategies, they may be less dependent on relational consideration of the fractions than before learning them. In contrast, second graders, lacking the fluency or procedural strategies, must grapple with the fraction problems using their general relational reasoning strategies or capacities. Future research that attempts to measure and isolate the effect of procedural fraction knowledge as well as relational reasoning ability on overall fraction competence could clarify this issue.

In our future work, we will aim to confirm the direction of causality and rule out hidden confounds in the relation between relational reasoning and fraction knowledge. The data from the current study were collected as part of the first wave of data in a longitudinal study. When coupled with later data points, we will be able to use temporal contiguity to assess causality and to chart growth curves. Future experimental studies might further unpack these causal relations by assigning participants to relational reasoning training groups to be contrasted with controls. Previous studies have shown that it is possible to improve relational reasoning skills (Mackey, Hill, Stone, & Bunge, 2011; Mackey, Park, Robinson, & Gabrieli, 2017), but it is unclear how far these training effects transfer. Examining construct validity would also be desirable, for example, by including additional subtests of the TORR Jr. or different (e.g. experimental)

measures of relational reasoning. It would be informative to know whether it is analogy alone or relational reasoning more broadly construed that is most important for fraction competence.

Finally, although the current study was not designed to address this question, future item-based analysis fraction knowledge assessments could clarify what types of fraction knowledge depends most on relational reasoning.

Cognizant of the fact that relational reasoning plays a role in fraction understanding, we may now be able to translate basic cognitive science findings on improving relational learning into methods that improve fraction learning. For example, comparison, contrast, relational language, and structural alignment are known to improve relational learning (Christie & Gentner, 2010, 2014; Gentner, 2010; Kurtz, Boukrina, & Gentner, 2013), and instructional methods that use these techniques may prove beneficial if adapted for use with fractions. However, there is much work yet to be done to determine how best to translate the findings from the lab to the classroom; for example, although introducing fraction division by analogy to whole number division has shown some benefits, some adjustments may need to be made to this approach before it is more widely effective (Sidney & Alibali, 2015; Sidney & Alibali, 2017).

More broadly, these results suggest a role for considering relational complexity in curricular knowledge components for mathematics. Relational complexity can be defined in terms of the dimensions of variation required to define or solve a problem (Halford, Wilson, & Phillips, 1998; Livins & Dumas, 2015). From this perspective, reasoning about whole numbers or performing whole number arithmetic differs from reasoning about fractions or fraction arithmetic because fractions are more relationally complex than whole numbers: fractions have a bipartite form, therefore at least binary complexity; whole numbers have a simpler, unary form, therefore a lower level of relational complexity. Behavioral and neuroimaging findings confirm

that, like other higher cognitive functions including executive function, relational reasoning ability increases across developmental time (Bazargani, Hillebrandt, Christoff, & Dumontheil, 2014; Crone et al., 2009; Dumontheil, 2014; Krawczyk, Holyoak, & Hummel, 2004; Rattermann & Gentner, 1998;) and that increasing relational complexity increases processing demands (Cocchi et al., 2014; Xiao et al., 2014) However, to date no work has been done to characterize the relational complexity of knowledge components in the mathematics curriculum. It will rarely be as straightforward as the difference in relational complexity between whole numbers and fractions, but nevertheless feasible methods for measuring relational complexity exist (Feldman, 2000; Halford, Andrews, & Jensen, 2002; Halford, Wilson, & Phillips, 1998). Characterizing relational complexity can guide choices in instructional strategy and curriculum design to make knowledge components developmentally appropriate in terms of relational complexity, and also to appropriate scaffolding (for example, reducing working memory load) when introducing more complex knowledge components.

Beyond quantifying relational complexity, systematic identification of the specific relational structures in curricular knowledge components could facilitate teaching and learning of these concepts. For example, if the relational structure of the knowledge component is known, then the instructor or curriculum designer can choose a familiar concept with a similar relational structure for learning by analogy. However, in cases where a familiar analog is not available—as when the relational structure is one the learner has not encountered before—basic research on induction of relational structures (Gick & Holyoak, 1983; Goldwater, Don, Krusche, & Livesey, 2018; Halford & Busby, 2007; Halford, Bain, Maybery, & Andrews, 1998) may provide insights for new instructional strategies.

The current results demonstrate the importance of relational reasoning for fraction understanding, and as a corollary, point to importance of recognizing relational complexity and relational structures in the mathematics curriculum. Addressing relational structures and relational complexity in instructional strategies and curriculum designs could provide a new route to improving teaching and learning of these concepts. For fractions in particular, helping students engage with the relational subconstructs could complement existing successful magnitude-based instructional methods and lead to greater overall mastery of fractions.

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