Reliability as Lindley Information

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Code accompanying this article is available at <https://osf.io/w6j2t/>. Prior versions of this manuscript have appeared at <https://psyarxiv.com/vgpfb/>.

Abstract

This paper introduces a definition of reliability based on Lindley information, which is the mutual information between an observed measure and latent attribute. This definition reduces to the traditional definition of reliability in the case of normal variables, but can be applied to any joint distribution of observed and latent variables. Importantly, unlike traditional definitions of reliability, this formulation of reliability applies to vector- or matrix-valued estimates and summaries of responses, and therefore generalizes reliability to sets of scores and estimates in addition to individual scores and estimates. This formulation also leads to new bounds for reliability, as well as newly reported relationships between reliability and the traditional Fisher information function familiar in item response theory literature. This form of reliability can be estimated using formulae, or methods used in Bayesian inference such as Markov Chain Monte Carlo (MCMC) depending on the case. Examples based on well-studied datasets are provided, as well as applications to drift-diffusion modeling and randomly-varying intraindividual covariance structures.

The purpose of this paper is to explicate a definition of reliability based on Lindley information, the mutual information between measurement observations and parameters. Unlike classical definitions of reliability based on error variance or closely related quantities such as Fisher information, this definition of reliability is based on shifts in measurement estimates from prior to posterior distributions. It provides an alternative perspective on measurement precision, one that integrates reliability theory with Bayesian inference and information theory. It coincides with classical definitions in important scenarios, such as when distributions are normal, but allows for applications in which classical definitions might be difficult or impossible to apply.

After reviewing classical reliability theory, I discuss the application of this definition of reliability to the case where measurement errors are normally distributed, showing that it reduces to the traditional definition of reliability in that case. I then discuss the properties of this form of reliability in increasingly general cases. I then discuss minimax bounds on reliability implied by findings in the information theory literature, which links the traditional Fisher information function of IRT to reliability. I end with a discussion of estimation and future directions.

Background: Classical Reliability Theory

Classical Definitions of Reliability

Classically, reliability has been defined in many ways, depending on context and sources of influence on measurements (e.g., Revelle and Condon, 2019). In general, however, these classical definitions quantify measurement precision in terms of a variance ratio: a ratio of total variance associated with measurands of interest, to the total variance in the measurement, including extraneous sources of variance such as error. As these are ratios of partial variance to total variance, they generally range from 0 to 1, and can also be construed of as correlations, such as the correlation between a latent trait estimate and its true value, or as the correlation between two parallel measures (Kim, 2012).

Assume that a trait estimate $\hat{\theta}$ (for example, a maximum likelihood or an expected a posteriori [EAP] estimate) can be obtained from an item response model with additive error: that is, $\hat{\theta} = \theta + e$. Note this assumption is not necessarily about the model of the response process per se, as in the sense of an item response function, but rather about the possibility of modeling an estimate implied by the response process model in terms of true value plus additive error — a continuous trait estimate can be obtained from a model of dichotomous item responses, for example, and that estimate can be modeled in terms of a true value plus error.

Then, in the case of unbiased estimates, the reliability can be given as (Kim, 2012, Mellenbergh, 1993)

$$\rho = \frac{\sigma_{E[\hat{\theta}|\theta]}^2}{\sigma_{\hat{\theta}}^2} = \frac{\sigma_{E[\hat{\theta}|\theta]}^2}{\sigma_{E[\hat{\theta}|\theta]}^2 + E[\sigma_{\hat{\theta}|\theta}^2]} = 1 - \frac{E[\sigma_{\hat{\theta}|\theta}^2]}{\sigma_{\hat{\theta}}^2} = 1 - \frac{\sigma_e^2}{\sigma_{\hat{\theta}}^2} \quad (1)$$

where E denotes the expectation, here taken with regard to the latent variable θ .

Different formulations of reliability will differ depending on components of nonerror variance included. For example, in the case of biased estimates, parallel measures and latent-true value formulations of reliability will differ slightly, as the parallel forms formulation will include bias variance to a greater extent. However, in practice even with bias the two formulations will generally produce similar estimates, becoming increasingly similar as test length increases (Kim, 2012).

Fisher information, the form of information commonly discussed in traditional item response theory literature (in the form of test or item information functions), is related to reliability in that the error variance at a given value of the latent variable can be approximated by the inverse of the Fisher information at that point. That is, for maximum likelihood estimates, denoting the Fisher information as $I(\theta)$, $\sigma_{\hat{\theta}|\theta}^2 \approx I(\theta)^{-1}$, which gives

$$\rho = 1 - \frac{E[I(\theta)^{-1}]}{\sigma_{\hat{\theta}}^2} \quad (2)$$

as another estimate of reliability .

Multidimensional extensions can be obtained from Equation 1 by substituting as appropriate. For example, if the observed score is given under an additive error model with, $\hat{\theta} = f(\boldsymbol{\theta}) + e$ where $\boldsymbol{\theta}$ is a vector of latent variables, then the expectation in Equation 1 becomes $E[\sigma_{\hat{\theta}|\boldsymbol{\theta}}^2]$, taken with regard to the vector-valued variable $\boldsymbol{\theta}$ (see e.g., Haberman & Sinharay, 2010). Depending on the nature of $f(\boldsymbol{\theta})$, model, and what is treated as residual or error variables, this can lead to different indexes of reliability corresponding to total reliable variance, or the variance associated with a particular factor or factors of interest (Revelle & Condon, 2019; Zinbarg, Revelle, & Yovel, 2005).

Limitations of Classical Reliability Theory

Although classical reliability can be applied in a wide range of scenarios, it does have important limitations. One is the assumption that the observed score or estimate can be modeled as a function of the latent variable with additive error. Although a wide variety of models meet this criterion, many important types of models do not. Latent class models are one such example: in latent class models, estimated true values can be nominal in scale and not even compatible with an additive function. To accommodate such models, reliability theory would have to be extended to scenarios with other error structures and scales of measurement.

Novel measurement designs have also allowed for new types of measurands that do not easily fit within classical reliability frameworks. Intensively longitudinal designs (e.g., ecological momentary assessments), for example, have shifted focus onto individual differences in covariances and other complex features of psychological processes. These may be matrix-valued (as is the case with intraindividual covariances or partial covariances) or take other forms, and in such cases it might be unclear how to define reliability within classical frameworks. Extending reliability theory to other measurands would provide a framework for quantifying measurement precision with regard to a wider variety of variables of interest.

Reliability as Lindley Information

Reliability can be generalized to other measurement scenarios by conceptualizing it as a rescaled form of Lindley information (Lindley, 1956). In this case reliability is given as

$$\rho = 1 - e^{-2\iota} \quad (3)$$

where ι is the Lindley information, which in this context is the mutual information $M(\mathbf{X}, \boldsymbol{\theta})$ (Cover & Thomas, 2006) between an observed variable $\mathbf{X}$ and a latent attribute $\boldsymbol{\theta}$:

$$\iota = M(\mathbf{X}, \boldsymbol{\theta}) = \int p(\mathbf{x})p(\boldsymbol{\theta}|\mathbf{x}) \ln \left[\frac{p(\boldsymbol{\theta}|\mathbf{x})}{p(\boldsymbol{\theta})} \right] d\mathbf{x} d\boldsymbol{\theta} = \int p(\mathbf{x}, \boldsymbol{\theta}) \ln \left[\frac{p(\mathbf{x}, \boldsymbol{\theta})}{p(\mathbf{x})p(\boldsymbol{\theta})} \right] d\mathbf{x} d\boldsymbol{\theta} \quad (4)$$

Either $\mathbf{X}$ or $\boldsymbol{\theta}$ could be multidimensional, and here the notations $\iota = \iota(\mathbf{X}) = M(\mathbf{X}, \boldsymbol{\theta})$ will be used.

Lindley information can be interpreted in various ways depending on inferential paradigm and perspective: as an index of the expected change in probability of $\mathbf{X}$ and $\boldsymbol{\theta}$ under knowledge of their joint distribution, relative to knowledge only of their independent marginal distributions; as the expected change in probability of $\boldsymbol{\theta}$ given the data, relative to the prior distribution of $\boldsymbol{\theta}$; as the expected per-observation log-likelihood ratio comparing a model of measure informativeness to a model of measure uninformativeness (Kinney & Atwal, 2014); or as the difference in codelength required to describe the data and $\boldsymbol{\theta}$ under a model of informativeness compared to a model of uninformativeness (Ebrahimi, Soofi, and Soyer, 2010 and Grunwald, 2007 discuss fundamentals of Lindley information and its interpretation).

Like traditional reliability, the definition of reliability implied by Equations 3 and 4 produces a statistic that generally ranges between 0 and 1, and reflects the measurement precision of an indicator, such that values closer to 1 indicate perfect precision and values closer to 0 indicate no precision. Importantly, it reduces to the familiar definition of reliability in cases where measures are normally distributed. This formulation of reliability

is applicable to any joint distribution of $\mathbf{X}$ and $\boldsymbol{\theta}$, including that of discrete observed and latent variables, and can apply to items, composites, and trait estimates. Some examples include unidimensional and multidimensional IRT trait estimates, class estimates from latent class analysis, mixture subpopulation estimates from finite mixture models, trait estimates from normal distribution factor analysis, or sum scores, if there was a model of the probability of a given sum score given latent trait or class value.

Importantly, unlike traditional definitions of reliability, this formulation of reliability applies to vector-or matrix-valued summaries of responses (for example, where randomly varying covariance matrices are the latent variables of interest), and therefore generalizes reliability to sets of scores and estimates in addition to individual scores and estimates. This formulation also leads to new bounds for reliability, as well as newly reported relationships between reliability and the traditional Fisher information function familiar in item response theory literature. It also provides a framework for understanding sample or population reliability as an extension of individual-specific reliabilities, as well as an interpretation of reliability as an index of parsimony in summarizing individual responses by latent variable estimates.

The definition of reliability given by Equation 3 is inspired by an equivalent generalized correlation examined by various authors, beginning with Linfoot (1957; Inaba & Shirahata, 1986; Joe, 1989; Kent, 1963). That correlation index has been shown to be useful across a number of settings where traditional indices of correlation are inappropriate or useless, due to complexities of multivariate dependency (Ebrahimi, Jalali, & Soofi, 2014). In the time a version of the current paper has been available (Markon, 2017), Johnson & Sinharay (2020) have illustrated the utility of Lindley-based reliability in diagnostic classification models, a scenario where traditional reliability methods are inapplicable. Together these suggest the definition of reliability in Equation 1 is likely to be useful in quantifying measurement precision across a breadth of scenarios, such as with complex dependency or data structures.

Closely related indices based on Lindley information have also been proposed for use and found utility in quantifying measurement precision in various psychometric contexts. Many of these are normalized forms of Lindley information, where Lindley information is rescaled by some upper bound, rather than be rescaled to a reliability as in Equation 3 (Chen, Liu, & Xu, 2018; Jonas, 2021; Markon, 2013). Unscaled, nonnormalized mutual information has also been examined in the context of quantifying measurement precision with psychometric models (Walsh & Holmes, 2019), and the benefits of mutual information as an index of dependency have been demonstrated in other contexts (Kinney & Atwal, 2014a, 2014b).

Normally Distributed Variables and Relationships with Classical Measurement Theory

The traditional variance-based definition of reliability can be derived from Equation 3 in the case of normally-distributed measures with normally distributed errors. This derivation follows from Lindley

information being the mutual information between the observed data and the latent variable, and from the decomposition of mutual information into entropy components.

Univariate Model

Assume an observed score, estimate, or response X_{ij} of person i to item j is univariate normally distributed (although the latent variable might be multidimensional) and conforms to the latent variable model

$$X_{ij} = \lambda_j \theta_i + v_j + \varepsilon_{ij} \quad (5)$$

with normally distributed, homoscedastic errors with variance σ_ε^2 and intercept v_j , that is, $X_{ij} | \theta_i \sim N(\lambda_j \theta_i + v_j, \sigma_\varepsilon^2)$. Note that this could correspond to a number of scenarios, such as a normal-distribution factor model, or to the IRT trait estimation model discussed above, assuming that $X_{ij} = \hat{\theta}_{ij}$, with unit loadings and zero intercepts, and normally distributed errors, which is true of many common IRT models as applied to sufficiently long tests (e.g., Chang & Stout, 1993).

Note that for two variables $\mathbf{Y}$ and $\mathbf{Z}$ mutual information can be decomposed as $M(\mathbf{Z}, \mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y} | \mathbf{Z})$, where $H(\mathbf{Y})$ is the entropy of $\mathbf{Y}$ and $H(\mathbf{Y} | \mathbf{Z})$ is the entropy of $\mathbf{Y}$ conditional on $\mathbf{Z}$ (Golden, Brockett, & Zimmer, 1990; McGill, 1954). Note also that the entropy of a normal distribution is $(1/2) \ln(2\pi e \sigma^2)$. Then the Lindley information of the test score X with regard to the latent variable θ is

$$I = H(X) - H(X | \theta) = \frac{1}{2} \ln(2\pi e \sigma_x^2) - \frac{1}{2} \ln(2\pi e \sigma_\varepsilon^2) = \frac{1}{2} \ln \left[\frac{\sigma_x^2}{\sigma_\varepsilon^2} \right] \quad (6)$$

Substituting into the generalized reliability definition in Equation 1, the reliability is

$$\rho = 1 - \exp \left[-2 \left(\frac{1}{2} \ln \left(\frac{\sigma_x^2}{\sigma_\varepsilon^2} \right) \right) \right] = 1 - \frac{\sigma_\varepsilon^2}{\sigma_x^2} \quad (7)$$

which is equivalent to the classical definition of reliability. In this sense, using Lindley information to quantify the reliability of observed variables conforming to the model in Equation 5 will result in the same conclusions as classical accounts of reliability.

Multivariate Model

For the case of multivariate normally distributed indicators, assume that

$$\mathbf{X}_i = \mathbf{\Lambda}\boldsymbol{\theta}_i + \mathbf{v} + \boldsymbol{\varepsilon}_i \quad (8)$$

where $\mathbf{X}_i$ is a vector of responses that, conditional on a latent vector $\boldsymbol{\theta}_i$, are multivariate normally distributed with multivariate normally distributed errors $\boldsymbol{\varepsilon}_i$ with covariance matrix $\boldsymbol{\Psi}$. That is, $\mathbf{X}_i|\boldsymbol{\theta}_i \sim \text{MVN}(\mathbf{\Lambda}\boldsymbol{\theta}_i + \mathbf{v}, \boldsymbol{\Psi})$, and the covariance matrix of $\mathbf{X}_i$, $\boldsymbol{\Sigma}$, is given by

$$\boldsymbol{\Sigma} = \mathbf{\Lambda}\mathbf{\Lambda}' + \boldsymbol{\Psi}. \quad (9)$$

In this case, using the entropy-mutual information relationship, and the entropy of a multivariate normal distribution, the Lindley information is (Payaro & Palomar, 2009)

$$\iota = \frac{1}{2} \ln(|2\pi e \boldsymbol{\Sigma}|) - \frac{1}{2} \ln(|2\pi e \boldsymbol{\Psi}|) = \frac{1}{2} \ln\left(\frac{|\boldsymbol{\Sigma}|}{|\boldsymbol{\Psi}|}\right) \quad (10)$$

where $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$. Substituting into Equation 1, the reliability is

$$\rho = 1 - \frac{|\boldsymbol{\Psi}|}{|\boldsymbol{\Sigma}|}. \quad (11)$$

This provides a form of reliability for vector-valued observed variables — for example, sets of test scores or other indicators — that are multivariate normally distributed.

It is important to note that multivariate Lindley information-based definition of reliability in Equation 11, based on the determinants of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$, gives the reliability of a vector of observations $\mathbf{X}$ with reference to the latent variables $\boldsymbol{\theta}$. This is different from the classical case, where the interest might be in the reliability of a scalar score X^* obtained from multivariate vector of indicators (e.g., subscales, items) $\mathbf{X}$ with reference to the latent variables $\boldsymbol{\theta}$. For example, if the reliability of a sum score X^* is of interest, with the constituent items $\mathbf{X}$ conforming to the model in Equation 5, the reliability of X^* is given by

$$1 - \frac{\mathbf{1}'\boldsymbol{\Psi}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}} \quad (12)$$

where $\mathbf{1}$ is a column of ones and Σ and Ψ are the observed and error covariances for the items $\mathbf{X}$ (e.g., Bentler, 2009). In this sense Equation 12 is a form of Equation 7, with $\mathbf{X} = \mathbf{X}^*$, rather than Equation 11.

The multivariate reliability in Equation 11 is equivalent to the coefficient of determination (Bollen, 1989; Joreskog & Sorbom, 1986), and generalizes the notion of reliability to *sets* of indicators. As Joreskog and Sorbom (1986) note, it indexes “how well the observed variables serve, separately or jointly, as measurement instruments for the latent variables.” The univariate expression in Equation 7, corresponding to classical reliability, provides the reliability of a single scalar observed variable (such as a single test score), and reflects the information that scalar observed variable provides about a latent variable or set of latent variables in a population, sample, or other distribution of individuals. In contrast, the multivariate reliability in Equation 11 provides the reliability for a possibly vector-valued observed variable, such as a set of observed item responses, trait estimates, or scores. It reflects the information a vector-valued observed variable, or set of observed variables, provides about a latent variable or set of latent variables in a population, sample, or other distribution of individuals. The difference between Equations 7 and 11 reflects the difference in information provided by single indicator based on the $\mathbf{X}_i$ of Equation 5 (a single test score, for example), and the maximal information provided by the $\mathbf{X}_i$ treated together but distinctly as a set, as in Equation 8 (a set of test scores, for example).

Parallels between classical reliability and the coefficient of determination have been noted (e.g., Dolan, Wicherts, and Molenaar, 2004), but a formal justification for their relationship has not been discussed. Here, it can be seen that both are forms of rescaled Lindley information. This provides a unifying framework for understanding the two quantities and their application. In this regard the coefficient of determination provides a foundation for multivariate reliability treatments such as those of Wittman (1988). It exhibits interesting parallels with existing reliability formulations, and coincides with them in certain cases. These parallels and coincidences help integrate certain previously disparate findings in the literature.

The coefficient of determination or multivariate reliability has received relatively little attention in the literature, although its importance has been recognized. Dolan, Wicherts, and Molenaar (2004), for example, showed that power in latent regression modeling is related to multivariate reliability, in that greater multivariate reliability of a set of indicators in latent regression increases power to test hypotheses about slope parameters. This is consistent with interpretation of reliability in classical theory, in that reliability of indicators is seen as a bound to their observed relationships (e.g., in the sense of attenuation of observed relationships due to unreliability).

Multivariate Normal Reliability as Set Correlation. Interestingly, the multivariate reliability in Equation 11 can be seen as equivalent to the squared set correlation (Cohen, 1982; Wilks, 1932) between the observed and

latent variables.¹ Just as the traditional univariate reliability can be seen as the squared bivariate correlation between the observed and latent variables, the generalized reliability for multivariate normal data can be seen as the squared set correlation between a set of observed variables and a set of latent variables.

The set correlation between a set of variables Θ and $\mathbf{X}$ is given as (Cohen, 1982; Cohen & Nee, 1984)

$$R_{\Theta, \mathbf{X}}^2 = 1 - \frac{|R_{\Theta \mathbf{X}}|}{|R_{\Theta \mathbf{X}}||R_{\mathbf{X} \mathbf{X}}|} \quad (13)$$

which can be rewritten as (Cohen, 1982; Rozeboom, 1965)

$$R_{\Theta, \mathbf{X}}^2 = 1 - \frac{|C_{\mathbf{X} \cdot \Theta}|}{|C_{\mathbf{X} \mathbf{X}}|} \quad (14)$$

where $R_{\Theta \mathbf{X}}$ refers to the matrix of correlations between a set of variables Θ and set of variables $\mathbf{X}$, $C_{\mathbf{X} \mathbf{X}}$ is the covariance among a set of variables $\mathbf{X}$, and $C_{\mathbf{X} \cdot \Theta}$ is the matrix of covariances between variables in $\mathbf{X}$ partialing Θ . The set correlation is also equivalent to $1 - \Lambda$, where Λ is Wilks lambda (Wilks, 1932).

Comparing Equations 11 and 14, setting $\mathbf{X}$ equal to the observed variables and Θ equal to the latent variables, it is evident that with multivariate normal data, reliability can be interpreted as a squared set correlation between the observed and latent variables. Specifically, using the notation of Equation 14, the partial covariances between $\mathbf{X}$ given Θ are given by (e.g., Anderson, 2003)

$$C_{\mathbf{X} \cdot \Theta} = C_{\mathbf{X} \mathbf{X}} - C_{\mathbf{X} \Theta} C_{\Theta \Theta}^{-1} C_{\Theta \mathbf{X}} \quad (15)$$

Then, given Equations 8-9 with orthogonal standardized latent variables, Equation 15 can be rewritten as

$$C_{\mathbf{X} \cdot \Theta} = (\Lambda \Lambda' + \Psi) - (\Lambda \Lambda') = \Psi \quad (16)$$

Substituting $C_{\mathbf{X} \mathbf{X}} = \Sigma$ and $C_{\mathbf{X} \cdot \Theta} = \Psi$ into Equation 14 gives the multivariate normal reliability in Equation 11.

Wittman (1988), discussing the use of multiple forms of measures in assessment (for example, self-report, informant-report, and observation) recognized conceptual parallels between the squared set correlation and squared bivariate latent-observed correlation in the context of reliability theory, and the potential utility of set correlation in quantifying multivariate reliability in particular. Lindley information provides a formal quantitative

derivation: when used to quantify observed-latent variable associations with normal data, both types of correlation are instantiations of Lindley information. It therefore extends the observed-latent variable correlation definition of reliability (Kim, 2012) to vector-valued observed variables in the multivariate normal case.

Sufficient Statistics and Optimal Scoring. Flexibility regarding the nature of $\mathbf{X}$ in the reliability (Equations 4, 7, 11) raises questions about the choice of observed score and its impact on reliability. Lindley-information-based reliability can be calculated with regard to scalar or vector-valued item responses as well as scalar or vector-valued composite scores or trait estimates, so it is reasonable to inquire how observed score choice affects reliability.

The data-processing inequality (Cover & Thomas, 2006; Kinney & Atwal, 2014a) guarantees that the Lindley information, and therefore, generalized reliability, of a measure cannot be increased by using a scoring or estimation method relative to the observed responses the scoring or estimation method is based on. According to the data-processing inequality, the Lindley information $\iota(\mathbf{X}) = M(\mathbf{X}, \boldsymbol{\theta})$ of a variable derived from another, either stochastically (as is the case for a derived variable $\mathbf{Y}$ in a Markov Chain $\boldsymbol{\theta} \rightarrow \mathbf{X} \rightarrow \mathbf{Y}$) or deterministically (where $\mathbf{Y}$ is a deterministic transformation of $\mathbf{X}$) is less than or equal to the Lindley information of the original variable (that is, for a transformation $\mathbf{Y} = g(\mathbf{X})$, $\iota(\mathbf{Y}) \leq \iota(\mathbf{X})$). Noting that a derived composite, score, or estimate is part of a chain $\boldsymbol{\theta} \rightarrow \mathbf{X} \rightarrow \hat{\boldsymbol{\theta}}(\mathbf{X})$, it can be seen that the generalized reliability of a derived composite, score, or estimate can only be less than or equal to that of the observed responses; that is, $\iota(\hat{\boldsymbol{\theta}}(\mathbf{X})) \leq \iota(\mathbf{X})$. Put differently, a scoring or trait estimation procedure can only maintain or lose information and reliability relative to the observed data, not create it.

Importantly, however, the Lindley information and generalized reliability of a scoring or estimation procedure is equal to that of the observed data when the score or estimate is a sufficient statistic for trait value. Sufficient statistics by definition are interchangeable with the observed data in trait estimation, in that $P(\boldsymbol{\theta} | \mathbf{T}, \mathbf{X}) = P(\boldsymbol{\theta} | \mathbf{T})$ (or equivalently, $P(\mathbf{X} | \boldsymbol{\theta}, \mathbf{T}) = P(\mathbf{X} | \mathbf{T})$ where $\mathbf{T}$ is a sufficient statistic). For a score or estimate that is a sufficient statistic, $\iota(\hat{\boldsymbol{\theta}}(\mathbf{X})) = \iota(\mathbf{X})$. In this regard, a sufficient statistic for trait value is a reliability-maximizing score or trait estimate, in that it does not lose whatever information about the trait is available from the data.

Sufficient statistics do not always exist, although they do for exponential family distributions, and the sufficient statistics with regard to trait value have been characterized for many many common latent variable models, such as common IRT models (Anderson, 1977; Lord & Novick, 1968) and the normal linear latent variable model. Bartholomew (Bartholomew, 1984; Bartholomew & Knott, 1999) has shown that, for the normal linear factor model in Equations 5 and 8, the sufficient statistic for trait value is given by

$$\hat{\theta} = \Lambda' \Psi^{-1} \mathbf{x} . \quad (17)$$

Knott and Bartholomew (1993) observed that this statistic is sufficient and maximizes reliability. Other authors have also recognized this as the reliability-maximizing scoring weight for the normal linear latent variable model (Equations 5 and 8), without drawing connections with sufficiency (e.g., Raykov, 2004).

By formulating reliability in terms of Lindley information, the current results generalize the literature on maximal reliability to show that the maximal reliability score is the sufficient statistic score, where it exists. This holds for any measurement model with regard to the generalized reliability presented here. The sufficient statistic for trait value is available for many common latent variable models, providing a scoring or estimation procedure that preserves information available in the response data.

Parallels with Minimum Trace Factor Analysis and the Greatest Lower Bound Reliability. A sizable literature involving classical reliability has focused on identifying bounds to the reliability, including what has been referred to as the greatest lower bound reliability (the greatest lower bound in the sense that the uniquenesses are maximized and may include reliable specific variance; Bentler, 2009; Jackson & Agunwamba, 1976; Woodhouse & Jackson, 1977). This literature has shown that the greatest lower bound is given by minimum trace factor analysis, which provides solutions to the factor analysis model (Equations 5 and 8) by maximizing $\text{tr}(\Psi)$ (or equivalently, minimizing $\text{tr}(\Sigma - \Psi)$) subject to certain identifying constraints (that Ψ and $\Sigma - \Psi$ be positive semidefinite). For diagonal error covariance matrices, the $\mathbf{1}'\Psi\mathbf{1}$ term in Equation 12 is equal to $\text{tr}(\Psi)$, the trace of Ψ ; maximizing the values of Ψ therefore minimizes the value of Equation 12, providing the minimum trace factor analysis solution and a greatest lower bound reliability estimate.

In work on a different but related topic, Knott (2005) noted implications of and parallels between the problem of maximizing $\text{tr}(\Psi)$ and that of maximizing $|\Psi|$, which correspond here to the classical and information-theoretic formulations of reliability as in Equations 12 and 11. Knott illustrated that although the two problems in general are different, they coincide in certain cases (e.g., of parallel indicators) and can be treated as special cases of a broader maximization problem.

Interestingly, McDonald (1979, 1985) demonstrated that maximizing the likelihood function in maximum likelihood factor analysis is equivalent to maximizing $\ln|\Psi|$ when $\Psi = \Sigma - \Lambda\Lambda'$ and is in a correlation metric. Knott (2005) did not explore in detail the problem of maximizing $|\Psi|$ as a factor model solution analogously to minimum trace factor analysis, but given McDonald (1979)'s results, it would be found in the global maximum likelihood solution, over all identified solutions, in the population for cases conforming to the factor analysis model in Equations 5 and 8. The results of Knott (2005), combined with those of McDonald (1979) and Equation 11 here, suggest that when Σ conforms to the factor analysis model, the global maximum likelihood solution

provides a greatest lower bound reliability estimate. However, instead of providing a greatest lower bound to the classical reliability (for example, Equation 12), maximizing $|\Psi|$ provides a greatest lower bound to Lindley-information-based reliability (Equation 11).

Structural Considerations in Interpretation

Bounds to and reexpressions of Equation 11 help characterize how structural features relate to the generalized reliability in the multivariate normal case. For example, assuming that Ψ is invertible, one can write $|\Sigma| = |\Psi + \Lambda\Lambda'| = |\Psi| |\mathbf{I} + \Psi^{-1}\Lambda\Lambda'|$, where $\mathbf{I}$ is an identity matrix with the same dimensions as Ψ . Then Equation 11 can be rewritten as

$$\rho = 1 - \frac{1}{|\mathbf{I} + \Psi^{-1}\Lambda\Lambda'|} \quad (18)$$

In the case of a unidimensional latent variable model, a similar application of the matrix determinant lemma results in

$$\rho = 1 - \frac{1}{1 + \lambda'\Psi^{-1}\lambda} \quad (19)$$

which is equivalent to the classical reliability for Gaussian factor models, assuming standardized latent variables (Mellenbergh, 1996). This confirms what is indicated by Equations 7 and 11, that the multivariate normal generalized reliability reduces to the classical reliability for unidimensional models.

Loadings and Simple Structure. In the multivariate normal case, Lindley-based reliability increases to the extent that the latent variables and their measures conform to simple structure or are separable, and decreases with cross-loadings. Specifically, Lindley information and reliability is maximized when there are no cross-loadings and each observed variable has a single loading from one latent variable.

This can be shown by noting that the matrix $\Lambda\Lambda'$ (Equation 9) can be modeled as a block-partitioned error-free covariance matrix having a structure

$$\begin{bmatrix} (\Lambda\Lambda')_A & (\Lambda\Lambda')_C' \\ (\Lambda\Lambda')_C & (\Lambda\Lambda')_B \end{bmatrix} = \begin{bmatrix} A & C' \\ C & B \end{bmatrix} \quad (20)$$

where A indicates the variance-covariance block comprising variables having greatest loadings from one factor, B indicates the variance-covariance block comprising variables having greatest loadings from another factor, and C indicates the block of covariances between sets of variables due to cross-loadings (the arguments here can be extended to more generalized block structures; Fallat & Johnson, 2000).

Then, by the Fischer inequality (Fallat & Johnson, 2000),

$$|A||B| \geq |\Lambda\Lambda'| \quad (21)$$

which can be written using the Schur determinant identity as

$$|A| |B| \geq |A| |B - CA^{-1}C'| \quad (22)$$

This implies that, for given A and B, $|\Lambda\Lambda'|$ is maximized—and therefore, the Lindley information and reliability is maximized (Equations 11 and 18)—when C is a matrix of zeros, which occurs when there are no cross-loadings and each observed variable has a single loading from one latent variable. The generalized reliability, at least in the multivariate normal case, therefore increases to the extent that the measures and latent variables conform to simple structure or are separable, and decreases with cross-loadings. This provides a foundation for the intuition that indicators reflecting a single latent variable are relatively informative in the sense that the source of their increases and decreases can be explained with less uncertainty than indicators reflecting multiple latent variables. In the multivariate setting, information may come from the structure of the indicators as much as the magnitude of loadings per se.

The phenomenon represented by Equations 20-22 might be more generalizable than the structure represented by Equation 20, or multivariate normal distributions per se. Hierarchical (i.e., bifactor) models, for example, have prominent cross-loading features and have been scrutinized for producing anomalous estimates. The use of hierarchical model variants comprising some pure indicators without cross-loadings has been advocated as a solution (Eid, Geiser, Koch, & Heene, 2017), suggesting that presence of cross-loadings per se might be less critical than absence of pure indicators. Research on paradoxical latent variable estimates in multidimensional IRT – in which incorrect responses can raise ability estimates – has similarly pointed to the critical role of cross-loadings in factor structures, and of factor correlations in latent variable priors (Hooker, 2010; Hooker, Finkelman, & Schwartzman, 2009; Jordan & Speiss, 2012, 2018). Both of these areas of research suggest that a more general account of the role of indicator-factor structure in multivariate reliability might be possible.

Error Covariance Structure. Generalized reliability in the multivariate normal case will also be affected by the error covariance structure. For example, by Hadamard's inequality, assuming that Ψ is positive definite, for a given set of residual variances, $|\Psi|$ is maximized, and therefore ρ is minimized, when Ψ is diagonal; that is, when there are no residual covariances. Although this might seem counterintuitive, note that the presence of residual covariances implies additional multivariate regularity in the data and that a covariance between two variables can be reexpressed as a squared loading held equal between the variables (that is, $\sigma^2_{12} = \lambda^2$) and therefore represented in terms of a factor.

Theoretically, the Lindley-based reliability discussed here corresponds with classical conceptions in which reliability represents systematic shared as well as unique variance (e.g., communality as well as specificity; Lord & Novick, 1967). That is, Equation 8 can be reformulated as

$$\mathbf{X}_i = \Lambda \boldsymbol{\theta}_i + \boldsymbol{\gamma} \boldsymbol{\xi}_i + \mathbf{v} + \boldsymbol{\varepsilon}_i \quad (23)$$

where $\boldsymbol{\xi}$ represents factors specific to each observed variable; that is, $\boldsymbol{\gamma}\boldsymbol{\gamma}'$ is diagonal. The generalized reliability theoretically treats both $\sigma^2_{\boldsymbol{\theta}}$ and $\sigma^2_{\boldsymbol{\xi}}$ as reliable, systematic non-error variance.

In practice, however, the specificity term (the second right-hand term of Equation 23) may be indistinguishable from error variance, folded into a general unique variance-covariance component, depending on the measurement design (e.g., repeated measurement designs may be able to estimate reliable specific variance separately from error variance). Note that the equation for the Gaussian reliability is based on the conditional entropy, $H(\mathbf{X}|\boldsymbol{\theta})$; in practice what constitutes error therefore is whatever determines the distribution of observed responses conditional on the latent variables being modeled. From a practical, if not theoretical perspective, what an estimate of ρ represents will vary based on design.

Note that in the multivariate case, under conditions under which $|\Psi| = 0$, such as when one or more variables have residual variances of 0, using Equation 11 would lead to estimates of $\rho = 1$ even though the error covariance matrix is positive semidefinite and the multidimensional error distribution is degenerate. Technically in these situations, the entropy decomposition of mutual information in Equations 8-10 is no longer valid, as the full error distribution is no longer multivariate normal, and the generalized reliability should be calculated using other approaches. One possible approach is to calculate the generalized reliability using a pseudodeterminant of Ψ , such as the product of the nonzero eigenvalues; similarly, inference could be refocused on the set of variables producing a positive definite Ψ , such as those with nonzero residual variance. Care is required in situations with zero residual variances among the variables, as it can lead to misleading conclusions about the set of variables as a whole.

General Distributions

Precisely characterizing the form of the relationship between Lindley information, reliability, and estimation error becomes more abstract in the general case of general, possibly nonnormally distributed, measures due to the breadth of possible scenarios. Nevertheless, existing literature provides some bounds and identities that help to characterize a Lindley-information-based formulation of reliability as it applies to more general settings.

Upper Bounds to Reliability for Nonnormal Variables

One implication of the entropy-mutual information relationship $M(\mathbf{Z}, \mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{Z})$ is that the Lindley information is upper bounded by the entropy of the observed variable distribution. However, that entropy in turn is maximized over distributions by a specific distribution, the maximum entropy distribution, whose identity depends on the model constraints involved. The uniform distribution, for example, maximizes entropy for distributions parameterized only in terms of a range, and the normal distribution maximizes the entropy for distributions parameterized in terms of means and covariances. For certain classes of constraints, the exponential family of distributions can be derived as maximum entropy distributions (e.g., Ebrahimi, Soofi, & Soyer, 2008; Grunwald, 2007). It follows that the reliability under an appropriate maximum entropy distribution represents an upper bound to the reliability.

The normal distribution reflects a special case of maximum entropy for continuous variables, because in general the maximum entropy distribution is not defined with additional model constraints via skew or kurtosis parameters (although it is in certain specific scenarios involving certain combinations of skew and kurtosis constraints; e.g., Rockinger & Jondeau, 2002). In other words, of models with parameterized means and covariances, the normal model is the model with the greatest entropy that can be specified parametrically in closed form. Specifically, the reliability definitions given in Equations 7 and 11 for the normal distribution case represent bounds for cases where the factor models in Equations 5 and 8 apply except for the assumption of normally distributed variables.

Lindley information can be rewritten (Palomar & Verdu, 2006; Pinsker, Prelov, & Verdu, 1995) as

$$\iota = M(\mathbf{X}, \boldsymbol{\theta}) = M_{\phi}(\mathbf{X}, \boldsymbol{\theta}) + D(p(\mathbf{X}|\boldsymbol{\theta}) \| \phi(\mathbf{X}|\boldsymbol{\theta})) - D(p(\mathbf{X}) \| \phi(\mathbf{X})) \quad (24)$$

where $M_{\phi}(\mathbf{X}, \boldsymbol{\theta})$ is the mutual information between two variables that are jointly normally distributed with the same means and covariances as $\mathbf{X}$ and $\boldsymbol{\theta}$, and $D(p(\mathbf{X}) \| \phi(\mathbf{X}))$ is the relative entropy between $\mathbf{X}$ and a normally distributed variable having the same mean and covariance values as $\mathbf{X}$:

$$D(p(\mathbf{X}) \parallel \phi(\mathbf{X})) = \int p(\mathbf{X}) \ln \left[\frac{p(\mathbf{X})}{\phi(\mathbf{X})} \right] d\mathbf{X} \quad (25)$$

with $\phi(\mathbf{X})$ indicating the density of $\mathbf{X}$ under a normal distribution with the same mean and covariance values as $p(\mathbf{X})$.

When Equation 8 holds with normally distributed errors, the second term on the right side of Equation 24 becomes zero, and the Lindley information can be expressed as

$$\iota = \frac{1}{2} \ln \left(\frac{|\Sigma|}{|\Psi|} \right) - D(p(\mathbf{X}) \parallel \phi(\mathbf{X})) \quad (26)$$

Given that $D(p(\mathbf{X}) \parallel \phi(\mathbf{X}))$ is always greater than or equal to 0, it follows that the Lindley information in Equation 26 is always less than that of a normal distribution (Equation 11), and that with increasing nonnormality, under conditions of normal errors but nonnormal latent or observed variables, the actual reliability will decrease relative to the reliability in Equation 11.

Nonparametric Scenarios

Use of Lindley information as a basis for reliability theory and estimation requires specification of a prior and likelihood relating $\mathbf{X}$ and $\boldsymbol{\theta}$; other characteristics follow from these two elements (Equations 3 and 4). As the prior and likelihood often derive from a parametric model, one might question how applicable Lindley-information based reliability is to nonparametric scenarios, where as little as possible is to be assumed about the joint distribution of observed and latent variables. This is a strength of classical reliability theory, in that it can be derived using relatively few assumptions about the measurement process.

Bayesian empirical likelihood is one possible method for use of information-based reliability theory in nonparametric settings. Empirical likelihood is a method of nonparametric likelihood inference where likelihoods are assigned to data values subject to moment constraints (Lazar, 2021; Owen, 2001). It can be seen as a form of estimating equation-based inference where model parameters are estimated through their implied moments, which consequently constrain assignment of likelihoods to observed data values. This allows nonparametric likelihood-based inference for moment-based models, such as structural equation models (Yuan, 2007).

Empirical likelihood can be used with priors to form the basis of Bayesian empirical likelihood formulations (Lazar, 2003). Lindley information has been examined in an empirical likelihood setting indirectly through studies of reference priors for empirical likelihood models (Clarke & Yuan, 2010), forming a possible basis for

nonparametric information-based reliability theory. In theory it should be possible to use moment-based models with empirical likelihood to establish nonparametric inference for Lindley-information-based reliability. Inference for such models would follow from the mean and covariance constraints implied by measurement models, but based on nonparametric likelihoods rather than parametric likelihoods.

Nonparametric likelihood estimation can be computationally challenging, however, especially in the multivariate case. As such, it is unlikely to provide a general solution for cases where the joint relationship between latent and observed variables is difficult to specify parametrically. Application to reliability in terms of Lindley information would currently be limited in many such cases. Another possibility, a step more parametric, is to use semiparametric methods, where the conditional probability of a response given the latent variable is estimated, but flexibly using minimal constraints on the form of the functional relationship (for example, Liu & Wang, 2022). These types of methods have the advantage of making relatively few assumptions of the data while still affording estimation of some form of latent-observed variable relationship.

Relationships Between Lindley Information and Estimation Error

It has been shown that increasing mutual information implies decreasing estimation error for a wide variety of definitions of statistical error and loss criteria (e.g., Guntuboyina, 2011; Palomar & Verdu, 2007; Yang & Barron, 1999). As such, for a very broad set of models and error criteria, choosing measures so as to increase the Lindley-based reliability (Equation 3) will increase measurement precision and decrease estimation error.

For example, in the case of continuous measures, a number of authors (Brunel & Nadal, 1998; Huang & Zhang, 2018; Kang & Sompolinsky, 2001; Rissanen, 1996; Wie & Stocker, 2016) have shown that, subject to weak assumptions depending on the model class (for example, that the latent variable distribution is relatively low-dimensional, and that measurement error is smooth or that latent variable maximum likelihood estimates obey the central limit theorem), the Lindley and Fisher information (that is, the traditional information function of IRT) are related by

$$I \approx H(\boldsymbol{\theta}) - \int p(\boldsymbol{\theta}) \frac{k}{2} \ln \left(\frac{2\pi e}{|I(\boldsymbol{\theta})|} \right) d\boldsymbol{\theta} \quad (27)$$

where $|I(\boldsymbol{\theta})|$ is the determinant of the Fisher information, $H(\boldsymbol{\theta})$ is the entropy of the latent variable distribution, and k is the dimensionality of the latent variable.

The relation in Equation 27 indicates that, for a variety of measurement models, as the average measurement information in a population increases, in the sense that the average Fisher information increases

and the average standard error of measurement decreases, the mutual information will increase, which will bring the generalized reliability to 1. Conversely, as the average measurement information decreases (and the average standard error increases), the mutual information bound will decrease, and the generalized reliability bound will tend toward 0. Note that the second term on the right-hand side of Equation 27 will not necessarily reach zero as the Fisher information decreases, as the Fisher information often has a nonzero minimum (Park, Serpedin, & Qaraqe, 2013; Stein & Nossek, 2017). For example, the second term will approach the entropy of a multivariate standard normal distribution as the Fisher information matrix approaches an identity matrix, bringing the value of the entire bound to zero if the latent variable distribution is also standard normal.

Wei and Stocker (2016) and Huang and Zhang (2018) have shown that the precise form of the relationship in Equation 27 depends on the scenario. For example, for many scenarios with monotonic relationships between the observed and latent variables and smooth measurement error, as the error approaches zero and at the same time becomes more normal in shape, the inequality becomes an equality (a straightforward extension of the Wei and Stocker, 2016 proof to the multidimensional case is provided in the appendix). Similarly, in cases where the latent variable distribution is high-dimensional, other approximations of Lindley information in terms of Fisher information are more accurate (Huang and Zhang, 2018). Huang and Zhang (2018) provide a detailed discussion of how relationships between Lindley and Fisher information change under various modeling scenarios, but further research is needed to better characterize their relationship in general.

For discrete parameters, which are not differentiable and therefore do not have a defined Fisher information matrix, Fano's inequality provides a similar lower bound relating Lindley information and estimation error (e.g., Cover & Thomas, 2006; Prasad, 2015):

$$I \geq H(\boldsymbol{\theta}) - [H(P_e) + P_e \ln(M-1)] \quad (28)$$

where P_e is the probability of misestimating the latent variable value — that is, $P_e = P(\hat{\boldsymbol{\theta}}(\mathbf{x}) \neq \boldsymbol{\theta})$, $H(P_e)$ is the binary entropy function associated with the probability of making an error or not, and M is the number of possible latent variable values. Again, as measurement error decreases, Lindley information — and therefore, the reliability — increases.

A number of authors have also shown that Lindley information is proportional to the product of the Fisher information and the squared differential of the latent variable. For example, consider a scenario where the prior and posterior have the same distribution except that they differ by an infinitesimal value of the latent variable $\boldsymbol{\theta}$ — that is, $P(\boldsymbol{\theta}|\mathbf{x}) = P(\boldsymbol{\theta} + \Delta\boldsymbol{\theta})$. Noting that mutual information is equal to the expected relative

entropy between the posterior and prior (the second expression in Equation 4), it can be shown that (Amari, 2001; Jeffreys, 1946; Kullback & Liebler, 1951) in the limit where $\Delta\theta$ approaches $\mathbf{0}$,

$$I \approx \frac{1}{2} E \left[\sum_{a,b=1}^k \Delta\theta_a \Delta\theta_b I(\theta_{ab}) \right] \quad (29)$$

where E denotes expectation, here with regard to $\mathbf{X}$, and the subscripts a and b are used to denote elements of the parameter vector $\Delta\theta$ and corresponding Fisher information matrix $I(\theta)$. Similar expressions have been derived for closely related scenarios (for example, Kostal & Lansky, 2010; Kostal, 2010; Lindley, 1961; Nirenberg, 1975; Prelov & van der Meulen, 1993; Wang, Chang, & Boughton, 2011 in an IRT context).

Measurement Potential and the Global Reliability Bound

For a given family of models, the Lindley information is maximized by a unique trait distribution, which, when interpreted as a Bayesian prior, is known as the *reference prior* (Berger, Bernardo, & Sun, 2009; Bernardo, 1979; Good, 1968). The reference prior places maximum weight on the information obtained from the data relative to that obtained from the prior, and can in that sense be thought of as maximally conservative. Reference priors have a number of important roles in inferential and information theory, including non-Bayesian inferential paradigms (Grunwald, 2007).

In a psychometric context, the reference prior can be interpreted from within a frequentist paradigm as well, as a reference distribution rather than reference prior (Markon, 2013). The reference distribution, from this perspective, is the population latent distribution that maximizes the utility of a given measure in an information theoretic sense. The Lindley information and therefore, the generalized reliability, will differ across population distributions, depending on the nature of the latent distribution. The reference distribution is the distribution that maximizes the Lindley information, and therefore, the reliability. It can be thought of as the distribution a measure is optimally suited to, in the sense of the distribution the measure provides the most information about.

Computation of the Reference Prior Distribution

Different methods for estimating the reference prior distribution have been developed. In some cases, the reference prior has been determined analytically; for other cases, methods have been developed for estimating the reference prior or for closely approximating it (e.g., Berger, Bernardo, & Sun, 2009; Nalisnick & Smyth, 2017).

In cases where the parameter estimate of interest—in this case, the estimate of a latent attribute—conforms to the central limit theorem (as is the case for many measurement models; e.g., Chang & Stout, 1993), the reference prior is known to be approximated by the Jeffreys prior:

$$\frac{\sqrt{|I(\boldsymbol{\theta})|}}{\int \sqrt{|I(\boldsymbol{\theta})|} d\boldsymbol{\theta}} \quad (30)$$

The Jeffreys prior is therefore proportional to the square root of the IRT information function, or the inverse of standard error of measurement, and can be motivated from the relationship represented in Equation 29 (e.g., Jeffreys, 1946; Lindley, 1961).

The Jeffreys approximation grows more accurate with measure length; the actual reference prior may be discrete for real finite measures (consider that the possible number of values of a latent variable estimate will be finite, related to the number of possible response vectors from the measure; Berger, Bernardo, & Mendoza, 1989; Mattingly, Transtrum, Abbott, & Machta, 2018). Clarke and Barron (1994) have also shown that for bounded discrete latent variables, the reference prior will often be approximated by the uniform prior.

Global Minimax Bounds on Reliability and Measurement Potential

The reference prior, in combination with lower bounds such as those in Equations 25 and 26, can be used to provide a reliability measure that indexes the “measurement potential” of a test (Jonas, 2021; Markon, 2013). Measurement potential can be thought of as the potential precision or information obtained when a measure is optimally suited to a particular latent distribution, in either a frequentist or Bayesian sense. If the reference prior is the latent distribution maximizing the Lindley information, and therefore reliability, and a lower bound can be defined (Equation 27), then one can define a global minimax bound that reflects the minimum information to be expected when a measure is being used optimally. In this case, one has

$$\rho^* = 1 - e^{-2\iota^*} \quad (31)$$

where ρ^* is the global minimax reliability, and ι^* is a lower bound estimate of the Lindley information under the reference prior distribution.

The global measurement potential bound can be obtained by applying lower bound formulas, such as those in Equations 27 and 28, with the reference prior for a particular measurement model. For measurement scenarios where the Jeffreys prior (Equation 30) is the reference prior, it can be shown (e.g., Clarke & Barron, 1994) that ι^* is given by

$$i^* = \frac{k}{2} \ln \left[\frac{1}{2\pi e} \right] + \ln \int \sqrt{|I(\boldsymbol{\theta})|} d\boldsymbol{\theta} \quad (32)$$

For discrete variables where the reference prior can be approximated by the uniform distribution, Clarke and Barron (1994) have shown that i^* is asymptotically (in the length of the measure) equal to the entropy of the uniform distribution (cf. Equation 28, where for very long informative measures, $H(P_e)$ and P_e both go to zero and the remaining term is the entropy of the latent variable). Tight upper bounds to the Lindley information are also available for discrete variables (in comparison with bounds such as that of Equation 28; Feder & Merhav, 1994; Prasad, 2015), suggesting that precise global bounds to the reliability might be obtained in cases poorly approximated by the uniform, by applying upper as well as lower bounds.

The global minimax reliability bound p^* provides an additional relationship between reliability and the Fisher information function, to complement those previously identified in the literature. Existing psychometric identities have related the Fisher information function to reliability through the average measurement error variance in a sample or subpopulation (e.g., Mellenberg, 1996). p^* extends the reliability-information relationship globally, across different possible subpopulation latent distributions, and to scenarios where Fisher information may not even be definable. Jonas (2021), focusing on multidimensional tests, provided preliminary evidence that this global bound provides useful additional information about the characteristics of tests in terms of their predictive validity.

Note that p^* is a global, person-independent reliability index, in the sense that it depends on indicator parameters and is independent of a particular observed sample. Unlike Cronbach's alpha and other traditional greatest lower-bound estimates, p^* is not a lower-bound estimate to the reliability in a particular sample or population; it is a lower bound to the expected information obtained from an optimally suited sample or population. p^* therefore blurs traditional boundaries between information and reliability, in that it is a reliability index, but is also person-independent in an important sense, like the IRT information function.

Estimation

Plugin Estimation

For cases where the latent and observed variable distributions follow defined distributions, such as a normal distribution, derived equations such as those in Equations 5-11 could be used to create plugin estimates of the generalized reliability. For instance, for multivariate normally distributed variables, estimates of $\boldsymbol{\Psi}$ could be obtained from standard structural equation modeling software, together with either the observed variable covariance matrix, or the model-predicted $\boldsymbol{\Sigma}$, together with Equation 11, to estimate the generalized reliability.

Examples: Normal Models of Covariance Matrices. Lindley reliabilities were calculated (using Equation 11) for a number of well-known published covariance and correlation matrices, in order to compare them to classical reliabilities. Specifically, reliabilities were calculated for covariance and correlation matrices as published by Gnambs and Schroeders (2018), Gnambs and Staufenbiel (2018), Gosling, Rentfrow, and Swann (2003), Harman (1967, p. 244, based on unpublished data from Holzinger), and Thurstone (1933, based on data from Brigham, 1932). Each matrix represents correlations between an array of variables of interest, including psychological distress (Gnambs & Staufenbiel, 2018), self-esteem (Gnambs & Staufenbiel, 2018), Big Five personality traits (Gosling, Rentfrow, & Swann, 2003), and cognitive ability (Harman, 1967; Thurstone, 1933). The Thurstone-Brigham and Harman-Holzinger matrices are all well-studied and have been used in a variety of psychometric texts and software libraries; the Gnambs and Schroeders (2018) and Gnambs and Staufenbiel (2018) matrices were meta-analytically estimated and correspond to widely used scales.

Models. In all cases except Gosling et al. (2003), a general factor was modeled together with specific factors in a bifactor structure. The matrices from Gnambs and Schroeders (2018) and Gnambs and Staufenbiel (2018), for example, were modeled in terms of general factors and positive and negative valence factors. The Thurstone-Brigham matrix was modeled in terms of a general factor and a numerical factor comprising the Arithmetic and Number Series subtests, and the Harman-Holzinger matrix was modeled in terms of a general factor as well as verbal and spatial factors. Gosling, et al. (2003), in contrast, was modeled in terms of five correlated factors. In cases where there were only two indicators per factor (Gosling, et al. 2003 and the Thurstone-Brigham matrix), the absolute values of the loadings involved were constrained to be equal.

Reliabilities. The calculated reliabilities are shown in Table 1. The value in the fourth column (p) is the Lindley reliability (Equation 1), computed with regard to the general factor, treating the remaining covariance as residual (given that it reflects the covariance conditional on a particular trait value). The fifth column (p_{tot}) is the generalized multivariate reliability, computed with regard to all the factors; this corresponds to the reliability of a set of sufficient statistic factor estimates. The sixth (ω) and seventh (ω_{tot}) columns are values of ω , calculated with regard to the general factor and all factors, respectively. The last column is Cronbach's α .

Comparing the columns is instructive. For example, the difference between p and p_{tot} can roughly be thought of as reflecting the difference in informativeness between a single sufficient statistic general factor estimate, and a set of sufficient statistic factor estimates, one for each factor in the model. This contrasts with the difference between ω and ω_{tot} , which reflects the difference in reliable variance in a single sum score when only the general factor, or all the factors, are treated as components of reliable variance.

Given that ω is the Lindley reliability for a sum score, the difference between ω and p can be thought of as the difference in reliability between using a simple sum score versus using a sufficient statistic estimate of the

general factor, respectively. ω_{tot} and ρ_{tot} differ in that the former is the reliability of a *single score* accounting for all the factors as part of the reliable score variance; the latter is the reliability of a *vector* of sufficient statistic estimates of all the factors.

The Lindley reliability can be thought of as being in the same model-based family of reliabilities as ω , but generalized to vector- or matrix-valued quantities. For univariate cases, ω and the generalized reliability are the same. When will ω and the Lindley reliability differ? Given that ω reflects the reliability of a unit-weighted composite, ω and the Lindley reliability will differ to the extent that unit weighting is distant from the optimal, sufficient statistic attribute estimate. To some extent, this is evident in Table 1: loadings in the model of the Harman-Holzinger matrix vary more across measures, and the Lindley reliability and ω differ more; in contrast, loadings in the model of the Thurstone-Brigham matrix are more similar across measures, and the Lindley reliability and ω are more similar.

The case of Gosling et al. (2003) is instructive, in that the matrix represents a scenario with five correlated factors (the Big Five), where there would typically be an interest in five estimated trait values and no single general factor estimate or score. None of the classical reliability estimates functions in this scenario well, in that they represent the nonerror variance in a scalar valued score or estimate and not a vector-valued set of estimates as in this case. The multivariate reliability in contrast reflects the overall information provided by the measure (the Ten-Item Personality Inventory; TIPI) as a whole about the measurands of interest (the Big Five traits).

The reliability values in Table 1 underscore a number of observations about applied use of reliability. First, they highlight the potential informativeness of a set of measures as opposed to a single score in summarizing diverse attributes. The relatively low measure-to-factor ratios represented by some of the scenarios in Table 1 may not be reflective of certain applied scenarios, but the values nevertheless point to the utility of Lindley reliability in quantifying loss of information from a single measure relative to a set of measures. Second, the Lindley reliabilities underscore potential gain in information obtained by using one scoring method relative to another.

Estimation Bias. Some caution is needed, as parameter estimates based on different scoring methods will probably not cross-validate equally well; the reliability of optimal scores relative to sum scores is susceptible to inflation due to capitalization on chance in estimation of parameters. Recent work has shown that uncorrected estimates of the maximal reliability – equivalent to ρ in Table 1 – are positively biased (Aguirre-Urreta, Rönkkö, & McIntosh, 2019), and that this bias is greater for smaller sample sizes, fewer indicators, and smaller loadings. This same work shows that uncorrected maximal reliability, and therefore Lindley reliability, is more positively biased than forms of reliability such as ω . The Lindley reliability values in Table 1 are likely overestimates in this

regard, especially for the smaller sample size cases; for the larger sample sizes the values, which are closer to ω , are probably more accurate.

Because the multivariate Lindley reliability is a new to the psychometric literature, existing reliability research has not explicated the role of bias in its estimation, although it is likely to be as great as is the case with univariate reliability, if not greater. Set correlations between observed variables demonstrate positive estimation bias (Cohen & Nee, 1984); analytic corrections have been developed, but it is unclear whether those would apply to Lindley reliability, which is a form of set correlation between latent and observed variables.

To preliminarily explore potential bias in multivariate reliability, I ran simulations consisting of multivariate standard normal populations with correlations following the linear factor model of Equation 8. Factor structures followed an orthogonal simple structure model, with either 3 or 5 latent variables and 3 observed variables per factor. Nonzero loadings were set equal at 0.35, 0.50, or 0.70; samples were generated at sizes of $N=150$, $N=500$, and $N=1000$. 1000 replications were generated per condition.

Results are given in Table 2. Consistent with the findings of Aguirre-Urreta, Rönkkö, & McIntosh (2019), bias was nonnegative in all conditions, usually positive, and smaller for larger sample sizes, greater number of factors, and larger loadings. Bias was substantial for small sample sizes and for smaller loadings, and approached zero for larger loadings and the largest sample size.

In general, the results suggest that the estimates in Table 1 should be approached with some caution due to positive bias. However, it is important to note that even in the population conditions represented in Table 2, the largest multivariate reliabilities were very large, at least 0.99. As these correspond to realistically encountered conditions for developed tests reflecting multiple factors, very large multivariate Lindley reliabilities might be expectable. Again, the results point to the potential gains with optimal estimation of latent variables in multidimensional scenarios. Further research is needed to improve estimates of multivariate reliability and reduce bias (cf. Cohen & Nee, 1984).

Markov Chain Monte Carlo Estimation

In cases where formulas for the Lindley information are unavailable, Markov Chain Monte Carlo (MCMC) methods can be used to estimate the Lindley information and reliability. Because the Lindley information can be written in terms of posterior and prior densities (Equation 4), MCMC methods can be used for estimation even when a frequentist interpretation of trait estimation and inference is used and the densities are interpreted in terms of joint latent-observed variable population distributions. When Bayesian trait estimation is assumed, estimating the generalized reliability using MCMC methods provides convenience, as the prior and posterior distributions would be specified and estimated as a matter of course.

Examples: Item Response Data. To provide an example, MCMC was used to estimate generalized reliabilities for a number of datasets, including: data on the Figure Classification and Debate item subtests of the Law School Admission Test (LSAT), originally published by Bock and Lieberman (1970); data from the 1984, 1994, 2004, and 2014 Vocabulary subtests of the General Social Survey, a large-scale ongoing longitudinal sociological and psychological population-representative study (GSS; Smith, Marsden, Hout, & Kim, 2015); and International Personality Item Pool (IPIP) Big Five personality data from the Synthetic Aperture Personality Assessment project (Revelle, Wilt, & Rosenthal, 2010; available in the R *psych* library, Revelle, 2017). Unidimensional 2-parameter logistic item response models were fit to the LSAT and GSS datasets using Bayesian estimation with No-U-Turn sampler in Stan (Carpenter, et al., 2017); a unidimensional generalized partial credit model was used for the IPIP data. θ values were estimated using a standard normal prior; loadings were estimated using a $N(0, 10)$ prior distribution truncated at 0; and intercepts were estimated using a $N(0, 10)$ prior (generalized reliabilities were also estimated with loadings and intercepts fixed at their ML estimates, but results were extremely similar).

The Lindley reliability for each subtest was estimated from the posterior and prior θ distributions using Equation 4; marginal observed variable distributions of Equation 4 (i.e., $p(\mathbf{X})$) were estimated using the empirical probabilities of each (giving p_e), as well as the model-predicted probabilities given the maximum likelihood parameter estimates (giving p_m). Item response theory-based reliabilities (ω_{mar} ; Mellenbergh, 1996), as well as reliabilities based on factor models of biserial (ω_{pear}) and polychoric correlations (ω_{poly}), were also calculated for comparison.

Estimated reliabilities are presented in Table 3. As is evident, the generalized reliabilities were similar to the other estimates of reliability, especially the marginal IRT-based reliabilities and the Pearson-based reliabilities. In general, the results of Tables 1 and 3 illustrate that the Lindley reliabilities coincide with other reliability formulations in situations where those alternate formulations exist.

Figure 1 gives the log-posterior, log-prior, and posterior density for two individuals (Persons 201 and 205) in the IPIP Neuroticism dataset. A similar plot could be presented for each individual, or for each response pattern. Lindley information is proportional to the difference in $\ln P(\theta|\mathbf{X})$ and $\ln P(\theta)$, averaged over the posterior density, and then averaged over response patterns (e.g., across individuals). Reliability is a rescaling of the Lindley information in a 0, 1 metric. Note that the difference in $\ln P(\theta|\mathbf{X})$ and $\ln P(\theta)$ will approach zero as θ approaches negative or positive infinity, as both the prior and posterior approach zero themselves. However, as the difference in $\ln P(\theta|\mathbf{X})$ and $\ln P(\theta)$ is weighted by the posterior, the contribution of the difference to the entire integral is focused around the posterior maximum.

The Lindley reliabilities did differ depending on whether marginal probabilities of the observed variables (i.e., $p(\mathbf{X})$) were estimated using the empirical or model-based probabilities of the response patterns. Usually the reliabilities based on the model-based probability estimates were smaller than those based on the empirical probability-based estimates, although this was not always the case, and the two different approaches varied in how similar they were to the IRT-based reliabilities. It is likely that the goodness-of-fit of the model contributes to these differences; as the model-predicted and empirical frequencies diverge, they will produce different estimates of reliability (note that similar issues arise in the literature on ω as well). Moreover, other model-based methods for calculating $p(\mathbf{X})$ are available (Chib, 1995); it may be those other methods produce more accurate results.

The polychoric reliabilities were consistently larger than the other estimates of reliability. The polychoric-based reliabilities do not reflect the reliability of observed variables with regard to the latent traits, but rather, the reliabilities of hypothetical intermediate variables underlying responses. As such, they overrepresent associations between the observed variables per se, and therefore between the observed and the generating latent variables (Revelle and Condon, 2019). The Lindley information-based definition (Equations 3 and 4), in contrast, emphasizes that reliability quantifies information about the latent trait that is available in the actual observed data at hand; it also provides a general framework for handling diverse models and scales of measurement, obviating some of the problems that motivate use of polychorics in reliability estimation.

It is important to note that changes in the latent variable models being used could change the reliability estimates in Table 3, as all the values in the table depend on the assumed model. For example, use of a bifactor rather than unidimensional model could change the estimated reliabilities due to issues related to misspecification or changes in representation of general versus specific but reliable variance.

Along these lines, it is also worth noting that the IPIP Big Five measure could also be considered as a single set of items as a whole, relative to the set of five factors measure estimating, together with a multidimensional model. Therefore, in addition to estimating reliabilities for each Big Five item set with reference to its target factor, one could estimate the multivariate reliability of the entire set of IPIP items with reference to the five factors, considered as a set. Due to the computational time involved, these were not calculated using MCMC for the present paper. However, using the Pearson correlation matrix and polychoric correlation matrix, together with Equation 11, one obtains estimates of 0.999 for both matrices.

Example: Drift-Diffusion Model Continuous Performance Task Estimates. Recent work on reliability of individual difference measures from cognitive tasks has highlighted the importance of reliability as well as the challenges sometimes encountered in its estimation. Cognitive task measures designed for experimental group comparisons often decrease individual variation, and by extension, reliability (Parsons, Kruijt, & Fox, 2019;

Rouder, Kumar, & Haff, 2019); the structure of such tasks — often involving blocks and other design features to isolate processes — can also create difficulties in estimating reliability, as individual trial responses may violate equivalence assumptions underlying many traditional methods of estimation (Green, et al., 2016).

Drift-diffusion models have been used with a wide variety of cognitive tasks to represent data having the form of a response time and dichotomous response (for example, responses of A versus B, true versus false, correct versus incorrect, along with the latency of response). Numerous versions of the drift-diffusion model have been proposed, but in general they represent responses in terms of a process by which information in favor of a response or decision is cumulated; this information cumulation follows a random walk from an initial state toward a decision or response threshold. Drift-diffusion models are parameterized in different ways but the most popular parameterizations involve a drift rate parameter v , which reflects the rate at which information cumulates toward a decision; a boundary parameter a , which reflects the “informational distance” between decision thresholds; a nondecision time parameter t , which reflects the amount of time spent on processes not involved in the decision per se (e.g., psychomotor or perceptual speed); and a bias parameter z , which reflects the degree of predisposition or bias toward one response or another prior to stimulus presentation.

Individual differences in diffusion model parameter estimates on continuous performance tasks have been associated with real-world variables of interest, such as attention-deficit psychopathology (Huang-Pollock, Karalunas, Tam, & Moore, 2012). Quantifying the reliability of such estimates is challenging, however, as little is known about the precise functional relationship between drift-diffusion model estimation error and underlying parameter values. Test-retest reliabilities have been reported (Lerche & Voss, 2017; Schubert, Frischkorn, Hagemann, & Voss, 2016), but this does not directly address the precision of estimates with regard to the time of response. Parameter recovery simulations have also been used to estimate parameter-estimate correlations analogous to reliabilities (Shahar, et al., 2019; White, Servant, & Logan, 2018), but these have not necessarily focused on individual differences per se, and depend on an assumption that true values of parameters are known, the falsity of which is the motivation for estimating reliability in the first place.

Littman and Takács (2017) presented data on a continuous performance task including both go-no-go and stop-signal trials, to explore differences in types of inhibitory processes. Details of the task are presented in their paper; respondents ($N=51$) were asked to press keys to indicate the appearance of either a framed circle or framed square (go trials); on some trials, however, the frames appeared as dashed lines, indicating that a key press should be inhibited (no-go and stop-signal trials). In the no-go trials, the inhibitory dashed frame appeared immediately; in stop-signal trials, the inhibitory dashed frame occurred after a delay following initial presentation of a go signal, when frames changed from solid to dashed lines. The task was organized such that

go stimuli comprised the majority of trials, to induce a predisposing “go” response bias. Reliability information was not presented in the paper.

To illustrate use of Lindley reliability, a drift diffusion model was fit to the Littman and Takács (2017) data, using the Bayesian HDDM Python library (Wiecki, Sofer, & Frank, 2013). All four parameters (a , v , t , z) were allowed to vary across respondents. The bias parameter z was maintained constant across trial types to reflect the overall probability of go trials (Gomez, Ratcliff, & Perea, 2007), but the boundary parameter a was varied between the no-go trials and the go and stop-signal trials (that is, the model comprised two a parameters, to reflect the greater uncertainty around initial presentation of a “go” stimulus, which might change in the stop-signal trials).

Results are given in Table 4. As can be seen, the reliabilities of the boundary and bias parameters (0.991, 0.995, and 0.990, respectively) were greater than those of the drift rate and nondecision time parameters (0.734 and 0.457 respectively). Although it is possible the large magnitudes of the boundary and bias parameter reliabilities are inflated, reflecting bias in reliability estimation, they may also be accurate: previous parameter recovery studies of diffusion model parameters have sometimes found similarly large estimates (Shahar, et al., 2019). The boundary and bias parameters are also related in that asymmetric boundaries can be parameterized in terms of symmetric boundaries with bias, and bias parameters are not always estimated (e.g., Wiecki, Sofer, & Frank, 2013).

To this author’s knowledge, this is the first time model-based reliabilities have been estimated for the drift-diffusion model, a key model in representation of cognitive processes. It is interesting to note that previous studies of cross-task and cross-time correlations of diffusion parameter estimates have found lower generalizability of nondecision relative to boundary parameter estimates (Lerche & Voss, 2017; Schubert, Frischkorn, Hagemann, & Voss, 2016). These findings have depended somewhat on the specific task and sample (the relative magnitudes of drift rate parameter generalizabilities, similarly, have varied across tasks and samples; Lerche & Voss, 2017; Schubert, Frischkorn, Hagemann, & Voss, 2016), but they are generally consistent with the reliability estimates here, suggesting that the lower cross-task and test-retest correlations observed for some parameters in previous work might be associated with poorer estimation quality of the parameters within task. These patterns are in line with increasing calls for greater attention to the importance of understanding reliability in interpreting findings in the cognitive literature (Green, et al., 2016; Parsons, Kruijt, & Fox, 2019; Rouder, Kumar, & Haff, 2019).

Example: Reliability of Daily Ratings as Measures of Intraindividual Correlation. The flexibility of Lindley reliability in representing information across a broad range of variables with diverse distributional characteristics suggests novel applications of measurement theory. Lindley reliability can be applied to vector-

valued variables, or even matrix-valued quantities or quantities of arbitrary dimension, with a variety of distributional properties, as long as a probability distribution can be defined.

Intraindividual covariance structures—where each individual has a different covariance structure, generally inferred from intensively longitudinal data—are of increasing interest in various literatures. Questions regarding reliability are relevant to estimates of these structures in the same way as other latent variable estimates, in that they constitute randomly varying inferences about randomly varying quantities of interest. Formulating the reliability of intraindividual covariance structures is difficult within traditional frameworks, because these intraindividual covariance estimates are matrix-valued with relatively complex distributional characteristics; the generalized reliability framework presented here, in contrast, provides a means for doing so.

Consider, for example, the National Survey of Midlife in the United States (MIDUS II) Daily Stress Project, which collected daily mood ratings over the course of eight days on a representative sample of adults in the US. These daily ratings in turn were used to construct daily positive and negative emotion scores (Ryff & Almeida, 2010).

A relatively simple model of these mood scores might assume that the scores for each person can be described using a multivariate normal distribution; specifically, a random effects multivariate normal model, where randomly varying intercept vectors describe the mean intraindividual scores, and a randomly varying covariance matrix describes the variability and relationships between positive and negative mood scores:

$$\mathbf{X}_{it} \sim \text{MVN}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \quad (33)$$

where $\mathbf{X}_{it}$ is the bivariate vector of positive and negative mood scores for person i on day t , and $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are the mean vector and 2x2 covariance matrix of the scores for person i .

In this model, $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are randomly varying across persons, so the reliability of the data or estimates with regard to these random intercepts and covariance matrices is conceptually definable. In traditional reliability theory, it would be difficult or impossible to define this reliability, because the pair of parameters $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ form a tuple of vectors and matrices whose variance is difficult to define. In contrast, because Lindley reliability is defined in terms of the probability distributions constituting the data model, this type of reliability can be defined.

For the purposes of illustration, the model in Equation 33 was applied to the MIDUS daily mood rating scores, and reliability was again estimated using MCMC in STAN. Complete data was available on 742 individuals over the course of the 8 days. For simplicity, the reliability was estimated with regard to the intraindividual correlation implied by the intraindividual covariance matrix, to avoid computing a 5-dimensional

integral for each person. Previous data on the MIDUS mood items (Mroczek & Kolarz, 1998) was used to create priors on the intraindividual intercepts and variances implied by Equation 33, and previous studies of correlations between negative and positive mood were used to create a prior for the intraindividual correlations (Diener & Emmons, 1984; Gill, Bos, Wit, & de Jonge, 2017; Goldstein & Strube, 1994; Watson, 1988; Zheng, Plomin, & von Stumm, 2016). The intraindividual intercepts and variances were assumed to be lognormally distributed ($\log N(2.22, 0.35)$ and $\log N(0.63, 0.57)$ for the positive mood scale intercepts and variances, and $\log N(0.33, 1.04)$ and $\log N(0.53, 0.64)$ for the negative mood scales); the intraindividual correlations were assumed to be normally distributed with a mean of -0.316 and a variance of 0.212 (a scaled beta distribution was tried and resulted in almost identical estimates).

Calculating the reliability of the correlation estimates in this way results in a value of 0.941. This value represents, in a rescaled form, the model-expected Lindley information across individuals, which in the sample ranged from 0 to 4.66, with a mean of 1.38 and variance of 0.455. If the individual Lindley information values are rescaled, one obtains individual-specific reliabilities, which in this sample ranged from 0 to 0.999, with a mean of 0.889 and variance of 0.012 (note that the overall reliability is a rescaling of mean Lindley information, and not equal to the mean individual reliability). These values suggest that in general, for most individuals, the daily measures of positive and negative mood provide adequate estimates of intraindividual correlations, at least relative to traditional reliability benchmarks. It also points to how reliability can be generalized to quantities traditionally outside the scope of classical reliability, including matrix-valued variables such as randomly varying covariance matrices.

Areas for Further Research on Estimation

Additional research is needed to develop estimation methods for information-based reliability estimates, characterize the sampling distribution of those estimates, and develop corrections for positive bias of estimators. The small set of simulations reported here (Table 2) indicate that in the multivariate normal case Lindley reliability estimates are upwardly biased, especially in smaller samples. This is consistent with related research showing that estimates of mutual information between observed variables are susceptible to positive bias (e.g., Paninski, 2003), as are set correlations between observed variables (Cohen & Nee, 1984), and maximal reliability estimates (Aguirre-Urreta, Rönkkö, & McIntosh, 2019). In the case of multivariate reliability, information contained in a set of indicators is likely to be significantly greater than that contained in a single indicator; however, it is also likely that estimates pertaining to a vector of indicators, involving more parameters, are likely to be more susceptible to overfitting than the single-indicator reliability estimates. For reasons such as these, characterizing the nature of any bias involved in the sampling distribution of Lindley reliability estimates is critical to interpreting their observed values in samples. Existing research has

characterized the distribution of closely related estimates, including bias corrections, but in the case of observed variables only (e.g., Cohen & Nee, 1984).

Development of explicit analytic solutions for estimates in specific model scenarios, akin to the explicit solutions for the linear latent variable model in Equation 9, is also important. Explicit analytic solutions might help improve the efficiency of estimation, and might also help better understand the properties of generalized reliability estimates in more diverse settings. Outside of the linear latent variable model, the estimates discussed here were obtained using general Bayesian estimation methods, which do not necessarily lend themselves to detailed characterization of the reliability estimates, their properties, or the consequences of using them. Analytic, closed-form solutions are needed to better address these challenges.

Scale

One important question is how the scale of Lindley reliability relates to that of traditional definitions of reliability under different scenarios, and what factors might influence this relationship. Also, other indices of measurement precision have been based on different scalings of Lindley information, most importantly normalized Lindley information, in which Lindley information is scaled relative to some bound (Chen, Liu, & Xu, 2018; Jonas, 2021; Markon, 2013). As they are all based on the same fundamental information-theoretic quantity, it is important to understand how these indices might differ in magnitude.

Comparisons here suggest that Lindley reliability can be interpreted in much the same way as traditional reliability. For example, the Lindley reliabilities in Table 3, although different from traditional reliability estimates, track the latter relatively closely, suggesting traditional interpretation of reliability size is appropriate for interpretation of information-based reliability as well. Nevertheless, understanding factors that influence differences between the two is important.

Various authors have attempted to characterize relationships between the scale of correlational and mutual information-based dependency measures (e.g., Herzel & Große, 1995; Kvålseth, 2017; Li, 1990; Smith, 2015). Kinney and Atwal (2014a) examined these issues in the context of a discussion of equitability, a criterion that an index of statistical dependency assigns “similar scores to equally noisy relationships of different types.” Kinney and Atwal suggest a formalization of equitability they term self-equitability, and note that mutual information satisfies this criterion. As an unnormalized rescaling of mutual information, reliability based on Lindley information would also satisfy this criterion.

Normalized Versus Unnormalized Lindley Information

Some authors (e.g., Joe, 1989) have suggested that mutual information should be normalized by the variables’ entropy in certain cases (for example, with nominal observed variables), to ensure that the upper bound is 1 regardless of the variables’ distributions (note that in the discrete case mutual information is upper

bounded by the minimum of the variables' entropies; that is, $M(\mathbf{Z}, \mathbf{Y}) \leq \min\{H(\mathbf{Y}), H(\mathbf{Z})\}$. In psychometric contexts, authors have suggested indices in which Lindley information is normalized by either the entropy of the observed or latent variables (Chen, Liu, & Xu, 2018; Jonas, 2021; Markon, 2013), or by a minimax entropy implied by the reference prior, and therefore, model (Jonas, 2021; Markon, 2013). These normalized mutual information indices can generally be interpreted as the expected reduction in uncertainty in observed value attained through knowledge of the latent variable value, expressed as a proportion of information.

The consequences of normalization in terms of statistical properties are somewhat unclear, however. In part this is because mutual information is invariant under transformations that normalizing factors (including entropy) are not necessarily invariant under. Kvålseth (2017), for example, discussed a number of normalizing methods, and illustrated that they can lead to problematic scaling outside the boundaries of 0 and 1. Kinney and Atwal (2014), as part of their discussion of self-equitability, similarly note that normalizing mutual information by entropy can lead to violations of the data processing inequality and incoherent orderings. Similar issues have been raised by other authors (e.g., Bell, 1962).

It is also important to consider the meaning of a value of 1 on a $[0,1]$ scale, and whether Lindley information should be scaled differently depending on the scale of the observed variables, as is implied by normalization. For example, a nominal variable with more values will have a greater possible entropy than one with fewer values, affording greater Lindley information, but those values will also be more difficult to guess correctly. The argument for normalizing Lindley information assumes that a value of 1 should index the strength of relationship or amount of information relative to the maximum information afforded by the variables involved, rather than information on some absolute scale.

It is important to note that there is precedent for indices of dependence that cannot always attain values of 1 depending on the distribution of the variables, kappa being an important example. In the case of kappa, practical upper bounds less than 1 occur with low base rate phenomena; although one could argue that kappa should always be upper bounded at 1, upper bounds less than 1 reflect the difficulty of predicting one low base rate variable from another, even when the functional relationship between the latent and observed variables is strong. Similar phenomena likely arise with Lindley information, in that the magnitude of Lindley information is determined not only by the likelihood function, but also the prior.

Overall, further research is needed to address questions about scaling with different scales of measurement, and normalizing of Lindley information. Compelling arguments have been made for and against normalization in certain scenarios (Joe, 1989; Kinney & Atwal, 2014), and more research is needed to understand the consequences of doing so. In an important sense, the Lindley-based reliability discussed here is expressed in absolute units, but the implications of this requires more study.

Univariate Versus Multivariate Reliability

Also important to address is the scale of univariate versus multivariate reliability. The values of multivariate reliability in Table 1 are relatively large compared to traditional univariate reliability standards. Although the results of the simulations in Table 2 indicate that there is positive estimation bias in the multivariate reliabilities, the population values also increased relatively quickly as well, suggesting that estimation bias cannot account for all of the relatively large magnitudes observed.

Another possibility is that increases in dimensionality affect the effective information attained during measurement. Figure 2 illustrates this effect of dimensionality: it presents the multivariate reliabilities for orthogonal factor models as the number of factors increases, for measures with 3 standardized observed indicators per factor, each having the same loading, for various values of loadings. As can be seen, the multivariate reliability increases as the number of factors increases. The limit of 1 is reached much more rapidly when loadings are larger, but the multivariate reliability increases to relatively high levels even when the loadings per factor are relatively small by traditional standards.

That large multivariate reliabilities might be obtained even when the loadings for each factor are relatively small might seem counterintuitive. However, it is important to remember that the distance between two points in latent space increases with each additional dimension, even if the marginal variance in each factor remains the same. For example, with 1 dimension, the expected Euclidean distance between two points in a standard normal distribution is 1.13; with 2 dimensions, 1.77; with 3 dimensions, 2.26; with 4 dimensions, 2.66. From this perspective, for a given level of marginal information about each dimension (in the form of loadings, for example), the ability to distinguish between two points in the multivariate space increases with increasing dimension, because the distance between the points is increasing.

These phenomena suggest that some care is needed in how multivariate reliability is interpreted vis-a-vis traditional univariate reliability guidelines. Reliability can be seen as an index of how distinguishable different points are in latent space via the observed measures (Myung, Balasubramanian, & Pitt, 2000). One influence on this is the magnitude of the loadings, generally speaking, but the structure of the latent space (e.g., Equations 20-22) and dimensionality of the space will also influence distinguishability. Further research is needed to clarify these issues.

Implications for Interpretation of Reliability and Measurement Precision

Traditionally, reliability, and measurement precision more generally, has been interpreted in terms of latent and observed variable variances and covariances: the ratio of latent to observed variance, observed variable variance conditional on latent value, and the covariance between observed and latent variables. However, by recasting reliability in terms of probabilities, which can be understood in multiple ways, the results presented

here suggest that reliability and measurement precision have a plurality of possible interpretations. Lindley information, and by extension, reliability, can be interpreted in novel ways within frequentist, Bayesian, and information-theoretic paradigms.

Frequentist Interpretation of Measurement Precision and Reliability

Frequentist inference has formed the basis for most interpretations of measurement precision and reliability. Different frequentist interpretations of measurement precision have been explicated (e.g., Holland, 1990) but measurement precision has generally been discussed in terms of variation in observed scores, either conditional on a latent variable, or relative to latent variable variance.

The identities discussed here (Equations 3-4, 9, 11) reinforce this type of interpretation, but also point to novel frequentist interpretations as well. Specifically, reliability can be thought of as a rescaled expected likelihood ratio test value. If the measurement model is true, the Lindley information is equal to the expected value of the log-likelihood ratio comparing a model of dependence between the observed and latent variables to a model of independence (Equation 4). This expected log-likelihood ratio is that of a single datum, and the reliability is a rescaling. For example, a reliability of 0.80 corresponds to an expected per-datum likelihood ratio of 2.24 in comparing the measurement model to an independence model.

Bayesian Interpretation of Measurement Precision: Latent Distributions as Priors

The examples discussed here reinforce that reliability can be interpreted within a Bayesian paradigm as well. Specifically, as a function of Lindley information, reliability reflects how much a measure has changed, or is expected to change, posterior trait estimates relative to the prior estimates. Reliability indexes the information provided by a measure about a trait distribution relative to that provided by a given prior; the global minimax bound in Equations 31-32 indexes the maximum such information one could expect from the measure.

Interpretation of the assumed latent distribution. Measurement theory has often been characterized by a certain dualism or ambiguity in interpreting latent trait densities as either priors or known distributions (for example, in the use of marginal maximum likelihood trait estimation, which has origins in Bayesian theory). This dualism is extended here, in that one can interpret the trait density $P(\theta)$ in the Lindley information as either a prior or assumed distribution. This points to certain parallels in the form of reliability discussed here and traditional reliability: both are proportional to a ratio, of an observed distribution after measurement, to an assumed distribution before measurement. In the traditional case, this ratio is of observed and latent variances; in the Bayesian case, this ratio involves posterior and prior probabilities.

Individual- or response pattern-specific reliability. A Bayesian interpretation of reliability also points to new potential applications of reliability. Historically there has been recurring interest in the notion of individual-specific reliabilities (i.e., conditional reliability; Nicewander, 2018; Raju, Price, Oshima, & Nering, 2007; Thissen,

2000), but this application has been complicated by frequentist interpretations of latent variance, in that the meaning of latent variance is unclear when referring to a specific person or trait value (at least with interindividual phenomena; intraindividual variation is different, but then pertains to intraindividual reliability).

Here, though, the reliability-relevant latent trait density can be seen as a prior, and by extension, so can the latent variance in traditional definitions of reliability. From this perspective, then, reliability can be seen to apply to trait estimation of individual persons (or, more precisely, response patterns), if the relevant latent distributional quantities are seen in terms of priors. Reliability could be used to index measurement precision in individual-specific measurement scenarios, such as computerized adaptive testing, similar to standard error of measurement. A response-pattern-specific reliability is easily derived from Equations 3 and 4 by considering the case where the probability of the observed response pattern is 1 (i.e., $p(\mathbf{X}) = 1$ in Equation 4). In contrast to the conditional standard error of measurement, this conditional reliability incorporates information from the prior, thereby providing an index of how much change in estimation has occurred, although depending on the scenario, this might be advantageous or disadvantageous. It is also conditional on an observed response pattern, which is known, rather than a latent trait value, which is unknown. Further research is needed to determine how well bounds such as those in Equation 27 would apply to the case of response-pattern-specific reliabilities, however, given unselected distributions implied by the prior that might not apply when response patterns are selected individually.

Descriptivist Interpretation of Measurement Precision

Couching reliability in terms of probabilities extends conceptualizations of measurement precision to information-theoretic terms as well. Specifically, it illustrates how measurement precision can be seen as the degree to which a trait estimate, as a model of responses, is expected to increase or decrease the parsimony with which data are represented.

In descriptivist accounts (Grunwald, 2007; Jonas & Markon, 2016; Rissanen, 2007), for instance, inference is seen as a process of identifying a model that is optimally parsimonious, in the sense that the data and model can be represented by with the shortest code possible. Consider a hypothetical dataset of 100 responses to a vocabulary item, for example, recorded as correct or incorrect. One possible representation of these responses is a binary code of 100 digits. This representation then comprises a perfectly accurate but complex model requiring substantial information to describe, especially if one considers that a different model would be required for any future sample. Another representation, however, could be the probability of giving a correct response, together with the number of responses. This would be less accurate as a representation, resulting in some misspecified data to be encoded uniquely, but the model itself would require very little data to encode, and would be transferable to new samples.

It can be shown that in order to minimize the total information required to encode data, the optimal codelength for a datum is proportional to the inverse of its probability, specifically $-\ln(p(\mathbf{x}))$. Maximizing the model likelihood or posterior probability also maximizes parsimony, in the sense that it minimizes the information required to represent the data.

Similarly, as an expected difference in log-probabilities, the Lindley information (Equation 4) can be interpreted as an expected difference in codelengths: specifically, it can be interpreted as the expected decrease in information required to encode data using the measurement model, as opposed to a model where the data and trait are unrelated.

In this paradigm, reliability can be seen as an index of how much a trait estimate, as part of a corresponding measurement model, is expected to increase parsimony of representation relative to simply representing the data without a trait. Greater reliability implies that the trait estimate is more parsimonious or efficient as a representation of data; lower reliability implies that the trait estimate is less parsimonious as a data representation.

This descriptivist account extends to interpretations of reliability in terms of repeated measurements or generalizability as well. To the extent that random observations under a latent variable model can be seen as repeated measurements of individuals (Holland, 1990), reliability reflects expected information savings or parsimony in representing individuals, using a particular latent variable estimate from a particular measure, over repeated measurements. This link between generalizability and parsimony has been noted in many contexts (e.g., Grunwald, 2007), and is hinted at in research linking sufficiency and maximal reliability using a repeated measurements paradigm (Knott & Bartholomew, 1993; Thompson, 1940; Zhang, Liao, & Sun, 2011).

The Lindley information-based reliability described here integrates these three perspectives on measurement precision—frequentist, Bayesian, and descriptivist—and provides a mechanism by which precision can be interpreted in different ways. It also provides novel interpretations and applications of reliability even as it links to traditional formulations.

Appendix

The inequality in Equation 27 can be shown to approach an equality in cases where the error approaches zero and is normal in distribution. Wei and Stocker (2016) give a proof for the unidimensional case that can be applied straightforwardly to the multidimensional case. Assume a generalized version of Equation 8,

$$\mathbf{X}_{ij} = f(\boldsymbol{\theta}_i) + \boldsymbol{\epsilon}_{ij} \quad (\text{A1})$$

where f is an invertible function, and the errors $\boldsymbol{\epsilon}$ are additive but not necessarily normally distributed, having smooth density $q(\boldsymbol{\epsilon})$. Write $f(\boldsymbol{\theta}) = \boldsymbol{\vartheta}$. Then $p(\mathbf{X}|\boldsymbol{\vartheta}) = q(\mathbf{X} - \boldsymbol{\vartheta})$, and the Fisher information with regard to $\boldsymbol{\vartheta}$,

$$I(\boldsymbol{\vartheta}) = \int [\nabla_{\boldsymbol{\vartheta}} \ln p(\mathbf{X}|\boldsymbol{\vartheta})]^2 p(\mathbf{X}|\boldsymbol{\vartheta}) d\mathbf{X} \quad , \quad (\text{A2})$$

can be expressed as

$$I(\boldsymbol{\epsilon}) = \int [\nabla_{\boldsymbol{\epsilon}} \ln q(\boldsymbol{\epsilon})]^2 q(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon} \quad . \quad (\text{A3})$$

Analogously, the entropy $H(\mathbf{X}|\boldsymbol{\vartheta})$ can be reexpressed as $H(\boldsymbol{\epsilon})$.

Stam's inequality (also known as the isoperimetric entropy inequality; Dembo, Cover, & Thomas, 1991; Stam, 1959) indicates that for a given Fisher information (or its determinant or trace, in the multidimensional case; Park, Serpedin, & Qaraqe, 2013), the entropy is minimized if and only if the variable is normally distributed (note, though, that given a set of first and second moments, the Fisher information is minimized by a normal distribution; Park, Serpedin, & Qaraqe, 2013). This implies

$$H(\boldsymbol{\epsilon}) \geq \frac{k}{2} \ln \left[\frac{2\pi e}{|I(\boldsymbol{\epsilon})|} \right] \quad (\text{A4})$$

and an index of the nonnormality of the error distribution can be defined as

$$D_0 = H(\boldsymbol{\epsilon}) - \frac{k}{2} \ln \left[\frac{2\pi e}{|I(\boldsymbol{\epsilon})|} \right] \quad (\text{A5})$$

with $D_0 \geq 0$, being 0 when the error is normal and increasing with deviations from normality.

Wei and Stocker (2016) note that, because f is invertible, the Lindley information between $\mathbf{X}$ and $\boldsymbol{\theta}$, $\iota(\mathbf{X}, \boldsymbol{\theta})$, is equal to $\iota(\mathbf{X}, \boldsymbol{\vartheta})$. Given this, and that the entropy $H(\mathbf{X}|\boldsymbol{\vartheta})$ can be reexpressed as $H(\boldsymbol{\epsilon})$, $\iota(\mathbf{X}, \boldsymbol{\theta})$ can be expressed as

$$\iota(\mathbf{X}, \boldsymbol{\theta}) = \iota(\mathbf{X}, \boldsymbol{\vartheta}) = H(\mathbf{X}) - \int p(\boldsymbol{\vartheta}) H(\boldsymbol{\epsilon}) d\boldsymbol{\vartheta} \quad (\text{A6})$$

Equation A6 can in turn can be reexpressed as

$$\iota(\mathbf{X}, \boldsymbol{\theta}) = \iota(\mathbf{X}, \boldsymbol{\vartheta}) = H(\mathbf{X}) - \int p(\boldsymbol{\vartheta}) \left[\frac{k}{2} \ln \left[\frac{2\pi e}{|I(\boldsymbol{\epsilon})|} \right] + D_0 \right] d\boldsymbol{\vartheta} \quad (\text{A7})$$

Substituting $I(\boldsymbol{\vartheta})$ for $I(\boldsymbol{\epsilon})$ (Equations A2 and A3), and rearranging, one has

$$\begin{aligned} \iota(\mathbf{X}, \boldsymbol{\theta}) &= \iota(\mathbf{X}, \boldsymbol{\vartheta}) \\ &= (H(\mathbf{X}) - H(\boldsymbol{\vartheta})) + H(\boldsymbol{\vartheta}) - \int p(\boldsymbol{\vartheta}) \frac{k}{2} \ln \left[\frac{2\pi e}{|I(\boldsymbol{\vartheta})|} \right] d\boldsymbol{\vartheta} - D_0 \end{aligned} \quad (\text{A8})$$

Using standard formulas for the entropy and Fisher information under a transformation, Wei and Stocker (2016) note that

$$H(\boldsymbol{\vartheta}) - \int p(\boldsymbol{\vartheta}) \left[\frac{k}{2} \ln \frac{2\pi e}{|I(\boldsymbol{\vartheta})|} \right] d\boldsymbol{\vartheta} = H(\boldsymbol{\theta}) - \int p(\boldsymbol{\theta}) \left[\frac{k}{2} \ln \frac{2\pi e}{|I(\boldsymbol{\theta})|} \right] d\boldsymbol{\theta} \quad (\text{A9})$$

Using this, Equation A8 can be reexpressed as

$$\begin{aligned} \iota(\mathbf{X}, \boldsymbol{\theta}) &= \iota(\mathbf{X}, \boldsymbol{\vartheta}) \\ &= (H(\mathbf{X}) - H(\boldsymbol{\vartheta})) + H(\boldsymbol{\vartheta}) - \int p(\boldsymbol{\vartheta}) \frac{k}{2} \ln \left[\frac{2\pi e}{|I(\boldsymbol{\vartheta})|} \right] d\boldsymbol{\vartheta} - D_0 \\ &= H(\boldsymbol{\theta}) - \int p(\boldsymbol{\theta}) \frac{k}{2} \ln \left[\frac{2\pi e}{|I(\boldsymbol{\theta})|} \right] d\boldsymbol{\theta} + C_0 - D_0 \end{aligned} \quad (\text{A10})$$

with $C_0 = H(\mathbf{X}) - H(\boldsymbol{\vartheta})$. $C_0 \geq 0$ and will increase with the magnitude of the error, as $C_0 = H(\mathbf{X}) - H(\boldsymbol{\vartheta}) = H[f(\boldsymbol{\theta}) + \boldsymbol{\epsilon}] - H[f(\boldsymbol{\theta})]$.

Note that C_0 represents the magnitude of the error, and D_0 represents the nonnormality of the error. As the error decreases and C_0 decreases to zero at the same time that the error distribution becomes more normal and D_0 decreases to zero, the inequality of Equation 27 implied by Equation A10 becomes an equality. In this way, for normal, low-error measurement scenarios, it is more accurate to treat the right side of Equation 27 as a sort of approximation to Lindley information, rather than a lower bound. Under these conditions, as the average measurement information increases, its relationships with Lindley information and reliability increase as well.

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Footnotes

1. I thank William Revelle for noting the connection to set correlation, as a reviewer.

Table 1

Reliabilities Based on Normal-Distribution Models of Published Covariance Matrices

Dataset	N	k	ρ	ρ_{tot}	ω	ω_{tot}	α
Gnambs & Staufienbiel	76473	12	0.873	0.957	0.866	0.867	0.849
Gnambs & Schroeders	109988	10	0.851	0.980	0.754	0.893	0.866
Gosling, et al.	1799	10	-	0.993	-	-	-
Thurstone-Brigham	4175	9	0.913	0.966	0.887	0.915	0.902
Harman-Holzinger	696	9	0.931	0.998	0.750	0.936	0.886

Note. Values in table are sample size (N); number of parameters in the factor model (k); the generalized reliability with regard to a general factor (ρ); and the generalized multivariate reliability computed with regard to all the factors (ρ_{tot}). The sixth and seventh columns are values of ω , calculated with regard to the general factor (ω) and all factors (ω_{tot}). The last column is Cronbach's α . Cells with hyphens are not applicable because of the model used for the covariance matrix of that row.

Table 2

Estimation Bias of Multivariate Normal Reliabilities

<i>N</i>	<i>3 Factors</i>			<i>5 Factors</i>		
	<i>0.35</i>	<i>0.50</i>	<i>0.75</i>	<i>0.35</i>	<i>0.50</i>	<i>0.75</i>
150	0.312	0.074	0.001	0.172	0.028	0.000
250	0.267	0.040	0.001	0.166	0.020	0.000
500	0.173	0.017	0.000	0.138	0.009	0.000
1000	0.063	0.007	0.000	0.074	0.004	0.000

Note. Values in tables are the mean difference between the population multivariate normal reliabilities (Equation 9) and the sample estimated multivariate reliabilities, for each of the sample size, loading, and numbers of factors conditions simulated. 1000 samples were simulated for each condition assuming a standard multivariate normal distribution. The true multivariate reliabilities were, for the 3-factor conditions, 0.650, 0.875, and 0.99 for factor loadings of 0.35, 0.50, and 0.75; and 0.826, 0.969, and 0.9996 for the 5-factor conditions.

Table 3

Reliabilities Based on MCMC Analysis of Item Data

	N	ρ_m	ρ_e	ω_{mar}	ω_{pear}	ω_{poly}
<i>LSAT</i>						
Figure Classification	1000	0.306	0.296	0.281	0.300	0.480
Debate	1000	0.422	0.458	0.378	0.460	0.640
<i>GSS</i>						
2014	1675	0.572	0.716	0.694	0.710	0.860
2004	1472	0.673	0.756	0.722	0.760	0.890
1994	1977	0.800	0.818	0.784	0.810	0.930
1984	1473	0.824	0.826	0.785	0.810	0.920
<i>IPIP Big Five</i>						
Agreeableness	2709	0.785	0.765	0.720	0.720	0.770
Conscientious	2707	0.714	0.732	0.722	0.730	0.770
Extraversion	2713	0.747	0.757	0.756	0.760	0.800
Neuroticism	2694	0.960	0.839	0.830	0.820	0.840
Openness	2726	0.691	0.720	0.635	0.620	0.680

Note. Values in tables are sample size (N), generalized reliability estimates calculated using model-based marginal probabilities (ρ_m , using model-based estimates of $p(\mathbf{X})$), generalized reliability estimates calculated using empirical marginal probabilities (ρ_e , using empirical frequencies of $\mathbf{X}$ to estimate $p(\mathbf{X})$), and ω calculated based on item response theory (ω_{mar}), as well as factor models of Pearson correlation matrices (ω_{pear}) and polychoric correlational matrices (ω_{poly}). The multivariate reliability for the entire IPIP item set, considered with reference to the set of five factors, was 0.999 using either the Pearson or polychoric correlation matrix.

Table 4

Reliabilities of Drift-Diffusion Individual Difference Parameters

	μ_{EAP}	σ^2_{EAP}	ρ
a_n	1.417	2.276	0.991
a_g	1.329	2.662	0.995
v	2.039	7.834	0.734
t	0.293	0.002	0.457
z	0.505	0.006	0.990

Note. Values in tables are mean (μ), variance (σ^2), and estimated Lindley reliability (ρ) of the expected a posteriori (EAP) estimates for the individually varying boundary parameters (a), drift rate (v), nondecision time parameter (t), and bias parameter (z). a_n and a_g refer to boundary parameters for trials with an initial no-go stimulus, and those with an initial go stimulus (go and stop-signal trials).

Figure 1. Log-posterior (black solid line), log-prior (grey solid line), weighted difference between the log-posterior and log-prior (dashed line), and posterior density (dotted line) for two individuals (Persons 201 and 205) in the IPIP Neuroticism dataset. The weighted log-posterior-log-prior difference was calculated by weighting the difference (or, alternatively, log-ratio) by the posterior as in the integrand of Equation 2. Note the difference in θ range in the two plots. Lindley information is proportional to the difference in $\ln P(\theta|\mathbf{X})$ and $\ln P(\theta)$, integrated over the posterior density, and then averaged over response patterns (e.g., across individuals). Reliability is a rescaling of the Lindley information in a 0, 1 metric. Lindley information is 0.646 for the individual on the left, and 1.293 for the individual on the right.

Figure 2. Multivariate Lindley reliability as a function of number of factors and loading size. Multivariate Lindley reliabilities were calculated assuming standardized multivariate normal latent and observed distributions with 1 to 10 orthogonal factors, and 3 observed indicators per factor. All loadings were equal across factors, set at values between 0.25 and 0.95 per condition.